Prépublications du Département de Mathématiques

Université de La Rochelle
Avenue Michel Crépeau
17042 La Rochelle Cedex 1
http ://www.univ-lr.fr/Labo/MATH

Stabilizing endogenous fluctuations by fiscal policies ; Global analysis on piecewise continuous dynamical systems

E. Augeraud-Veron et L. Augier

Janvier 2001

2001/01

Stabilizing endogenous fluctuations by fiscal policies ; Global analysis on piecewise continuous dynamical systems

E. Augeraud-Veron* and L. Augier‡

19 janvier 2001

Résumé

This paper considers a one-sector OLG model with production. We observe that, under standart assumptions on preferences and technology, the perfect-foresight equilibrium violate positivity constraints for large sets of initial conditions. The consideration of the positivity constraints of the consumer enable to defined degenerate equilibrium, which are perfect-foresight equilibrium that remains on the trivial steady state from certain time. To correct this unsatisfactory situation, we introduce a mechanism redistributing resources among generations by levying taxes.

*Department of mathematics, University of La Rochelle

†Department of mathematics, University of La Rochelle, ERUDITE-LR

‡We would like to thank A. Fruchard, E. Benoit, F. Cathala and referees for their useful discussions and comments.

This study is supported by an allocation of the Communaute de Ville (CDV) of La Rochelle

This paper was presented at the Eight Annual Symposium of the Society of Nonlinear Dynamics and Economics (March 16-17, 2000).

The aim of this study is to present the problem of definition of perfect-foresight equilibrium in an overlapping generations model with production.

The model we use is the one proposed in Reichlin's paper (Reichlin (1986)). The modelling gives rise to a two-dimensional dynamical system, depending on a technological parameter.

Reichlin proved that there exists, in the neighborhood of the non trivial economy steady state, an attractive or repelling invariant closed curve. This result, based on the Neimark-SÙcker theorem, is local : the invariant curve lies in the neighborhood of the fixed point and the theorem states its existence only for parameters which lie in the neighborhood of the bifurcation value.

We provided (Augeraud and Augier (1999)) a global analysis of the same model. We gave mathematical proofs of the variation of the initial conditions basin leading to reasonable long term economic equilibrium under variations of the technological parameter. The phenomenon that happened corresponds to *blue sky catastrophe* (Medio (1992)). Studying the basin of attraction in economy is mainly done when there are multiple steady states (Soliman (1998)).

The global analysis shows that perfect-foresight equilibrium is degenerate ; it means that variables are null after a certain finite time.

After having defined the model (section (2)), we shall then explain in the modelling which hypothesis is responsible for the not satisfying long-term behavior of the economy (section (3)). Then to avoid this problem, we will propose some modifications in the modelling, based on government fiscal policy, .

The systems we obtained are piecewise continuous dynamical systems. The mathematical tools we use (section (4)) have been developed by Mira (1964) and are called critical lines. These curves are useful tools to study non invertible systems or non differentiable two-dimensional systems.

Details of the study of the dynamics are to be found in section (5).

2 The economic model

We assume that the size of the population is constant. The model is a two-period overlapping generations economy with production (Diamond (1965), Reichlin (1986)).

The consumers work during the first period of their lives and consume during the second. The representative agent born at time t supplies a quantity of labor l_t, receives wage income saves it when young and consumes c_{t+1} when old.

We assume that the utility function of the representative agent is equal to $u(c_{t+1}) - v(l_t)$[1].

The following standard hypothesis on utility can be made :

[1] This contains the following assumptions : First, utility is separable in labor and good, then disutility of labor is directly taken into account.

– *u is strictly increasing and concave over* $\mathbb{R}^{+*}$ *;*
– *v is strictly increasing and convex over* $\mathbb{R}^{+*}$.

Under perfect-foresight hypothesis, the agent's decision problem is the following :

$$\left\{\begin{array}{l} \max_{c_{t+1},l_t} u(c_{t+1}) - v(l_t) \\ \qquad\qquad \text{subject to} \left\{\begin{array}{l} s_t = w_t l_t \\ c_{t+1} = s_t R_{t+1} \\ c_{t+1} \geq 0, l_t \geq 0 \end{array}\right. \end{array}\right. \tag{1}$$

where w_t is the real wage and R_{t+1} the real rate of return[2] on savings.

Under assumption (1) and with $w_t > 0$ and $R_{t+1} > 0$, the decision problem has a unique solution satisfying $c_{t+1} > 0$ and $l_t > 0$. Futhermore, consumption demand and labor supply are the unique solution of the system :

$$\left\{\begin{array}{l} c_{t+1}u'(c_{t+1}) = l_t v'(l_t) \\ c_{t+1} = R_{t+1} w_t l_t \end{array}\right. \tag{2}$$

Let us write[3] $U(x) = xu'(x)$ and $V(x) = xv'(x)$.

As in Reichlin's paper the following hypothesis is done :

Hypothesis 2. $U' > 0$ *and* $\lim_{x\to 0} U(x) = 0, \lim_{x\to+\infty} U(x) = +\infty$.

This hypothesis enables to consider the function $h(x) = U^{-1} \circ V(x)$, and to rewrite system (2) as

$$\left\{\begin{array}{l} c_{t+1} = h(l_t) \\ c_{t+1} = R_{t+1} w_t l_t \end{array}\right. \tag{3}$$

We can take the following CRRA utility function as an example :

$$U(c_{t+1}, l_t) = \frac{c_{t+1}^{1-\alpha}}{1-\alpha} - \frac{l_t^\gamma}{\gamma} \tag{4}$$

with $0 < \alpha < 1$ and $\gamma > 1$. Hypotheses (1) and (2) are verified.

Let δ be $\dfrac{\gamma}{1-\alpha}$.

We write Y_t the quantity produced at time t. The technology is described by a Leontief production function. The two production inputs K_t (stock of capital) and L_t (labor) are used in fixed proportions in the following way :

$$Y_t = \min(\frac{L_t}{a_0}, \frac{K_t}{a_1}) \tag{5}$$

We assume that capital depreciate, so that the profit function of producers rewrites $Y_t - R_t K_t - w_t L_t$.

[2] $R_{t+1} = 1 + r_{t+1}$ where r_{t+1} is the interest rate.
[3] These notations are the one used by Reichlin (1986).

$$Y_t = \frac{L_t}{a_0} = \frac{K_t}{a_1}$$

The market clearing assumptions are given by hypothesis (3) :
Hypothesis 3. $K_t = s_{t-1}$ *and* $L_t = l_t$

A consequence is that capital stock is the only asset in the model and labor demand and supply are equals.

3 The problem

We assume that perfect foresight hypothesis is done. Benhabib and Laroque (1988) give the definition of an intertemporal perfect-foresight equilibrium. *Perfect foresight expectations are such that agent's expectation should be the actuel future sequence predicted by the model* (Kehoe and Levine (1985)).
Property 3.1. *In Reichlin's model, the perfect-foresight equilibrium is associated with a sequence* $(s_t)_t$ *satisfying*

$$s_{t+1} = \frac{1}{a_1} s_t - h(\frac{a_0}{a_1} s_{t-1}) \tag{6}$$

if $s_{t+1} > 0$.
If there exists t_1 *such that* $s_{t+1} \leq 0$*, then* $s_t = 0$ *for all* $t \geq t_1$.

Proof :
The definition of intertemporal equilibrium gives $Y_{t+1} = c_{t+1} + s_{t+1}$. The equation (5) and market clearing conditions give $Y_{t+1} = \frac{1}{a_1} s_t$ and $L_t = \frac{a_0}{a_1} s_{t-1}$. The resolution of consumer program, in case constraints are not saturated, enables one to prove the property.
If s_{t+1} is negative, we have to come back to expectation formation and take into account the positivity constraints of the consumer program.
The presentation of (Bðhm and Wenzelburger (1999)) enables one to express the dynamics according to expectations $(\widehat{R_{t+1}})_t$. The economic law is the solution of consumer and firm programs and market's constraints.
Lemma 3.1. *The economic law, for* $\widehat{R_{t+1}} > 0$ *is given by*

$$\begin{cases} s_t = \frac{h(\frac{a_0}{a_1} s_{t-1})}{\widehat{R_{t+1}}} \\ R_t = \frac{1}{a_1} - \frac{h\left(\frac{a_0}{a_1} s_{t-1}\right)}{\widehat{R_{t+1}} s_{t-1}} \end{cases} \tag{7}$$

Proof : The resolution of the consumer program gives that $\widehat{R_{t+1}} s_t = h(l_t)$. As the producer's maximization and market's constraints give $l_t = \dfrac{a_0}{a_1} s_{t-1}$, we obtain the first equation of the system (7).

$Y_t = R_t K_t + w_t L_t$. As the technology is Leontief's, the result is : $R_t = \frac{1}{a_1} - \frac{a_0}{a_1} w_t$. The following condition ($\widehat{R_{t+1}} w_t l_t = h(l_t)$) obtained according to the consumer program enables one to prove lemma (3.1).□

Lemma 3.2. *When perfect-foresight hypothesis is done, expectation is given by*

$$\widehat{R_{t+1}} = \frac{h(\frac{a_0}{a_1} s_{t-1})}{\frac{1}{a_1} s_{t-1} - h(\frac{a_0}{a_1} s_{t-2})}$$

Proof : The economic law (7) gives

$$\widehat{R_{t+1}} = \frac{h(\frac{a_0}{a_1} s_{t-1})}{s_t}$$

So $\widehat{R_t} = \frac{h(\frac{a_0}{a_1} s_{t-2})}{s_{t-1}}$. Then the second equation of system (7) gives R_t.
As $\widehat{R_t} = R_t$ under the perfect expectation hypothesis, we can then conclude.□

Remark 3.1. *Note, here, that the expectation function is characterized by a memory of order 2 with variables s_{t-1} and s_{t-2}. This order two memory can give rise to an indetermination problem (Hahn (1966), Laitner (1982)).*

Lemma 3.3. *If there exists t_1 such that $s_{t_1} = 0$, then for all $t > t_1$, $s_t = 0$.*

Proof : If the expression of s_t given by equation (6) is negative, it means, according to lemma (3.2) that $\widehat{R_{t+1}} < 0$. As this has no economic meaning as $R = 1 + r$, the agent then expects $\widehat{R_{t+1}} = 0$. Resolution of consumer program gives $s_t = 0$. Leontief's technology then gives $L_{t+1} = K_{t+1} = Y_{t+1} = 0$. Then prices R_{t+1} and w_t are null. Then agents who are young at time t have saved nothing, $c_{t+1} = 0$, so $s_{t+1} = 0$. An immediate recurrence enables one to conclude. □

Definition 3.1. *We call a degenerate equilibrium a sequence $(s_t)_t$ such that*
- *$(s_t)_t$ is associated with a perfect-foresight equilibrium ;*
- *there exists t_1 such that for all $t \geq t_1$ $s_t = 0$.*

3.1 Reminder of local dynamical properties of the system

Reichlin (1986) has studied local properties of system (6). He proved that for $a_1 < 1$, the system has two steady states. When utility is the CRRA function given by equation (4), we have :
- $(0, 0)$ is a saddle with eigenvalues 0 and $\frac{1}{a_1}$;
- $\left(\left(\frac{1}{a_1} - 1\right)^{\frac{1}{\delta-1}} \left(\frac{a_1}{a_0}\right)^{\frac{\delta}{\delta-1}}, \left(\frac{1}{a_1} - 1\right)^{\frac{1}{\delta-1}} \left(\frac{a_1}{a_0}\right)^{\frac{\delta}{\delta-1}}\right)$ which is an attractive focus for

$\left(\frac{1}{a_1}, \delta\right) \in \left\{\left(\frac{1}{a_1}, \delta\right), \left(\frac{1}{a_1}\right)^2 - 4\delta\left(\frac{1}{a_1} - 1\right) < 0 \text{ and } \delta\left(\frac{1}{a_1} - 1\right) < 1\right\}$ and a repelling focus for $\left(\frac{1}{a_1}, \delta\right) \in \left\{\left(\frac{1}{a_1}, \delta\right), \left(\frac{1}{a_1}\right)^2 - 4\delta\left(\frac{1}{a_1} - 1\right) < 0 \text{ and } \delta\left(\frac{1}{a_1} - 1\right) > 1\right\}$.

Reichlin then considered bifurcation according to technological parameter a_1.

He proved, by applying the *Neimark-saker* bifurcation theorem (Hale and Kocak (1991)) that there locally exists, an invariant attractive closed curve. The parameter for which the bifurcation occurs is $a_1 = \dfrac{\delta}{\delta + 1}$.

3.2 Reminder of glocal dynamical properties of the system

It has been proved (Augeraud and Augier (1999)) that there exists a value a_1^* of parameter a_1, such that for all $a_1 < a_1^*$, there exists a value $s_{min}(a) \in \mathbb{R}^+$ such that for all $s_0 > s_{min}(a)$, for all $s_1 \in \mathbb{R}^+$, then (s_t) is a degenerate equilibrium.

Figure (1) illustrates this situation for a peculiar value of a_1 .

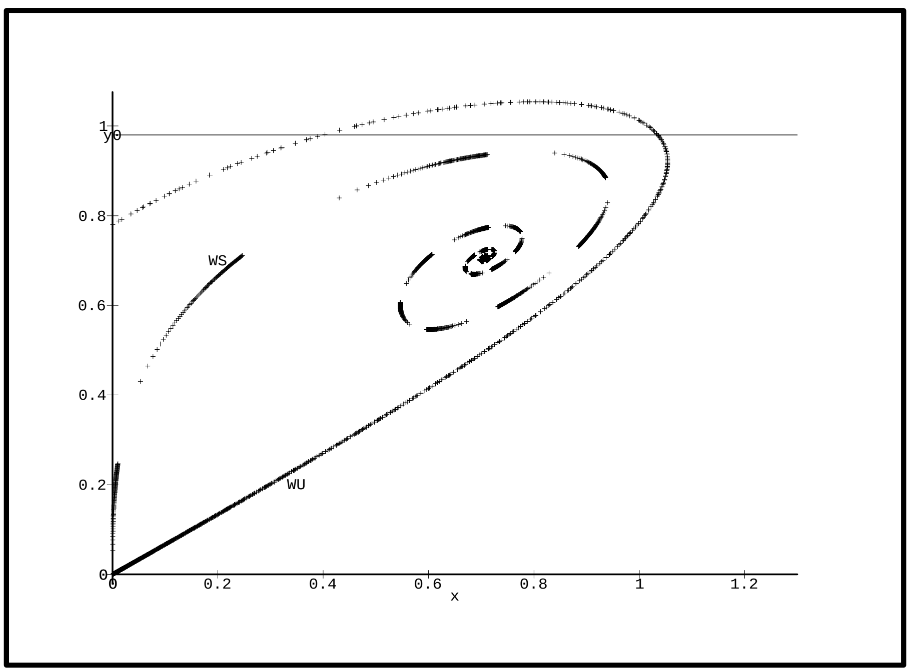


FIG. 1 – Invariant manifolds of the trivial fixed point for $h(x) = x^2$, $a = 1.5$.

Critical lines enable us to give a geometrical characterization of the attractors. In this section, we describe the tools we use in a general framework. These tools were first introduced by Gumowski and Mira in 1964.

4.1 Setting of the general framework

Let F_a be a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$F_a : (x_n, y_n) \in \mathbb{R}^2 \rightarrow (x_{n+1}, y_{n+1}) \in \mathbb{R}^2$$

Just notice that notation F_a indicates that the mapping depends on a parameter a. When we are not interested with this dependence, the mapping would only be written F.
Hypothesis 4. *F is proper (that is if the inverse image of a compact is compact). This hypothesis is done so that each point in $\mathbb{R}^2$ has a finite number of preimages.*

The systems on which we would apply the critical lines method have the following form :

- Type 1,

$$F_a \begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}$$

  where f and g are $\mathcal{C}^k$ functions ($k \geq 1$)
- Type 2, A piecewise dynamical system for which each part is defined with systems of type 1

4.2 Definition of critical lines

Properties using the topological nature of the system :
Definition 4.1. *1. Let Z_i be the set of points having i preimages of order 1*

2. Let say that X_0 satisfies the property $\mathcal{P}$ iff
There exist a neighborhood $\mathcal{U}$ of X_0 and a neighborhood $\mathcal{V}$ of $F(X_0)$, such that F is a bijection from $\mathcal{U}$ to $\mathcal{V}$

3. We call set of critical points the curve LC_{-1} defined by

$$LC_{-1} = \{X_0, \ \textit{which do not satisfy the property } \mathcal{P}\}$$

4. Let LC be

$$F(LC_{-1}) = LC$$

LC may intersect Z_i's interior. Let us consider f such that :

$$f : x \to x(x^2 - 1)^2 \tag{8}$$

The system which is defined by f has a set Z_1 which is defined by :
$Z_1 =]-\infty, -\sqrt{\frac{1}{5}}\left(\frac{16}{25}\right)[\cup]\sqrt{\frac{1}{5}}\left(\frac{16}{25}\right), \infty[$ and a set $Z_3 =]-\sqrt{\frac{1}{5}}\left(\frac{16}{25}\right), \sqrt{\frac{1}{5}}\left(\frac{16}{25}\right)[$.

The critical lines are as follows : $LC_{-1} = \{-1\} \cup \{-\sqrt{\frac{1}{5}}\} \cup \{\sqrt{\frac{1}{5}}\} \cup \{1\}$, and
$LC = \{-\sqrt{\frac{1}{5}}\left(\frac{16}{25}\right)\} \cup \{0\} \cup \{\sqrt{\frac{1}{5}}\left(\frac{16}{25}\right)\}$.

Properties based on the differentiability of the system :
Property 4.1.

$$LC_{-1} \subseteq J_0$$

$$\textit{where } J_0 = \{(x, y), |Jac(F)(x, y)| = 0\}$$

The proof of this property is based on the local inversion theorem.

Just notice that the previous relation can be a strict inclusion ; by analogy with what can be done in dimension 1, we have to eliminate from J_0 all points corresponding to inflection points.

When F is of type 2, we have
Property 4.2.

$$LC_{-1} \subseteq J_0 \cup ND$$

$$\textit{where } ND = \{(x, y), F(x, y) \textit{is not differentiable}\}$$

Properties using the algebraic nature of the system :
Remark 4.2. Points in LC have preimages whose multiplicity is strictly superior to 1.

5 Redistribution of resources

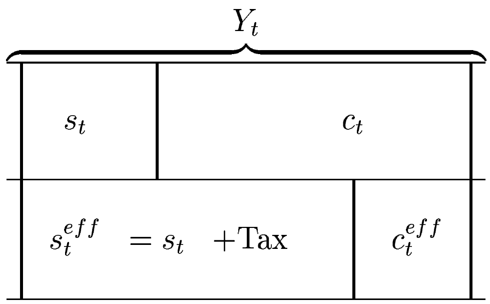
We have showed that if positivity constraints are not satisfied at time t, the quantities chosen by consumers after time t are trivial. The economic interpretation of this result is clear : a young agent's savings are insufficient to keep the economy on a feasible growth path. It follows that the government levies a tax on the older generation and uses the

[4]The following example is defined on $\mathbb{R}$ for simplicity reasons. If we considered F_a for which f is the one given in example (8) and g is such that $g(x, y) = y$, we can show that the result we have remains true in $\mathbb{R}^2$.

consumption (resp. the saving) at time t shall be written c_t^{eff} (resp. s_t^{eff}), where eff stands for effective.

We assume that the government policy is continuous according to the set of economic variables. Such an hypothesis eliminates peculiar cases where the fiscal policy is implemented through lump-sum tax schemes or taxation on one state variable. An example of such a case is given by a proportionnal tax policy on consumption.

We shall study several tax schemes : a tax scheme in a function of s_{t-1} and another tax scheme depending on s_{t-2}.



The government distributes wealth as it would have been distributed in a competitive market. The fiscal policy appears as an alternative adjustment mechanism.

In the models we present here, the government levies tax at different times t. This action has to be isolated, because of problems of time inconsistency (Kydland and Prescott (1977)). Indeed, individual consumers are not able to predict government action.

5.1 Tax producing a system depending on s_{t-1} only

Government levies a tax $T(s_{t-1}, s_{t-2})$ at time t, such that

$$as_{t-1} - h(s_{t-2}) + T(s_{t-1}, s_{t-2}) = \overline{w}s_{t-1}$$

where $\overline{w}$ is given *a priori*.

Remark 5.1. Tax T could be interpreted as a way of creating minimum wages $\overline{w}$.

We assume that the government applies this fiscal policy when

$$as_{t-1} - h(s_{t-2}) < s_{t-1}\overline{w}$$

This tax policy's conditions imply that the economic dynamical system is defined by piecewise countinuous functions.

Remark 5.2. The government has to take into account the following wealth constraint :

$$T(s_{t-1}, s_{t-2}) < h(s_{t-2})$$

This last inequality means that the government cannot levy a tax higher than the older generation's income.

It implies that $\overline{w}$ must be inferior to a.

Let $x_t = s_t$ and $y_t = s_{t-1}$
The dymanics (S) is given by map $F_{a,\overline{w}}$ from $\mathbb{R}^2_+$ to $\mathbb{R}^2_+$ such that :

$$F_{a,\overline{w}} : (x_{t-1}, y_{t-1}) \longrightarrow \begin{cases} \text{If } \frac{ax_{t-1}-h(y_{t-1})}{x_{t-1}} > \overline{w}, & \\ & \begin{cases} x_t = ax_{t-1} - h(y_{t-1}) \\ y_t = x_{t-1} \end{cases} \\ \text{If } \frac{ax_{t-1}-h(y_{t-1})}{x_{t-1}} < \overline{w}, & \\ & \begin{cases} x_t = \overline{w}x_{t-1} \\ y_t = x_{t-1} \end{cases} \end{cases} \tag{9}$$

Map $F_{a,\overline{w}}$ is continuous and piecewise differentiable.
Let

$$\begin{array}{rcl} F_1 : \mathbb{R}^2_+ & \to & \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} ax - h(y) \\ x \end{pmatrix} \end{array} \quad \text{et} \quad \begin{array}{rcl} F_2 : \mathbb{R}^2_+ & \to & \mathbb{R}^2_+ \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} \overline{w}x \\ x \end{pmatrix} \end{array}$$

Let (S_1) (resp. (S_2)) be the dynamical system defined by F_1 (resp. F_2).

Description of critical lines We can consider the following sets :

1. ND set of points on which $F_{a,\overline{w}}$ is not differentiable

$$ND = \{(x,y), (a-\overline{w})x - h(y) = 0\}$$

   Notations : ND partitions $\mathbb{R}^2_+$ into $\mathcal{S}_1$ (resp. $\mathcal{S}_2$), defined by :

$$\mathcal{S}_1 = \{(x,y), \tfrac{ax-h(y)}{x} > \overline{w}\} \quad (\text{ resp. } \mathcal{S}_2 = \{(x,y), \tfrac{ax-h(y)}{x} < \overline{w}\})$$

   on which (S_1) (resp. (S_2)) is defined.
   ND is the border between $\mathcal{S}_1$ and $\mathcal{S}_2$.

2. and J_0 where the jacobian of $F_{a,\overline{w}}$ is defined and vanishes.

$$J_0 = \{(x,y) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+, h'(y) = 0\} \cup \mathcal{S}_2$$

   As utility is CRRA, $y = 0$ is the unique solution of $h'(y) = 0$.

Let LC^2 be the image of ND and LC^1 the image of J_0. Let LC be $LC^1 \cup LC^2$.
Calculation gives LC^1 and LC^2 :

$$\begin{aligned} LC^2 &= \{(x,y), x = \overline{w}y\} \\ LC^1 &= \{(x,y), x = ay\} \end{aligned}$$

Illustration (2) presents these curves.

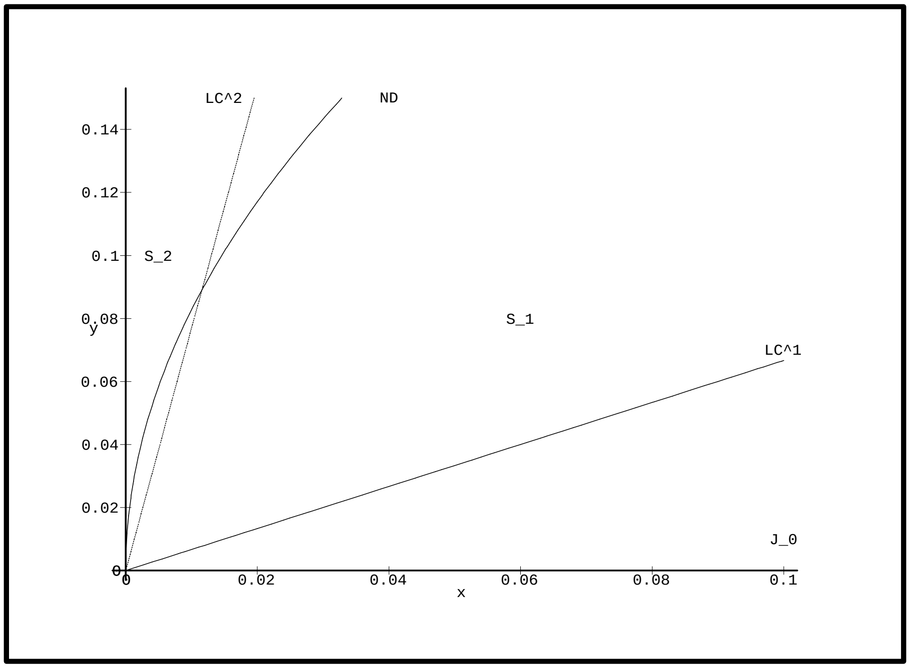


FIG. 2 – Critical lines for $h(x) = 2x^2$, $a = 1.5$ and $w = 0.13$

Remark 5.3. The image of $\mathcal{S}_2$ by F_2 is line LC^2.
Remark 5.4. *Let M be a point of ND and $\mathcal{U}$ be a neighborhood of M. Let $\mathcal{V}$ be the image of $\mathcal{U}$ by F. Then $F|_{\mathcal{U}} : \mathcal{U} \to \mathcal{V}$ is not a bijection.*

We can use the following property :
Property 5.1. *Each half-line given by $y = \alpha x$ (with $\alpha > 0$ and $x \geq 0$) intersects ND in two points : The trivial point and one belonging to $\mathbb{R}^+ \times \mathbb{R}^+$.*

Proof : Let $\varphi : x \to (a - \overline{w})x - h(\alpha x)$. The solutions of equation $\varphi(x) = 0$ give the abscissa of intersection points between ND and the half-line.
As utility is given by a CRRA-function, we have $h(0) = 0$ and $\lim\limits_{x\to\infty} h(x) = \infty$.

$$\varphi'(x) = (a - \overline{w}) - \alpha h'(\alpha x)$$

As h' is strictly increasing from $[0, \infty[$ to $[0, \infty[$ and unbounded, there exists a unique v_0 such that $\varphi'(v_0) = 0$. The graph of φ is the following :

$$\begin{array}{c|ccccc}
x & 0 & & v_0 & & \infty \\
\hline
\varphi(x) & & \nearrow & & \searrow & \\
 & 0 & & & & -\infty
\end{array}$$

Remark 5.5. The line $\overline{w}y = x$ intersects ND in $(0,0)$ and another point in $\mathbb{R}^+ \times \mathbb{R}^+$; we call $a_0 = (x^*, y^*)$.

The study of the order-1 images of $\mathcal{S}_1$ and $\mathcal{S}_2$ gives the following property :
Property 5.2. $Z_0 = \{(x,y), \overline{w}y > x, x \in \mathbb{R}^+\}$ *has no preimage.*

Proof Just consider the images by F_1 (resp. F_2) of $\mathcal{S}_1$ (resp. $\mathcal{S}_2$).
- F_2 maps $\mathcal{S}_2$ on the line $\overline{w}y = x$. Points belonging to Z_0 have no preimages by F_2.
- Moreover, let $(x,y) \in \mathcal{S}_1$. By definition of $\mathcal{S}_1$, we have $ax > \overline{w}x + h(y)$. The image (x_1, y_1) of (x,y) satisfies $(x_1 > \overline{w}x, y_1 = x)$. So $x_1 > \overline{w}y_1$. This means that $(x_1, y_1) \notin Z_0$. □

Remark 5.6. If there exist attractors, they do not belong to Z_0.

5.2.1 Remarks on parameters

Remark (5.2) says that parameters belong to set $\{(a,\overline{w}), (a,\overline{w}) \in]1,\infty] \times [0,a]\}$.
Property 5.3. *The steady state P of (S_1) belongs to the domain of definition of (S_1) if and only if $\overline{w} < 1$.*

Proof It is very easy to prove this property; We just have to notice that P belongs to $y = x, x \in \mathbb{R}^+$ and to find conditions of intersection of this half-line and ND. □
Property 5.4. *If $\overline{w} > 1$, for all $(x_0, y_0) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, trajectories are unbounded.*

Proof
- If $x_t \in \mathcal{S}_1$, then $ax_t - f(y_t) > \overline{w}x_t$. But $x_{t+1} = ax_t - f(y_t)$. So $x_{t+1} > \overline{w}x_t$.
- If $x_t \in \mathcal{S}_2$, $x_{t+1} = \overline{w}x_t$.

So, for all $t \geq 0$, we have $x_{t+1} \geq \overline{w}x_t$.

Then $(x_t)_t$ is strictly increasing and goes to infinite.□
Remark 5.7. The study of the case $\overline{w} \geq 1$ shows that the economy has a growth rate equal to $\overline{w}$ from a given date t_0. The tax then becomes predictable, this invalidating so the model we have chosen.

The previous remark leads us to consider parameter $\overline{w} < 1$. We must restrict the domain of the parameter a more, by using the following hypothesis :
Hypothesis 5. *We assume there exists t such that a_t (image of order t of a_0 by $F_{a,\overline{w}}$) that does not belong to $\mathcal{S}_2$.*

This hypothesis is observed numerically.

So the domain of the parameters which we shall take into account is :

$$\{(a,\overline{w}), (a,\overline{w}) \in]a', a_3[\times[0,1[\}$$

The methodology which we are going to use in order to describe the dynamics is based on the study of the critical lines.

Lemma 5.1. *There exists a set globally invariant whose dimension is at least 1.*

Proof : We are going to prove this lemma by building this set.

- Let us consider $a_0 = (x^*, y^*)$ (which is the non trivial intersection of ND and its image by $F_{a,\overline{w}}$). Such an intersection exists and is unique, according to property (5.1).
  - Let a_i be the point such that $a_i = F^i(a_0)$ and $\widetilde{a_i a_{i+1}}$ the image of order i, of the part of LC^2 between a_0 and a_1.
    As $\overline{w} < 1$, a_1 belongs to $\mathcal{S}_1$.
  - We map F_1 until there exists t such that $a_t \notin \mathcal{S}_1$ (such a t exists according to hypothesis (5)). Let b_0 be the first intersection point between $\widetilde{a_{t-1} a_t}$ and ND. Let b_1 be the image of b_0 by F.
    Only two cases can occur :
    - Either $a_{t+1} \in \{(x,y), \overline{w}y = x, x > x^*\}$;
    - Or $a_{t+1} \in \{(x,y), \overline{w}y = x, \overline{w}x^* < x < x^*\}$.

  Remark 5.8. a_{t+1} cannot belong to $\{(x,y), \overline{w}y = x, 0 < x < \overline{w}x^*\}$, which would mean that a_t belongs to $\mathcal{S}_2 \cap Z_0$.
  - Consider the case where $a_{t+1} \in \{(x,y), \overline{w}y = x, x > x^*\}$. The two following situations can occur :
    - Either $b_1 \in [a_1, a_{t+1}]$;
      There exists $n \geq 0$ such that $F_2^n(a_{t+1}) \in \widetilde{a_0 a_1}$. Indeed, as $\overline{w} < 1$ the abscissa of $F_2^n(a_{t+1})$ decrease with n. Furthermore, if one considers three points of abscissas decreassing strictly, the abscissas of their images are in the same order.
      So curve $\mathcal{C}_1$ defined by $\cup_{i=0..t}\widetilde{a_i a_{i+1}} \cup \widetilde{a_{t+1} a_0}$ is globally invariant.
    - Or $a_{t+1} \in [a_1, b_1]$. As previously, there exists $n \geq 0$ such as $F_2^n(a_{t+1}) \in \widetilde{a_0 a_1}$. So curve $\mathcal{C}_1'$ defined by $\cup_{i=0..t}\widetilde{a_i a_{i+1}} \cup [a_{t+1} b1] \cup \widetilde{a_{t+1} a_0}$ is globally invariant.
  - Let us consider the case where $a_{t+1} \in \{(x,y), \overline{w}x^* < x < x^*\}$.
    - Either $b_1 \in [a_1, a_{t+1}]$ or $b_1 \in [a_{t+1}, a_0]$;
      curve $\mathcal{C}_2$ being defined by $\cup_{i=1..t-1}\widetilde{a_i a_{i+1}} \cup \widetilde{a_t a_{t+1}} \cup [a_{t+1}, a_0]$ is globally invariant.
    - Or $b_1 > a_0$;
      curve $\mathcal{C}_2'$ being defined by $\cup_{i=1..t-1}\widetilde{a_i a_{i+1}} \cup \widetilde{a_t a_{t+1}} \cup [a_{t+1}, b_1]$ is globally invariant.

We shall call $\mathcal{C}$ the constructed curve. $\square$

Remark 5.9. We should now consider the conditional character of the intervention of the government. We should reject, and consider as not acceptable, the situations for which the government has to intervene over two consecutive periods, because then the tax becomes predictable by the agents. This remark in parallel to remark (5.7)

Remark 5.10. $\mathcal{C} = \cup_{n=0}^{t} F^n(\widetilde{a_0 a_1})$.

Property 5.5. $\mathcal{C}$ *is an attracting set.*

Preuve We have to show that there exists an open set U of $\mathcal{C}$ such that, for all $M \in U$, $\lim\limits_{n \to \infty} F^n(M) \to \mathcal{C}$.

Let V be a neigbourhood of $\widetilde{a_t a_{t+1}}$. Let us consider $V_0 = \cup_{n=0..t} F^{-n}(V)\mathbb{R}^+ \times \mathbb{R}^+ \cup$

such that $F^n(y) \in \mathcal{S}_2 \setminus Z_0$. So there exists $m \geq n$ such that $F^n(y) \in \mathcal{C}$.□

Dynamics on the attractor We won't take into account dynamics on the curve $\mathcal{C}$. De Vilder (1995) shows with an example of the same nature that various behaviors can be envisaged according to the nature of $\mathcal{C}$:

- If $\mathcal{C}$ is homeomorphic to a circle (case where $\mathcal{C}$ is $\mathcal{C}_1$) then, according to the value of the number of rotations, the dynamics can have a periodic or dense orbit ; (Guckenheimer and Holmes (1983)).
- otherwise the dynamics on $\mathcal{C}$ may be chaotic.

5.3 Tax producing a system depending only on s_{t-2}

We assume that the government levies a tax of amount $T(s_{t-1}, s_{t-2})$, such that

$$as_{t-1} - h(s_{t-2}) + T(s_{t-1}, s_{t-2}) = Ts_{t-2}$$

where T is given a priori. As previously, we assume for continuity reasons, that the government levies the tax when

$$as_{t-1} - h(s_{t-2}) < Ts_{t-2}$$

We have to take into account the wealth constraint :

$$T(s_{t-1}, s_{t-2}) < h(s_{t-2})$$

This constraint can be written as :

$$Ts_{t-2} < as_{t-1}$$

The dynamical system in which one is interested is given by the following map F :

$$F_{a,T} : (s_{t-1}, L_{t-1}) \longrightarrow \begin{cases} \text{If } as_{t-1} - h(L_{t-1}) \geq TL_{t-1}, & (S_1) F_1 : \begin{cases} s_t = as_{t-1} - h(L_{t-1}) \\ L_t = s_{t-1} \end{cases} \\ \text{If } as_{t-1} - h(L_{t-1}) \leq TL_{t-1}, & (S_2) F_2 : \begin{cases} s_t = TL_{t-1} \\ L_t = s_{t-1} \end{cases} \end{cases}$$

Map F is continuous, piecewise differentiable.

Description of critical lines As previously, we are interested in the two following sets :

1. Set ND of points where $F_{a,T}$ is not differentiable.

$$ND = \{(x, y), ax - h(y) = Ty\}$$

$$J_0 = \{(x,y), y = 0\}$$

The images of these sets are called LC^1 and LC^2. They make up a partition of $\mathbb{R}^+ \times \mathbb{R}^+$ into three areas.

LC^1 and LC^2 are the following sets :

$$LC^1 = \{(x,y), ay = x + h(\frac{x}{T})\}$$

$$LC^2 = \{(x,y), ay = x\}$$

If$(x,y) \in \{(x,y), ay > x + h(\frac{x}{T})\}$	, then (x,y) has no preimage by$F$
If $(x,y) \in \{(x,y), ay > x + h(\frac{x}{T}), y > \frac{x}{a}\}$	, then (x,y) has more than one preimage
If$(x,y) \in \{(x,y), y < \frac{x}{a}\}$	, then (x,y)has exactly one preimage

In this paragraph, we can apply a method similar to the one we used in the previously. We can work on the following example, assuming that f is defined by $f(x) = 2x^2$.

The critical lines of order 0, LC^1 and LC^2 make up a partition of $\mathbb{R}^+ \times \mathbb{R}^+$ into three areas $(Z_0 - Z_3 - Z_1)$.

$$LC^1 = \{(x,y), ay = x + 2\left(\frac{x}{T}\right)^2\}$$

$$LC^2 = \{(x,y), ay = x\}$$

If $(x,y) \in \{(x,y), ay > x + 2\left(\frac{x}{T}\right)^2\}$	, then(x,y) has no preimage by$F$
If $(x,y) \in \{(x,y), ay > x + 2\left(\frac{x}{T}\right)^2, y > \frac{x}{a}\}$	, then (x,y) has three preimage
If $(x,y) \in \{(x,y), y < \frac{x}{a}\}$	, then (x,y) has exactly one preimage

Property 5.6. *LC^1 cuts the curve ND into two points : the trivial point and a non trivial we call a_0.*

Proof : A point (x,y) belonging to LC^1 satisfies $y = \dfrac{x + \left(\frac{x}{T}\right)^2}{a}$. In replacing this formula in the equation of ND we have :

$$x\left(a - \frac{T}{a}\right) - x^2 2\left(\frac{1}{a^2} + \frac{1}{aT}\right) - x^3\left(\frac{8}{a^2T^2}\right) - x^4\left(\frac{4}{a^2T^4}\right)$$

So the trivial point belongs to LC^1 and ND.

Let φ be the map such that $\varphi : x \to \left(a - \frac{\ldots}{a}\right) - 2x\left(\frac{\ldots}{a^2} + \frac{\ldots}{aT}\right) - x^{\ldots}\left(\frac{\ldots}{a^2T^2}\right) - x^{\ldots}\left(\frac{\ldots}{a^2T^4}\right)$.

The variation diagram of φ is the following :

x	0		∞
$\varphi''(x)$		$-$	
$\varphi'(x)$	< 0	$\searrow$	$-\infty$
$\varphi(x)$	> 0	$\searrow$	$-\infty$

So there is a unique solution to the equation $\varphi(x) = 0.\square$

5.3.1 Study in the phase plane

As previously, we shall construct a curve $\mathcal{C}$.

The algorithm is the same than the one described previously. The only difference lies in the fact that $\mathcal{C}$ is not necessarily an invariant curve.

The following property is given to Barugola (1984).

Property 5.7. *$\mathcal{C}$ is the border of a region of the plane, called d' such that :*
- *d' is a compact set ;*
- *$F_{a,T}(d') \subseteq d'$;*
- *for all neighborhood $\mathcal{U}$ of d', for all $M \in \mathcal{U}\backslash d'$ there exists $n > 0$ such that $F^n_{a,T}(M) \in d'$. This last property ensures that d' is superabsorbing (Abraham (1997)).*

Such a region is called an absorbing area (Abraham (1997)).

Figure 3 illustrating the previous construction has been realized for $A = 1.8$, $T = 0.5$.

The previous example allows us to show an annular zone which looks[5] chaotic, in which a hole W appears. It would be of interest to investigate the existence of an annular shape for the attractor.

Use of critical lines for the determination of the value of bifurcation of the parameter for which the attractor is or not annular One considers a variation of parameter T. One is interested in the analytical determination of the value of T for which hole W does not exist any more. The determination of this value of bifurcation is made by means of two theorems presented in Barugola and Cathala (Barugola (1986)).

The two theorems are as following :

Let P be a fixed point situated inside hole W.

[5] The presence of chaos has not been proved analitically here, but there exists certain similarities with other situations that we know to be truly chaotic (Guckenheimer and Holmes (1983))

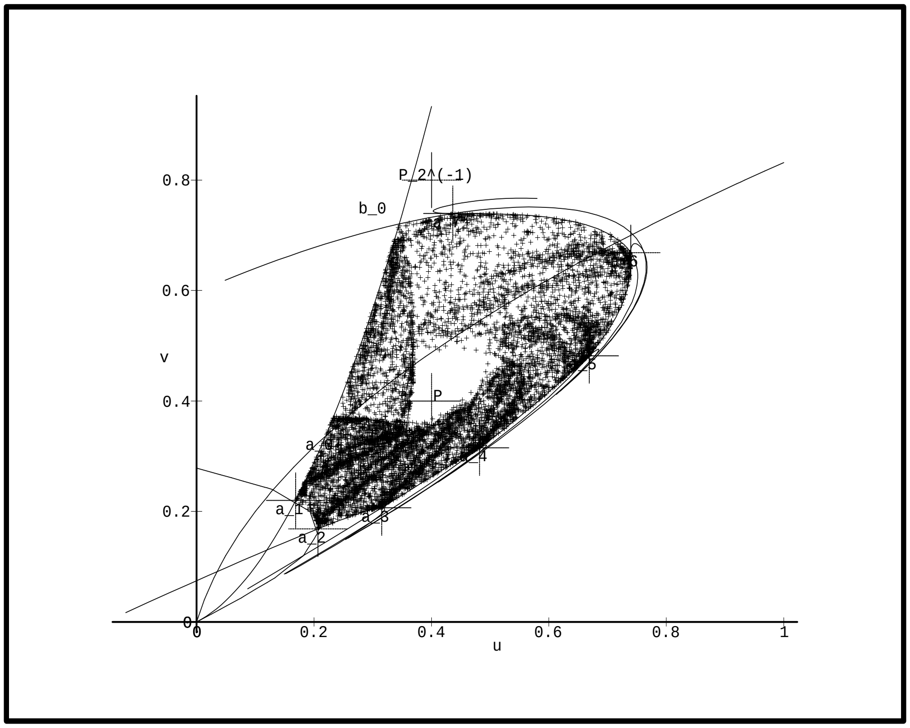


FIG. 3 – Absorbing area for $a = 1.8$ and $T = 0.5$

Theorem 5.1. *If all the preimages of P (others than P) do not belong to d', then the hole W containing P exists.*
Theorem 5.2. *If at least one antecedent of P (other than P) belongs to d', then the hole W containing P does not exist any more.*

The idea of the proof of these two theorems is as follows : If hole W belongs to a zone Z_n with $n \neq 0, 1$, W admits multiple preimages, in particular, there are preimages W^{-1} of W not included in W. Let us assume that W contains a repelling fixed point P. W^{-1} contains a preimage P^{-1} of the point P. Let us suppose that there is a zone W and that P^{-1} belongs to the annular chaotic zone. Then the image of W^{-1} also belongs to the chaotic zone, which is contradiction with the existence of hole W.

These two theorems allow us to determine the value of bifurcation for which the chaotic zone does not contain any more repelling internal part.

Let us come back to the previous example and let us calculate the value of bifurcation for which the chaotic attracteur is full.

As the fixed point $P = (\frac{a-1}{2}, \frac{a-1}{2})$ belongs to Z_3, it has three images :
– itself $P = (\frac{a-1}{2}, \frac{a-1}{2})$
– another preimage by F_1 : $P_1^{-1} = (\frac{a-1}{2}, -\frac{a-1}{2})$
– a preimage by F_2 : $P_2^{-1} = (\frac{a-1}{2}, \frac{a-1}{2T})$
Only point P_2^{-1} can meet the absorbing domain. To calculate the value of T corresponding

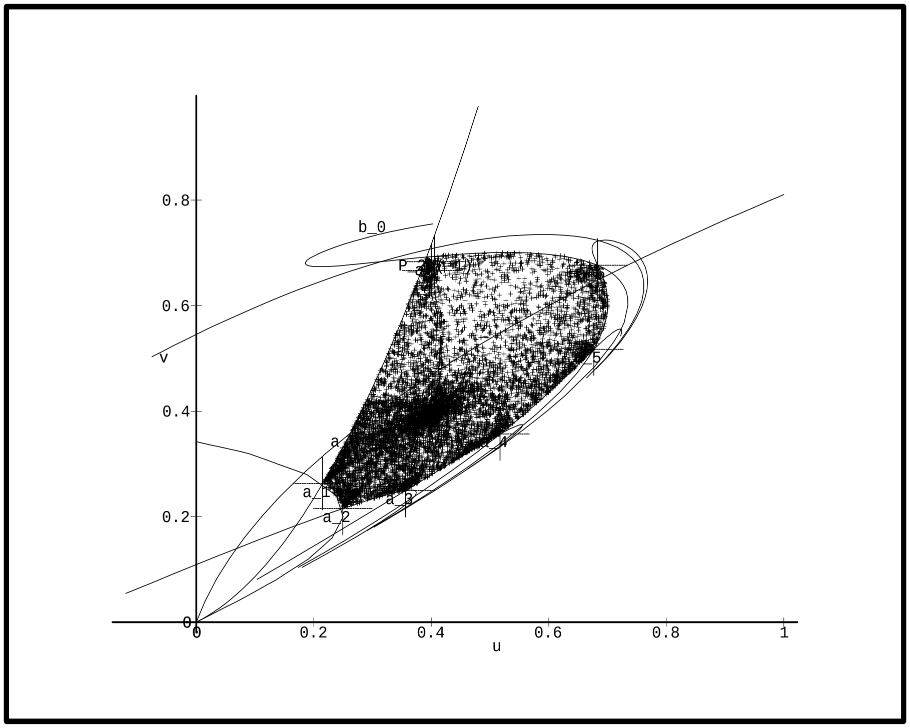


FIG. 4 – Absorbing area for $a = 1.8$ and $T = 0.6$

5.3.2 Return on the modelling

In the study of the previous dynamical system, we did not check if the constraint of function of the tax (tax lower than the wealth of the older people) was satisfied. We are going to make this study a posteriori and clarify the choice of the parameters which allows to satisfy this constraint.

Let us recall that the constraint of wealth is set

$$CW = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+, Ty < ax\}$$

Property 5.8. *If* $\frac{a^2}{T} - 1 > 0$*,* CW *intersects* LC^1 *into two points : the trivial point and a non trivial one that we call* I*.*

Proof The equation of LC^1 is $ay - x - h\left(\frac{x}{T}\right)$. Let (x, y) be the intersection point of LC^1 and CW. x satisfies $x\left(\frac{a^2}{T} - 1\right) - h\left(\frac{x}{T}\right) = 0$.

If b_0 belongs to the part of LC^1 contained between $(0,0)$ and I, then the wealth's constraint is satisfied, otherwise it is not and the range of parameter do not allow the government to implement this tax.

5.4 Time inconsistency problem

As the tax does not appear in the consumer program, it means that agents can not predict it. The character of non predictability should be verified a posteriori.

We can observe that if b_0 is closed to ND, then the tax seems not to be predictable.

6 Conclusion

This study raises serious doubts about regarding the global stability properties of OLG model *à la Reichlin*. The perfect-foresight hypothesis is not accurate enough to keep the economy on a stable growth path. It seems that further research on the analysis of global stability should examine alternative expectations mechanisms. This question goes beyond the scope of our work. Nervertheless, we can give an answer through the government stabilizing fiscal policy. The redistributive policy among generations appears as an interesting stabilizing adjusment mechanism. Despite this stabilizing fiscal policy, the economic dynamics is characterized by weak stability properties.

A small change of parameters induces a large variation of the qualitative behaviour of the dynamics. This property reflects perhaps the intrinsic instability generate by spatial and intertemporel interactions among agents. In this sense, mathematical tools are not an obstacle to the economic anaysis (McCallum (1983)) but on the contrary they reveals the intrinsec instability of a society based on competitive markets.

Reference

Abraham,R. H. and L. Gardini, C. Mira (1997) : *Chaos in Discrete Dynamical Systems.* Springer Verlag.

Augeraud, E., and L. Augier (1999) : "Basin's varaition in an OLG model with production," Mimeo.

Azariadis, C. (1993) : *Intertemporal Macroeconomics.* Blackwell.

Barugola, A. (1984) : "Quelques propriétés des lignes critiques d'une récurrence du second ordre à inverse non unique. Détermination d'une zone absorbante," *RAIRO analyse numérique*, vol 18, 2,137-151.

Barugola, A., and J.C. Cathala (1986) : "Annular chaotic areas," *Non Linear Analysis, Theory and Application*, vol 10, 11,1223-1236.

Benhabib J. and G. Laroque (1988) : "On competitive cycles in productive economies," Journal of Economic Theory, 45 :145-170.

dynamical system approach," *Macroeconomic Dynanic*, 3, 2, 167-186.

De Vilder, R. (1995) : "Endogenous business cycles," *Ph.D thesis University of Amsterdam.*

Diamond,P. (1965) : "National Debt in a Neoclassical Growth Model," *American Economic Review*, 55, 1126-1150.

Gale, D. (1973) : "Pure exchange equilibrium of dynamic Economic Models," , *Journal of Economic Theory,* 6, 12-36.

Guckenheimer, J. and P. Holmes (1983) : *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields.* Springer Verlag.

Gumowski I. and C. Mira (1964) : "Sur un alogorithme de dÝtermination du domaine de stabilitÝ d'un point double d'une rÝcurrence non linÝaire du deuxiÜme ordre Õ variables rÝelles ;" CRAS, sÝrieA, 260 :6524-6527.

Hahn, F. H. (1966) : "Equillibrium Dynamics with Heterogeneous Capital Goods," *Quarterly Journal of Economics*, 80.

Hale J. and H. Kocak. (1991) : *Dynamics and Bifurcations*, volume 3 of *Texts in Applied Mathematics*, New-York :Springer-Verlag.

Kehoe T. and D. Levine (1985) : "Comparative statics and perfect foresight", *Econometrica*, 53 :433–454.

Kydland, F. and E. Prescott (1977) : "Rules rather than discretion : the inconsistency of optimal plans," *Journal of Political Economy*, 85.

Laitner J. (1982) :"The definition of stability in models with perfect foresight", Journal of Economic Theory, 28 :347–353.

McCallum, B. (1983) : "On non uniqueness in rational expectations Models : An attempt at perspective," *Journal of Monetary Economics,* 11, 139-168.

Medio, A. (1992) : *Chaotic dynamics.* Cambridge University Press.

Mira, C. and Gardini, Barugola, Cathala (1996) : "Chaotic dynamics in two dimensional noninvertible maps," *World Scientific Series on Nonlinear Science,* series A, vol.20.

Reichlin, P. (1986) : "Equilibrium cycles in an overlapping generations economy with production," *Journal of Economic Theory,* 40, 89-102.

Soliman, A. (1997) : "The loss of predictability of monetary policy in a macrodynamic system : highly intertwined absolute and transient basins of attraction," *International Journal of Bifurcation and Chaos*, 7, 39-70.

99-1 Monique Jeanblanc et Nicolas Privault. A complete market model with Poisson and Brownian components. A paraître dans *Proceedings of the Seminar on Stochastic Analysis, Random Fields and Applications*, Ascona, 1999.

99-2 Laurence Cherfils et Alain Miranville. Generalized Cahn-Hilliard equations with a logarithmic free energy. A paraître dans *Revista de la Real Academia de Ciencias.*

99-3 Jean-Jacques Prat et Nicolas Privault. Explicit stochastic analysis of Brownian motion and point measures on Riemannian manifolds. *Journal of Functional Analysis*, Vol. **167**, pp. 201-242, 1999.

99-4 Changgui Zhang. Sur la fonction q-Gamma de Jackson. A paraître dans *Aequationes Math.*

99-5 Nicolas Privault. A characterization of grand canonical Gibbs measures by duality. A paraître dans *Potential Analysis.*

99-6 Guy Wallet. La variété des équations surstables. A paraître dans *Bulletin de la Société Mathématique de France.*

99-7 Nicolas Privault et Jiang-Lun Wu. Poisson stochastic integration in Hilbert spaces. *Annales Mathématiques Blaise Pascal*, Vol. **6**, pp. 41-61, 1999.

99-8 Augustin Fruchard et Reinhard Schäfke. Sursabilité et résonance.

99-9 Nicolas Privault. Connections and curvature in the Riemannian geometry of configuration spaces. *C. R. Acad. Sci. Paris, Série I, t. 330, pp. 899-904, 2000.*

99-10 Fabienne Marotte et Changgui Zhang. Multisommabilité des séries entières solutions formelles d'une équation aux q-différences linéaire analytique. A paraître dans *Annales de l'Institut Fourier*, 2000.

99-11 Knut Aase, Bernt Øksendal, Nicolas Privault et Jan Ubøe. White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance. *Finance and Stochastics*, Vol. **4**, pp. 465-496, 2000.

00-01 Eric Benoît. Canards en un point pseudo-singulier nœud. A paraître dans *Bulletin de la Société Mathématique de France.*

00-02 Nicolas Privault. Hypothesis testing and Skorokhod stochastic integration. *Journal of Applied Probability*, Vol. **37**, pp. 560-574, 2000.

00-03 Changgui Zhang. La fonction théta de Jacobi et la sommabilité des séries entières q-Gevrey, I. *C. R. Acad. Sci. Paris, Série I, t. 331, pp. 31-34, 2000.*

00-04 Guy Wallet. Déformation topologique par changement d'échelle.

00-05 Nicolas Privault. Quantum stochastic calculus for the uniform measure and Boolean convolution. A paraître dans *Séminaire de Probabilités* **XXXV**.

00-06 Changgui Zhang. Sur les fonctions q-Bessel de Jackson.

00-07 Laure Coutin, David Nualart et Ciprian A. Tudor. Tanaka formula for the fractional Brownian motion. A paraître dans *Stochastic Processes and their Applications.*

00-08 Nicolas Privault. On logarithmic Sobolev inequalities for normal martingales. A paraître dans *Annales de la Faculté des Sciences de Toulouse.*

01-01 Emanuelle Augeraud-Veron et Laurent Augier. Stabilizing endogenous fluctuations by fiscal policies ; Global analysis on piecewise continuous dynamical systems. A paraître dans *Studies in Nonlinear Dynamics and Econometrics*