Prépublications du Département de Mathématiques

Université de La Rochelle
Avenue Michel Crépeau
17042 La Rochelle Cedex 1
http://www.univ-lr.fr/Labo/MATH

Tanaka formula for the fractional Brownian motion

Laure Coutin, David Nualart et Ciprian A. Tudor

Septembre 2000

Classification: 60G15, 60H07, 60H05, 60J65, 60F25
Mots clés: Fractional Brownian motion, local time, Tanaka formula.

2000/07

Tanaka formula for the fractional Brownian motion

Laure Coutin*
Laboratoire de Statistique et de Probabilités
Universite Paul Sabatier
118 route de Narbonne
31062, Toulouse, France

David Nualart*,†
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via 585, 08007, Barcelona
Spain

Ciprian A. Tudor
Départament de Mathématiques
Université de La Rochelle
Avenue Michel Crépau
17042 La Rochelle Cedex 1
France

September 6, 2000

1 Introduction

The fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B = \{B_t, t \geq 0\}$ with the covariance function (see [13])

$$E(B_t B_s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{1}$$

*Supported by the grant ERBF MRX CT960075A from the European Union
†Supported by the grant PB960087 from the DGES

There has been a recent development in the stochastic calculus with respect to this process. Different approaches have been used to define stochastic integrals and to establish change-of-variable formulas (see, among others, [1, 2, 4, 5, 6, 7, 10, 12, 19]). The purpose of this paper is to establish the following version of Tanaka's formula for the fractional Brownian motion, assuming $H > \frac{1}{3}$:

$$|B_t - a| = |a| + \int_0^t \operatorname{sign}(B_s - a) dB_s + L_t^a. \tag{2}$$

The stochastic integral appearing in this formula coincides the divergence operator with respect to the fBm, and L_t^a is the density of the occupation measure $\Gamma \mapsto 2H \int_0^t 1_\Gamma(B_s) s^{2H-1} ds$. This result extends the classical Tanaka's formula for the Wiener process ($H = \frac{1}{2}$), where L_t^a is the local time of the Brownian motion, and the stochastic integral is an Itô integral.

The paper is organized as follows. Section 2 contains some preliminaries about the stochastic calculus with respect to the fractional Brownian motion. In Section 3 we show that the local time, defined formally as

$$L_t^a = 2H \int_0^t \delta_a(B_s) s^{2H-1} ds,$$

exists in $L^2(\Omega)$, and we compute its Wiener chaos expansion. Section 4 is devoted to establish Tanaka's formula and its application to a generalization of Itô's formula to convex functions.

2 Preliminaries

Let $B = \{B_t, t \in [0, T]\}$ be the fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$, that is, B is a zero mean Gaussian process with the covariance function

$$R(t, s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t - s|^{2H}\right).$$

We know that B admits an integral representation of the form

$$B_t = \int_0^t K(t, s) dW_s,$$

where $W = \{W_t, t \in [0,T]\}$ is a Wiener process, and $K(t,s)$ is the kernel (see [2, 6])

$$K(t,s) = c_H\,(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right), \tag{3}$$

c_H being a constant and

$$F_1\,(z) = c_H\left(\frac{1}{2} - H\right)\int_0^{z-1} \theta^{H-\frac{3}{2}}\left(1 - (\theta+1)^{H-\frac{1}{2}}\right) d\theta.$$

This kernel satisfies the condition (see [13]):

$$\frac{\partial K}{\partial t}(t,s) = c_H(H - \frac{1}{2})(\frac{s}{t})^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}. \tag{4}$$

It is possible to construct a stochastic calculus of variations with respect to the Gaussian process B, which will be related to the Malliavin calculus with respect to the Wiener process W. We refer to [2] for a complete exposition of this subject. We recall here the basic definitions and results of this calculus.

Let $I_n(f_n)$ denote the multiple stochastic integral of a symmetric kernel $f_n \in L^2([0,T]^n)$ with respect to the Wiener process W. Given a square integrable random variable $F \in L^2(\Omega)$ with the Wiener chaos expansion $F = \sum_{n=0}^{\infty} I_n(f_n)$, we consider the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n=0}^{\infty} n I_n(f_n).$$

If $p \in (1,\infty)$ and $\alpha \in \mathbb{R}$ we define the Sobolev spaces $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(I-L)^{\frac{\alpha}{2}} F\|_{L^p(\Omega)}.$$

We denote by D the derivative operator, defined on multiple integrals as $D_t(I_n(f_n)) = nI_{n-1}(f_n(\cdot,t))$. The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}(L^2([0,T]))$. The adjoint of D is called the divergence operator, denoted by δ, and it is continuous from $\mathbb{D}^{\alpha,p}(L^2([0,T]))$ into $\mathbb{D}^{\alpha-1,p}$. We denote by Domδ the domain of the divergence in $L^2(\Omega)$. The operator δ defined in Domδ coincides with an extension of the Itô stochastic integral to anticipating processes introduced by Skorohod in [18]. For this reason, the operator δ is also called the Skorohod integral, and denoted by

$$\delta(u) = \int_0^T u_s dW_s.$$

We can also develop a stochastic calculus for the fBm B. Now the basic Hilbert space $L^2([0,T])$ will be replaced by the Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}$ of the fBm, defined as the closure of the linear span of the indicator functions $\{1_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s)$.

We denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Consider the linear operator K_T^* from $\mathcal{E}$ to $L^2([0,T])$ defined by

$$(K_T^* \varphi)(s) = \varphi(s) K(T,s) + \int_s^T [\varphi(t) - \varphi(s)] \frac{\partial K}{\partial t}(t,s) dt.$$

In the case $H > \frac{1}{2}$, this operator can be simply written as

$$(K_T^* \varphi)(s) = \int_s^T \varphi(t) \frac{\partial K}{\partial t}(t,s) dt.$$

This operator satisfies the duality relationship (see Lemma 1 in [2])

$$\int_0^T (K_T^* \varphi)(t) h(t) dt = \int_0^T \varphi(t) (Kh)(dt),$$

for all $\varphi \in \mathcal{E}$ and $h \in L^2([0,T])$, where $(Kh)(t) = \int_0^t K(t,s) h(s) ds$.

As a consequence, the RKHS $\mathcal{H}$ can be represented as the closure of $\mathcal{E}$ with respect to the norm $\|\varphi\|_{\mathcal{H}} = \|K_T^* \varphi\|_{L^2([0,T])}$, and the operator K_T^* is an isometry between $\mathcal{H}$ and a closed subspace of $L^2([0,T])$, that is,

$$\mathcal{H} = (K_T^*)^{-1}(L^2([0,T])). \tag{5}$$

This isometry allows us to establish relationships among the derivative and divergence operators with respect to the processes W and B. We will add a superindex (or subindex) B to denote the spaces and operators for the process B. More precisely, we have:

(i) $\mathbb{D}^{\alpha,p} = \mathbb{D}_B^{\alpha,p}$, and $K_T^* D^B F = DF$, for any $F \in \mathbb{D}^{\alpha,p}$.

(ii) $\mathrm{Dom}\delta^B = (K_T^*)^{-1} (\mathrm{Dom}\delta)$, and $\delta^B(u) = \delta(K_T^* u)$ for any $\mathcal{H}$-valued random variable u in $\mathrm{Dom}\ \delta^B$.

We will make use of the notation $\delta^B(v) = \int_0^T v_s dB_s$ for any $v \in$Dom δ^B. Hence, if $u \in$Dom δ^B, then

$$\int_0^T u_s dB_s = \int_0^T \left(K_T^* u\right)_s dW_s. \tag{6}$$

The random variable $\delta^B(v)$ can be interpreted as a stochastic integral defined as the limit of Riemann sums constructed using Wick products (see [2, 10]).

We recall the following basic lemma on the divergence operator:

Lemma 1 *Let u_n be a sequence of elements in Dom δ^B which converges to u in $L^2(\Omega; \mathcal{H})$. Suppose that $\delta^B(u_n)$ converges in $L^2(\Omega)$ to some square integrable random variable G. Then u belongs to the domain of δ^B and $\delta^B(u) = G$.*

We will denote by c a generic constant that may be different from one formula to another one. Moreover, by convention $K(t,s) = 0$ if $s > t$.

3 The local time of the fBm

Let $B = \{B_t, t \in [0,T]\}$ be the fBm with Hurst parameter $H \in (0,1)$. We define the local time L_t^a of the the process B as the density of the occupation measure

$$m_t(\Gamma) = 2H \int_0^t 1_\Gamma(B_s) s^{2H-1} ds.$$

It is well-known (see Berman [3] and Geman and Horowitz [9]) that the occupation measure $\Gamma \mapsto \int_0^t 1_\Gamma(B_s^H) ds$ has a density λ_t^a which has a continuous version in the variables a and t . More precisely (see Table 2 in [9]), λ_t^a has Hölder continuous paths of order $\delta < 1 - H$ in time, and of order $\gamma < \frac{1-H}{2H}$ in the space variable, provided $H \geq \frac{1}{3}$. Moreover, λ_t^a is absolutely continuous in a if $H < \frac{1}{3}$, it is continuously differentiable if $H < \frac{1}{5}$, and its smoothness in the space variable increase when H decreases. The local time L_t^a appearing in formula (2) is related with λ_t^a by the equality $L_t^a = 2H \int_0^t s^{2H-1} \lambda^a(ds)$, and the Hölder continuity properties of λ_t^a can be transferred to L_t^a. In fact, integrating by parts we can write

$$L_t^a = 2Ht^{2H-1}\lambda_t^a - 2H(2H-1) \int_0^t s^{2H-2} \lambda_s^a ds.$$

Moreover, being the density of the occupation measure, L_t^a is a nondecreasing function of t, and the measure $L^a(dt)$ is concentrated on the level set $\{s : B_s = a\}$.

Let

$$p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}} \tag{7}$$

be the heat kernel with variance $\varepsilon > 0$. We denote by H_n the nth Hermite polynomial defined for $n \geq 1$ by

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}),$$

and $H_0(x) = 1$. Following the arguments of [16], in this section we will show the following result:

Proposition 2 *For each $a \in \mathbb{R}$, and $t \in [0, T]$, the random variables*

$$2H \int_0^t p_\varepsilon(B_s - a) s^{2H-1} ds$$

converge to L_t^a in $L^2(\Omega)$ as ε tends to zero. Furthermore the local time L_t^a has the following Wiener chaos expansion

$$L_t^a = 2H \sum_{n=0}^{\infty} \int_0^t s^{(2-n)H-1} p_{s^{2H}}(a) H_n(\frac{a}{s^H}) I_n\left(K(s,\cdot)^{\otimes n}\right) ds. \tag{8}$$

Proof: Let us first compute the Wiener chaos expansion of $p_\varepsilon(B_s - a)$, for any $s > 0$. Stroock's formula (see [15]) says that any random variable F belonging to the space $\bigcap_{k \geq 1} \mathbb{D}^{k,2}$ has the chaos expansion

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n[E(D^n F)], \tag{9}$$

where D^n denotes the nth iteration of the derivative operator D. We have

$$D^n \left[p_\varepsilon(B_s - a)\right] = p_\varepsilon^{(n)}(B_s - a) K(s,\cdot)^{\otimes n}. \tag{10}$$

By the semigroup property of the heat kernel,

$$E\left(p_\varepsilon(B_s - a)\right) = p_{s^{2H}+\varepsilon}(a). \tag{11}$$

As a consequence, using the reccurence formula for the Gaussian kernel

$$p_\epsilon^{(n)}(x) = (-1)^n n! \epsilon^{-\frac{n}{2}} p_\epsilon(x) H_n(\frac{x}{\sqrt{\epsilon}}),$$

we obtain

$$\begin{aligned} E\left(p_\varepsilon^{(n)}(B_s - a)\right) &= (-1)^n \frac{\partial^n}{\partial a^n} E\left(p_\varepsilon(B_s - a)\right) \\ &= (-1)^n p_{s^{2H}+\varepsilon}^{(n)}(a) \\ &= n! \left(s^{2H} + \epsilon\right)^{-\frac{n}{2}} p_{s^{2H}+\varepsilon}(a) H_n(\frac{a}{\sqrt{s^{2H}+\varepsilon}}). \end{aligned} \quad (12)$$

Hence, applying (9) to $F = p_\epsilon(B_s - a)$ and using (10) and (12) yields

$$p_\epsilon(B_s - a) = \sum_{n=0}^{\infty} \beta_{n,\varepsilon}(s) I_n(K(s,\cdot)^{\otimes n}), \quad (13)$$

where

$$\beta_{n,\varepsilon}(s) = \left(s^{2H} + \epsilon\right)^{-\frac{n}{2}} p_{s^{2H}+\varepsilon}(a) H_n(\frac{a}{\sqrt{s^{2H}+\varepsilon}}). \quad (14)$$

From (13) we deduce the Wiener chaos expansion

$$\int_0^t p_\varepsilon(B_s - a) s^{2H-1} ds = \sum_{n=0}^{\infty} \int_0^t \beta_{n,\varepsilon}(s) I_n\left(K(s,\cdot)^{\otimes n}\right) s^{2H-1} ds.$$

We will prove that this expression multiplied by the factor $2H$ converges in $L^2(\Omega)$, as ε tends to zero to the right-hand side of (8). It is clear that for any $n \geq 0$, the multiple stochastic integral in the above expression converges in $L^2(\Omega)$, as ε tends to zero, to

$$\int_0^t s^{-nH} p_{s^{2H}}(a) H_n(\frac{a}{s^H}) I_n\left(K(s,\cdot)^{\otimes n}\right) s^{2H-1} ds.$$

Set

$$\alpha_{n,\varepsilon} = E\left(\int_0^t \beta_{n,\varepsilon}(s) I_n\left(K(s,\cdot)^{\otimes n}\right) s^{2H-1} ds\right)^2.$$

Then, it suffices to show that

$$\lim_{N\to\infty} \sup_{\varepsilon>0} \sum_{n\geq N} \alpha_{n,\varepsilon} = 0. \quad (15)$$

We have

$$\begin{aligned}\alpha_{n,\varepsilon} &= n!\int_0^t\int_0^t E\left(I_n(K(u,\cdot)^{\otimes n})I_n(K(v,\cdot)^{\otimes n})\right) \\ &\quad \times\beta_{n,\varepsilon}(u)\beta_{n,\varepsilon}(v)(uv)^{2H-1}dvdu \\ &= 2n!\int_0^t\int_0^u R(u,v)^n\beta_{n,\varepsilon}(u)\beta_{n,\varepsilon}(v)(uv)^{2H-1}dvdu. \qquad (16)\end{aligned}$$

We recall that

$$H_n(y)e^{-y^2/2} = (-1)^{[\frac{n}{2}]}2^{n/2}\frac{2}{n!\sqrt{\pi}}\int_0^\infty s^n e^{-s^2}g(ys\sqrt{2})ds,$$

where $g(r) = \cos r$ for n even, and $g(r) = \sin r$ for n odd. Majorizing $|g|$ by 1, we obtain the estimate

$$\begin{aligned}\left|H_n(y)e^{-y^2/2}\right| &\leq \frac{2^{\frac{n}{2}+1}}{n!\sqrt{\pi}}\int_0^\infty s^n e^{-s^2}ds \\ &\leq \frac{c}{n(n-2)(n-4)...} \\ &\leq \frac{c}{2^{\frac{n}{2}}[\frac{n}{2}]!}.\end{aligned}$$

Substituting this estimate in (14) yields

$$|\beta_{n,\varepsilon}(s)| \leq \frac{c}{2^{\frac{n}{2}}[\frac{n}{2}]!}s^{-(n+1)H},$$

and from (16) we can estimate the term $\alpha_{n,\varepsilon}$ as follows

$$\alpha_{n,\varepsilon} \leq c_n \int_0^t\int_0^u R(u,v)^n\,(uv)^{(1-n)H-1}\,dvdu,$$

where $c_n = \frac{cn!}{2^n\left([\frac{n}{2}]!\right)^2}$.

By the scaling property of the fBm we have $R(u,v) = R(1,\frac{v}{u})u^{2H}$. Hence, making the change of variables $\frac{v}{u} = z$ yields

$$\begin{aligned}\int_0^t\int_0^u R(u,v)^n\,(uv)^{(1-n)H-1}\,dvdu &= \int_0^t\int_0^1 R(1,z)^n\,u^{2H-1}z^{(1-n)H-1}dzdu \\ &= \frac{t^{2H}}{2H}\int_0^1 R(1,z)^n\,z^{(1-n)H-1}dz.\end{aligned}$$

Suppose $H < \frac{1}{2}$. We claim that

$$\frac{R(1,z)}{z^H} = \left(\frac{1 + z^{2H} - (1-z)^{2H}}{2z^H}\right) \le z^H.$$

Indeed, the function $f(z) = 1 - z^{2H} - (1-z)^{2H}$ is nonpositive in $[0,1]$, because $f(1) = f(0) = 0$ and f is increasing in $[1/2,1]$ and decreasing in $[0,1/2]$. Hence,

$$\alpha_{n,\varepsilon} \le \frac{c_n}{2(n+1)H},$$

and (15) holds because, by the Stirling formula, c_n behaves as $\frac{1}{\sqrt{n}}$.

Suppose now $H > 1/2$. In order to show (15) it suffices to check that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_0^1 R(1,z)^n \ z^{(1-n)H-1} dz < \infty. \tag{17}$$

Notice that for any positive numbers $a < 1$ and $1 < p < 2$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a^n \le c_p \left(\sum_{n=1}^{\infty} a^{np}\right)^{1/p} \le c_p \frac{1}{(1-a^p)^{1/p}}.$$

Hence,

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_0^1 R(1,z)^n \ z^{(1-n)H-1} dz \\ \le \ & c_p \int_0^1 \left[1 - \left(R(1,z) z^{-H}\right)^p\right]^{-1/p} z^{H-1} dz < \infty, \end{aligned}$$

provided $p > 2H$, because in a neighbourhood of 1 the function

$$\left[1 - \left(R(1,z) z^{-H}\right)^p\right]^{-1/p}$$

behaves as $p(1-z)^{-2H/p}$, and (17) holds.

Finally we have to show that the limit of $2H \int_0^t p_\varepsilon(B_s - a) s^{2H-1} ds$, denoted by Λ_t^a is the local time L_t^a. The above estimates are uniform in $a \in \mathbb{R}$. Therefore, we can conclude that the convergence of $2H \int_0^t p_\varepsilon(B_s - a) s^{2H-1} ds$ to Λ_t^a holds in $L^2(\Omega \times \mathbb{R}, P \times \mu)$, for any finite measure μ. As a consequence, for any continuous function g in $\mathbb{R}$ with compact support we have that

$$2H \int_{\mathbb{R}} \left(\int_0^t p_\varepsilon(B_s - a) s^{2H-1} ds\right) g(a) da$$

converges in $L^2(\Omega)$ to $\int_{\mathbb{R}} \Lambda_t^a g(a)da$. But, this expression also converges to

$$2H\int_0^t g(B_s)s^{2H-1}ds.$$

Hence,

$$\int_{\mathbb{R}} \Lambda_t^a g(a)da = 2H\int_0^t g(B_s)s^{2H-1}ds,$$

which implies that $\Lambda_t^a = L_t^a$.□

Remark 1

In the particular case $a=0$, we obtain the Wiener chaos expansion

$$L_t^0 = \sum_{k=0}^{\infty}\int_0^t s^{(1-2k)H-1}\frac{\sqrt{2}H(-1)^k}{\sqrt{\pi}2^k k!} I_{2k}\left(K(s,\cdot)^{\otimes 2k}\right) ds.$$

Remark 2

As in the paper [16] we can introduce the composition $\delta_a(B_s)$, which is a distribution on the Wiener space in the sense of Watanabe. Actually it belongs to the negative Sobolev space $\mathbb{D}^{-1,2}$, and it has the Wiener chaos expansion

$$\delta_a(B_s) = \sum_{n=0}^{\infty} s^{-nH} p_{s^{2H}}(a) H_n(\frac{a}{s^H}) I_n\left(K(s,\cdot)^{\otimes n}\right).$$

Then, the local time can be formally written as

$$L_t^a = 2H\int_0^t \delta_a(B_s)s^{2H-1}ds.$$

Remark 3

If $H<\frac{1}{2}$, the proof of Proposition 1.1 shows that $\int_0^t \delta_a(B_s)s^{2H-1}ds$ belongs to the space $\mathbb{D}^{\alpha,2}$, for any $\alpha < 1/2$.

4 Tanaka formula for the fractional Brownian motion

Using the stochastic calculus for the fractional Brownian motion developed in the paper [2] we are able to deduce a Tanaka formula. We will consider first the case where the Hurst parameter is larger than $\frac{1}{2}$.

4.1 Case $H > \frac{1}{2}$

Theorem 3 *Let $B = \{B_t, t \in [0,T]\}$ a fBm with parameter $H > \frac{1}{2}$. Then*

$$|B_t - a| = |a| + \int_0^t \operatorname{sign}(B_s - a) dB_s + L_t^a. \tag{18}$$

Proof: Consider the heat kernel $p_\varepsilon(x)$ introduced in (7) and define

$$F'_\varepsilon(x) = 2\int_{-\infty}^x p_\varepsilon(y)dy - 1,$$

and

$$F_\varepsilon(x) = \int_0^x F'_\varepsilon(y)dy.$$

Notice that

$$F'_\varepsilon(x) = \left(1 - 2\int_{|x|/\sqrt{\varepsilon}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz\right) \operatorname{sign} x. \tag{19}$$

Then $F'_\varepsilon(x)$ converges to $\operatorname{sign}(x)$, and $F_\varepsilon(x)$ converges to $|x|$ as ε tends to zero. By Itô's formula of [2] we can write, for any fixed $a \in \mathbb{R}$

$$\begin{aligned} F_\varepsilon(B_t - a) &= F_\varepsilon(-a) + \int_0^t K_t^*\left[F'_\varepsilon(B_s - a)\right] dW_s \\ &\quad + 2H\int_0^t p_\varepsilon(B_s - a) s^{2H-1} ds, \end{aligned} \tag{20}$$

where

$$K_t^*\left[F'_\varepsilon(B_s - a)\right] = \int_s^t F'_\varepsilon(B_r - a)\frac{\partial K}{\partial r}(r,s)dr.$$

Clearly $F_\varepsilon(B_t - a)$ converges to $|B_t - a|$ as ε tends to zero in $L^2(\Omega)$. We claim that

$$\left\{K_t^*\left[F'_\varepsilon(B_s - a)\right], s \in [0,t]\right\}$$

converges in $L^2([0,t] \times \Omega)$ to $\{K_t^*[\operatorname{sign}(B_s - a)], s \in [0,t]\}$. In fact, for all $a \in \mathbb{R}$, and for all $t \geq s$ we have, by Jensen's inequality and the estimate (4)

$$
\begin{aligned}
& E\int_0^t \left(\int_s^t \left|F_\varepsilon'(B_r-a)-\mathrm{sign}(B_r-a)\right| \left|\frac{\partial K}{\partial r}(r,s)\right| dr\right)^2 ds \\
\leq \;& cE\int_0^t \left(\int_s^t \left|F_\varepsilon'(B_r-a)-\mathrm{sign}(B_r-a)\right|^2 (r-s)^{H-\frac{3}{2}} dr\right) ds \\
\leq \;& cE\int_0^t \left|F_\varepsilon'(B_r-a)-\mathrm{sign}(B_r-a)\right|^2 dr,
\end{aligned}
$$

which clearly converges to zero as ε tends to zero.

The proof follows taking the limit in $L^2(\Omega)$ as $\epsilon \to 0$ in the equation (20), and using Lemma 1 and Proposition 2. □

In the same way we can prove the following additional versions of Tanaka's formula:

$$
(B_t-a)^+ = (-a)^+ + \int_0^t 1_{(a,\infty)}(B_s)dB_s + \frac{1}{2}L_t^a, \tag{21}
$$

$$
(B_t-a)^- = (-a)^- - \int_0^t 1_{(-\infty,a)}(B_s)dB_s + \frac{1}{2}L_t^a. \tag{22}
$$

4.2 Case $\frac{1}{3} < H < \frac{1}{2}$

We begin with the following technical estimation.

Lemma 4 *Fix $0 < s < t \leq T$ and $a \in \mathbb{R}$. Then*

$$
P(B_t < a, B_s > a) \leq C_{T,a}(t-s)^H s^{-2H}, \tag{23}
$$

where $C_{T,a} = \left(\frac{T^H}{\sqrt{2\pi}} + \frac{\sqrt{2}|a|}{\sqrt{\pi}}\right)$.

Proof: We will make use of the decomposition

$$
B_t = \frac{R(t,s)}{\sqrt{R(s,s)}}X + \sigma Y,
$$

where $X = \frac{B_s}{\sqrt{R(s,s)}}$ and

$$
\sigma^2 = \frac{R(t,t)R(s,s)-R(t,s)^2}{R(s,s)}.
$$

Notice that X and Y are independent $N(0,1)$ random variables. As a consequence,

$$P(B_t < a, B_s > a) = \frac{1}{2\pi} \int_{\mathbb{R}^2} 1_{\{\sqrt{R(s,s)}x > a\}} 1_{\{\sigma y + \frac{R(t,s)}{\sqrt{R(s,s)}} x < a\}} e^{-\frac{x^2+y^2}{2}} dxdy.$$

We make the change of variables

$$x = r\cos\theta + \frac{a}{\sqrt{R(s,s)}},$$

$$y = r\sin\theta + \frac{a}{\sigma}\left(1 - \frac{R(s,t)}{R(s,s)}\right)$$

and we obtain

$$P(B_t < a, B_s > a) = \frac{1}{2\pi} \int_0^\infty r\ dr \int_{-\frac{\pi}{2}}^{\arg\tan - \frac{R(s,t)}{\sigma\sqrt{R(s,s)}}} e^{-\frac{1}{2}P(r,\theta)} d\theta, \qquad (24)$$

where

$$P(r,\theta) = \left(r\cos\theta + \frac{a}{\sqrt{R(s,s)}}\right)^2 + \left(r\sin\theta + \frac{a}{\sigma}\left(1 - \frac{R(s,t)}{R(s,s)}\right)\right)^2.$$

Let

$$\rho^2 = \frac{1}{R(s,s)} + \frac{1}{\sigma^2}\left(1 - \frac{R(s,t)}{R(s,s)}\right)^2,$$

and consider a $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$\begin{aligned} \rho\cos\psi &= \frac{1}{\sqrt{R(s,s)}}, \\ \rho\sin\psi &= \frac{1}{\sigma}\left(1 - \frac{R(s,t)}{R(s,s)}\right). \end{aligned}$$

With this notation we can write

$$\begin{aligned} P(r,\theta) &= r^2 + a^2\rho^2 + 2ar\rho\cos(\theta - \psi) \\ &\geq (r - |a|\rho)^2, \end{aligned}$$

and we obtain

$$\int_0^\infty re^{-\frac{1}{2}P(r,\theta)}dr \leq \int_0^\infty re^{-\frac{1}{2}(r-|a|\rho)^2}dr \qquad (25)$$
$$\leq e^{-\frac{1}{2}(|a|\rho)^2} + |a|\rho\sqrt{2\pi} =: c_\rho.$$

Substituting (25) into (24) yields

$$\begin{aligned} P(B_t &< a, B_s > a) \leq \frac{c_\rho}{2\pi}\left(\arg\tan - \frac{R(s,t)}{\sigma\sqrt{R(s,s)}} + \frac{\pi}{2}\right) \\ &= \frac{c_\rho}{2\pi}\left(\frac{\pi}{2} - \arg\tan\frac{R(s,t)}{\sigma\sqrt{R(s,s)}}\right). \end{aligned}$$

Taking into account that for any $x > 0$ we have

$$\frac{\pi}{2} - \arg\tan\frac{1}{x} \leq x,$$

we can deduce the following estimate:

$$\begin{aligned} P(B_t < a, B_s > a) &\leq \frac{c_\rho\sigma\sqrt{R(s,s)}}{2\pi R(s,t)} \\ &= \frac{c_\rho}{2\pi}\sqrt{\frac{R(t,t)R(s,s)}{R(s,t)^2} - 1}. \end{aligned} \qquad (26)$$

We have, using the decomposition $B_t = B_s + B_t - B_s$

$$\begin{aligned} R(t,t)R(s,s) - R(s,t)^2 &= E(B_t^2)E(B_s^2) - (E(B_tB_s))^2 \\ &= E\left((B_t - B_s)^2\right)E(B_s^2) - \left(E\left((B_t - B_s)B_s\right)\right)^2 \\ &\leq E\left((B_t - B_s)^2\right)E(B_s^2), \end{aligned}$$

and

$$R(s,t) \geq \frac{1}{2}s^{2H}.$$

Hence,

$$\sqrt{\frac{R(t,t)R(s,s)}{R(s,t)^2} - 1} \leq \sqrt{2}(t-s)^H s^{-H}. \qquad (27)$$

On the other hand, a simple computation yields

$$\rho^2 = \frac{E\left((B_t - B_s)^2\right)}{R(t,t)R(s,s) - R(s,t)^2},$$

and, consequently,

$$\rho\sqrt{\frac{R(t,t)R(s,s)}{R(s,t)^2} - 1} = \frac{\sqrt{E\left((B_t - B_s)^2\right)}}{R(s,t)} \leq 2(t-s)^H s^{-2H} \qquad (28)$$

Substituting (27) and (28) into (26) we get

$$P(B_t < a, B_s > a) \leq \left(\frac{T^H}{\sqrt{2\pi}} + \frac{\sqrt{2}|a|}{\sqrt{\pi}}\right)(t-s)^H s^{-2H},$$

which completes the proof of the lemma. □

Remark 4

In the case $a = 0$ the proof of the above lemma yields the explicit expression

$$P(B_t < 0, B_s > 0) = \frac{1}{4} - \frac{1}{2\pi} \arg\tan \frac{R(s,t)}{\sqrt{R(t,t)R(s,s) - R(t,s)^2}}.$$

Proposition 5 *Suppose that* $\frac{1}{3} < H < \frac{1}{2}$. *Then, the process*

$$\left\{\int_s^t \left[F_\varepsilon'(B_r - a) - F_\varepsilon'(B_s - a)\right] \frac{\partial K}{\partial r}(r,s)dr, 0 \leq s \leq t\right\}$$

converges in $L^2([0,t] \times \Omega)$ *to the process*

$$\left\{\int_s^t \left[\text{sign}(B_r - a) - \text{sign}(B_s - a)\right] \frac{\partial K}{\partial r}(r,s)dr, 0 \leq s \leq t\right\}$$

as ε *tends to zero.*

Proof: Notice first that

$$E\int_0^t \left(\int_s^t |\text{sign}(B_r - a) - \text{sign}(B_s - a)| \left|\frac{\partial K}{\partial r}(r,s)\right| dr\right)^2 ds < \infty. \qquad (29)$$

Indeed, using the relationship

$$\text{sign}(B_r - a) - \text{sign}(B_s - a) = 2\left(1_{\{B_r > a, B_s < a\}} - 1_{\{B_r < a, B_s > a\}}\right),$$

we obtain that the expectation in (29) is bounded by

$$
\begin{aligned}
&4\int_0^t \left(\int_s^t P(B_r < a, B_s > a)^{1/2}\left|\frac{\partial K}{\partial r}(r,s)\right| dr\right)^2 ds \\
&+4\int_0^t \left(\int_s^t P(B_r > a, B_s < a)^{1/2}\left|\frac{\partial K}{\partial r}(r,s)\right| dr\right)^2 ds, \qquad (30)
\end{aligned}
$$

and applying the estimates (23) and (4), and using that $-B$ has the same distribution than B, we get that (30) is bounded by

$$
c_{T,a}\int_0^t \left(\int_s^t (r-s)^{\frac{3H}{2}-\frac{3}{2}} s^{-H}\, dr\right)^2 ds,
$$

which is finite because $H > \frac{1}{3}$. Thus (29) holds.

Using equation (19) we can write on the set $\{B_r < a, B_s > a\}$

$$
\begin{aligned}
\left[F_\varepsilon'(B_r - a) - \operatorname{sign}(B_r - a)\right] &= 2\int_{|B_r-a|/\sqrt{\varepsilon}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz, \\
\left[F_\varepsilon'(B_s - a) - \operatorname{sign}(B_s - a)\right] &= -2\int_{|B_s-a|/\sqrt{\varepsilon}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz
\end{aligned}
$$

Similarly, on the set $\{B_r > a, B_s < a\}$ we have

$$
\begin{aligned}
\left[F_\varepsilon'(B_r - a) - \operatorname{sign}(B_r - a)\right] &= -2\int_{|B_r-a|/\sqrt{\varepsilon}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz, \\
\left[F_\varepsilon'(B_s - a) - \operatorname{sign}(B_s - a)\right] &= 2\int_{|B_s-a|/\sqrt{\varepsilon}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz\ .
\end{aligned}
$$

Finally, on the set $\{B_r < a, B_s < a\} \cup \{B_r > a, B_s > a\}$ the difference $\operatorname{sign}(B_r - a) - \operatorname{sign}(B_s - a)$ cancels and we have by equation (19)

$$
\left|F_\varepsilon'(B_r - a) - F_\varepsilon'(B_s - a)\right| \le \frac{2}{\sqrt{2\pi}}\left|\int_{\frac{(B_s-a)}{\sqrt{\epsilon}}}^{\frac{(B_r-a)}{\sqrt{\epsilon}}} e^{-\frac{z^2}{2}} dz\right|.
$$

Notice that in the set $\{B_r < a, B_s < a\} \cup \{B_r > a, B_s > a\}$ the quantities $B_r - a$ and $B_s - a$ have the same sign. As a consequence, to complete the proof of the proposition it suffices to show that the following expressions converge to zero as ε tends to zero:

$$\int_0^t \left(\int_s^t \left\| 1_{\{B_r<a,B_s>a\}} \int_{\min(|B_r-a|,|B_s-a|)/\sqrt{\varepsilon}}^{\infty} e^{-\frac{z^2}{2}} dz \right\|_2 \right.$$
$$\left. \times \left| \frac{\partial K}{\partial r}(r,s) \right| dr \right)^2 ds, \tag{31}$$

$$\int_0^t \left(\int_s^t \left\| 1_{\{B_r>a,B_s>a\}} \int_{\frac{(B_s-a)}{\sqrt{\epsilon}}}^{\frac{(B_r-a)}{\sqrt{\epsilon}}} e^{-\frac{z^2}{2}} dz \right\|_2 \left| \frac{\partial K}{\partial r}(r,s) \right| dr \right)^2 ds. \tag{32}$$

The convergence to zero of (31) follows from the dominated convergence theorem. In order to show the convergence to zero of the term (32), we can write

$$\int_s^t \left\| 1_{\{B_r>a,B_s>a\}} \int_{\frac{(B_s-a)}{\sqrt{\epsilon}}}^{\frac{(B_r-a)}{\sqrt{\epsilon}}} e^{-\frac{z^2}{2}} dz \right\|_2 (r-s)^{H-\frac{3}{2}} dr$$
$$\leq \int_s^t \left(\int_{\mathbb{R}} P(a < B_r < z\sqrt{\varepsilon}+a, B_s > \max(z\sqrt{\varepsilon}+a, a))^{1/2} e^{-\frac{z^2}{2}} dz \right)$$
$$\times (r-s)^{H-\frac{3}{2}} dr.$$

Clearly, for every $z \in \mathbb{R}$ and every $s < r$, the probability

$$P(a < B_r < z\sqrt{\varepsilon}+a, B_s > \max(z\sqrt{\varepsilon}+a, a))^{1/2} \tag{33}$$

converges to zero as ε tends to zero. By the dominated convergence theorem the integral of this probability with respect to the measure $e^{-\frac{z^2}{2}}(r-s)^{H-\frac{3}{2}} dr dz$ tends to zero as ε tends to zero. Indeed, from the estimate (23) we obtain that (33) is bounded by

$$c_T \left(1 + |a| + |z|\sqrt{\varepsilon}\right)^{1/2} (r-s)^{\frac{H}{2}} s^{-H} \quad .$$

A further application of the dominated convergence theorem yields the convergence to zero of (32) as ε tends to zero. The proof of the proposition is complete. □

The next theorem provides a Tanaka formula in the case for $1/3 < H < 1/2$.

Theorem 6 *Let $B = \{B_t, 0 \leq t \leq T\}$ be a fBm with parameter $\frac{1}{3} < H < \frac{1}{2}$. Then*

$$|B_t - a| = |a| + \int_0^t \operatorname{sign}(B_s - a) dB_s + L_t^a \tag{34}$$

Proof: By the Itô formula established in [2] when $H > \frac{1}{4}$, we deduce that (20) holds for each $\varepsilon > 0$, where here

$$\begin{aligned} K_t^* \left[F_\varepsilon'(B_s - a) \right] &= K(t,s) F_\varepsilon'(B_s - a) \\ &\quad + \int_s^t \left[F_\varepsilon'(B_r - a) - F_\varepsilon'(B_s - a) \right] \frac{\partial K}{\partial r}(r,s) dr. \end{aligned}$$

Then the result follows by taking the limit as ε tends to zero in $L^2(\Omega)$, using Lemma 1 and Proposition 2. □

As in the case $H > \frac{1}{2}$, we can also establish the formulas (21) and (22).

4.3 Itô formula for convex functions

We will assume in this section that $\frac{1}{3} < H < 1$. We recall that if a function f is convex, its second derivative f'' in the sense of distributions is a positive measure. Tanaka's formula for the fBm can be applied to derive a generalization of Itô's formula to a convex function f.

Proposition 7 *Suppose that f is a convex function such that the right derivative f'_+ is uniformly bounded. Then,*

$$f(B_t) = f(0) + \int_0^t f'_-(B_s) dB_s + \frac{1}{2} \int_{\mathbb{R}} L_t^a f''(da). \tag{35}$$

Proof: By our assumption on the derivative, the measure f'' has support contained in a compact interval J. We will use the following decomposition

$$f(x) = \alpha x + \beta + \frac{1}{2} \int_J |x - a| f''(da), \tag{36}$$

where α and β are real numbers. This implies, for all x out of a countable set in $\mathbb{R}$

$$f'_-(x) = \alpha + \frac{1}{2} \int_J \operatorname{sign}(x - a) \; f''(da). \tag{37}$$

Applying formula (36) to the fractional Brownian motion yields

$$f(B_t) = \alpha B_t + \beta + \frac{1}{2} \int_J |B_t - a| f''(da).$$

Then we apply now the Tanaka formula, obtaining

$$\begin{aligned}
f(B_t) &= \alpha B_t + \beta \\
&+ \frac{1}{2}\int_{\mathbb{R}} \left(|a| + \int_0^t \operatorname{sign}(B_s - a) dB_s + L_t^a \right) f''(da) \\
&= \alpha B_t + f(0) \\
&+ \frac{1}{2}\int_{\mathbb{R}} \left(\int_0^t \operatorname{sign}(B_s - a) dB_s \right) f''(da) \\
&+ \frac{1}{2}\int_{\mathbb{R}} L_t^a f''(da)
\end{aligned}$$

Using Fubini's theorem for the Skorohod integral (see exercice 3.2.8 of [15]) and the relation (37), the equation becomes

$$f(B_t) = f(0) + \int_0^t f'_-(B_s) dB_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a f''(da).$$

□

It is possible to extend formula (35) to any convex function by means of a localization argument. In fact, for any $k > 0$ define $G_k = \{\sup_{0\leq s\leq T} |B_s| < k\}$ and let $f^{(k)}$ be a convex function such that $f^{(k)} = f$ on $[-k, k]$, and such that $f^{(k)''}$ vanishes outside $[-k, k]$. By the above proposition we know that

$$f^{(k)}(B_t) = f(0) + \int_0^t f_-^{(k)\prime}(B_s) dB_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a f^{(k)''}(da),$$

which gives the desired formula on the set G_k, provided we define the stochastic integral $\int_0^t f'_-(B_s) dB_s$ on this set as $\int_0^t f_-^{(k)\prime}(B_s) dB_s$. Letting k tend to infinity we deduce the Itô formula for a general convex function f. However, notice that the stochastic integral $\int_0^t f'_-(B_s) dB_s$ may change if we use a different localization procedure, because we only know that the processes $f_-^{(k)\prime}(B_s)$ belong to the domain of the divergence operator δ^B, and the local property for this operator is known only on $\mathbb{D}_B^{1,2}$.

References

[1] E. Alòs, O. Mazet and D. Nualart, 2000. Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than $\frac{1}{2}$. *Stoch. Proc. Appl.*, **86**:*121-139.*

[2] E. Alòs, O. Mazet and D. Nualart, 1999. *Stochastic calculus with respect to Gaussian processes.* Preprint.

[3] S. Berman, 1973. Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.,* **23**:*69-94.*

[4] P. Carmona and L. Coutin, 1998. *Stochastic integration with respect to fractional Brownian motion.* Preprint.

[5] W. Dai and C. C. Heyde, 1996. Itô's formula with respect to fractional Brownian motion and its application. *Journal of Appl. Math. and Stoch. An.,* ***9****:439-448.*

[6] L. Decreusefond and A. S. Üstünel, 1998. Stochastic analysis of the fractional Brownian motion. *Potential Analysis,* **10**:*177-214.*

[7] T. E. Duncan, Y. Hu and B. Pasik-Duncan, 2000. Stochastic calculus for fractional Brownian motion I. Theory. *Siam J. Control Optim.* **38(2)**:*582-612.*

[8] D. Feyel and A. de la Pradelle, 1996. Fractional integrals and Brownian processes. *Potential Analysis,* **10**:*273-288.*

[9] D. Geman and J. Horowitz, 1980. Occupation densities. *Annals of Probability,* **8**:*1-67.*

[10] Y. Hu and B. Øksendal, 1999. *Fractional white noise calculus and applications to finance.* Preprint.

[11] M. L. Kleptsyna, P. E. Kloeden and V. V. Anh, 1996. *Existence and uniqueness theorems for fBm stochastic differential equations.* Preprint.

[12] S. J. Lin, 1995. Stochastic analysis of fractional Brownian motions. *Stochastics and Stoch. Reports,* ***55****:121-140.*

[13] B. B. Mandelbrot and J. W. Van Ness, 1968. Fractional Brownian motions, fractional noises and applications. *SIAM Review,* ***10****(4):422-437.*

[14] I. Norros, E. Valkeila and J. Virtamo. An elementary approach to a Girsano formula and other analytical resuls on fractional Brownian motion, 1999. *Bernoulli,* **5**:*571-587.*

[15] D. Nualart, 1995. *The Malliavin calculus and related topics.* Prob. and Appl., 21, Springer.

[16] D. Nualart, J. Vives, 1992. Chaos expansions and local times. *Publicacions Matemàtiques* **36**:*827–836.*

[17] F. Russo and P. Vallois, 1993. Forward, backward and symmetric stochastic integration. *Probab. Theory Rel. Fields,* **97**:*403-421.*

[18] A. V. Skorohod, 1975. On a generalization of a stochastic integral. *Theory Probab. Appl.,* ***20****:219-233.*

[19] M. Zähle, 1998. Integration with respect to fractal functions and stochastic calculus I. *Prob. Theory Rel. Fields,* ***111****(3):333-374.*

Liste des prépublications

99-1 Monique Jeanblanc et Nicolas Privault. A complete market model with Poisson and Brownian components. A paraître dans *Proceedings of the Seminar on Stochastic Analysis, Random Fields and Applications*, Ascona, 1999.

99-2 Laurence Cherfils et Alain Miranville. Generalized Cahn-Hilliard equations with a logarithmic free energy. A paraître dans *Revista de la Real Academia de Ciencias.*

99-3 Jean-Jacques Prat et Nicolas Privault. Explicit stochastic analysis of Brownian motion and point measures on Riemannian manifolds. *Journal of Functional Analysis*, Vol. **167**, pp. 201-242, 1999.

99-4 Changgui Zhang. Sur la fonction q-Gamma de Jackson. A paraître dans *Aequationes Math.*

99-5 Nicolas Privault. A characterization of grand canonical Gibbs measures by duality. A paraître dans *Potential Analysis.*

99-6 Guy Wallet. La variété des équations surstables. A paraître dans *Bulletin de la Société Mathématique de France.*

99-7 Nicolas Privault et Jiang-Lun Wu. Poisson stochastic integration in Hilbert spaces. *Annales Mathématiques Blaise Pascal*, Vol. **6**, pp. 41-61, 1999.

99-8 Augustin Fruchard et Reinhard Schäfke. Sursabilité et résonance.

99-9 Nicolas Privault. Connections and curvature in the Riemannian geometry of configuration spaces. *C. R. Acad. Sci. Paris, Série I, t. 330, pp. 899-904, 2000.*

99-10 Fabienne Marotte et Changgui Zhang. Multisommabilité des séries entières solutions formelles d'une équation aux q-différences linéaire analytique. A paratre dans *Annales de l'Institut Fourier*, 2000.

99-11 Knut Aase, Bernt Øksendal, Nicolas Privault et Jan Ubøe. White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance. A paraître dans *Finance and Stochastics.*

00-01 Eric Benoît. Canards en un point pseudo-singulier nœud. A paraître dans *Bulletin de la Société Mathématique de France.*

00-02 Nicolas Privault. Hypothesis testing and Skorokhod stochastic integration. A paraître dans *Journal of Applied Probability.*

00-03 Changgui Zhang. La fonction théta de Jacobi et la sommabilité des séries entières q-Gevrey, I. *C. R. Acad. Sci. Paris, Série I, t. 331, pp. 31-34, 2000.*

00-04 Guy Wallet. Déformation topologique par changement d'échelle.

00-05 Nicolas Privault. Quantum stochastic calculus for the uniform measure and Boolean convolution.

00-06 Changgui Zhang. Sur les fonctions q-Bessel de Jackson.

00-07 Laure Coutin, David Nualart et Ciprian A. Tudor. Tanaka formula for the fractional Brownian motion.