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Abstract

We define a class of distributions on Poisson space which allows to iterate a modification of the gradient of [1]. As an application we obtain, with relatively short calculations, a formula for the chaos expansion of functionals of jump times of the Poisson process.

Key words: Poisson process, stochastic analysis, chaotic decompositions.
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1 Introduction

Let $(N_t)_{t \in \mathbf{R}_+}$ be a standard Poisson process with jump times $(T_k)_{k \geq 1}$, and $T_0 = 0$. The underlying probability space is denoted by $(\Omega, \mathcal{F}, P)$, so that $L^2(\Omega, \mathcal{F}, P)$ is the space of square-integrable functionals of $(N_t)_{t \in \mathbf{R}_+}$. Any $F \in L^2(\Omega, \mathcal{F}, P)$ can be expanded into the series

$$F = E[F] + \sum_{n=1}^{\infty} \frac{1}{n!} I_n(f_n) \qquad (1.0.1)$$

where $I_n(f_n)$ is the iterated stochastic integral

$$I_n(f_n) = n! \int_0^{\infty} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) d(N_{t_1} - t_1) \cdots d(N_{t_n} - t_n)$$

of the symmetric function $f_n \in L^2(\mathbb{R}_+^{\circ n})$ (stochastic integrals are taken in the Itô sense, thus diagonal terms have no influence in the above expression), with the isometry

$$\langle I_n(f_n), I_m(g_m) \rangle_{L^2(\Omega)} = n! 1_{\{n=m\}} \langle f_n, g_m \rangle_{L^2(\mathbf{R}_+, dt)^{\circ n}}, \quad f_n \in L^2(\mathbb{R}_+, dt)^{\circ n}, \ g_m \in L^2(\mathbb{R}_+, dt)^{\circ m}.$$

If f_n is not symmetric we let $I_n(f_n) = I_n(\tilde{f}_n)$, where $\tilde{f}_n$ denotes the symmetrization of f_n in n variables, hence (1.0.1) can be written as

$$F = E[F] + \sum_{n=1}^{\infty} I_n(f_n 1_{\Delta_n})$$

where

$$\Delta_n = \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n \ : \ 0 \le t_1 < \cdots < t_n\}.$$

Let $D : L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{R}_+)$ denote the linear unbounded operator defined on multiple stochastic integrals as

$$D_t I_n(f_n) = n I_{n-1}(f_n(*, t)), \quad a.e. \ t \in \mathbb{R}_+.$$

The formula of Y. Ito [3] (Relations (7.4) and (7.5), pp. 26-27), allows in principle to compute f_n as

$$f_n(t_1, \ldots, t_n) = E[D_{t_1} \cdots D_{t_n} F], \quad a.e. \ t_1, \ldots, t_n \in \mathbb{R}_+.$$

Given the probabilistic interpretation of D as a finite difference operator (cf. [3] and [6]), we have for $F = f(T_1, \ldots, T_d)$:

$$D_t F = \sum_{k=1}^{k=d} 1_{]T_{k-1}, T_k]}(t) \big(f(T_1, \ldots, T_{k-1}, t, T_k, \ldots, T_{d-1}) - f(T_1, \ldots, T_d) \big), \quad t \in \mathbb{R}_+,$$

thus $D_{t_1} \cdots D_{t_n} F$ is well defined and explicit computations can be carried out but may be complicated due to the recursive application of a finite difference operator, cf. [4]. See [8] for an elementary approach using only orthogonal expansions in Charlier polynomials.

On the other hand, the gradient $\tilde{D}$ of [1] (see also [2]), defined as

$$\tilde{D}_t = -\sum_{k=1}^{k=d} 1_{[0, T_k]}(t) \partial_k f(T_1, \ldots, T_d),$$

has some properties in common with D, namely its adapted projection coincides with that of D, and in particular we have

$$E[D_t F] = E[\tilde{D}_t F], \quad t \in \mathbb{R}_+.$$

Since the operator $\tilde{D}$ has the derivation property it is easier to manipulate than the finite difference operator D in recursive computations. Its disadvantage is that it

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can not be iterated in L^2 due to the non-differentiability of $1_{[0,T_k]}(t)$ in T_k, thus an expression such as $E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F]$ makes a priori no sense, moreover $E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F]$ may differ from $E[D_{t_1} \cdots D_{t_n} F]$ for $n \geq 2$ (see Relations (6.0.15) and (6.0.16) below).

In [7] the combined use of D^n and $\tilde{D}$ in L^2 sense has led to the computation of the expansion of the jump time T_d, $d \geq 1$. A direct calculation using only the operator D can be found in [5], concerning a Poisson process on a bounded interval.

In this paper we show that the gradient $\tilde{D}$ can be iterated in a precise sense of distributions on Poisson space, to be introduced in Sect. 3. For example we have for $(t_1, \ldots, t_n) \in \Delta_n$:

$$
\begin{aligned}
\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d) &= (-1)^n f^{(n)}(T_d) 1_{[0,T_d]}(t_1 \vee \cdots \vee t_n) \\
&\quad + (-1)^n 1_{\{0 \leq t_1, \ldots, t_{n-1} < t_n\}} \sum_{j=1}^{n-1} f^{(n-j)}(t_n) \delta_{t_n}^{(j-1)}(T_d),
\end{aligned}
$$

where $\delta_{t_n}(T_d)$ is a generalized functional, i.e. the composition of the Dirac distribution δ_{t_n} at t_n with the jump time T_d, cf. Prop. 4.0.3, $f^{(n)}$ denotes the n-th derivative of the function $f \in \mathcal{C}_b^\infty(\mathbb{R}_+)$, and $t_1 \vee \cdots \vee t_n = \max(t_1, \ldots, t_n)$. Moreover we obtain the equality

$$
E[D_{t_1} \cdots D_{t_n} F \mid \mathcal{F}_a] = E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F \mid \mathcal{F}_a], \quad 0 \leq a < t_1 < \cdots < t_n, \ n \geq 2,
$$

where we make sense of the conditional expectation $E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F \mid \mathcal{F}_a]$ using the pairing $\langle \cdot, \cdot \rangle$ between distributions and test functions. This implies.

$$
f_n(t_1, \ldots, t_n) = E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F], \quad 0 < t_1 < \cdots < t_n.
$$

This gives an expression for the decomposition of $f(T_1, \ldots, T_n)$, $f \in \mathcal{C}_b^\infty(\Delta_d)$, with relatively short computations, cf. Prop. 6.0.6, for example

$$
f(T_d) = \sum_{n=0}^{\infty} I_n(h_n 1_{\Delta_n}),
$$

with

$$
\begin{aligned}
h_n(t_1, \ldots, t_n) &= E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d)] \\
&= (-1)^n \int_{t_n}^{\infty} f^{(n)}(t) p_{d-1}(t) dt + (-1)^n \sum_{j=1}^{j=n-1} (-1)^{j-1} f^{(n-j)}(t_n) p_{d-1}^{(j-1)}(t_n),
\end{aligned}
$$

$0 < t_1 < \cdots < t_n$, where $p_{d-1}(t) = \frac{t^{d-1}}{(d-1)!} e^{-t}$, $t \in \mathbb{R}_+$, $d \geq 1$.

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2 Integration by parts

In this section we review the definition of the three main gradient operators on Poisson space, and present an elementary derivation of integration by parts formulas. All $\mathcal{C}^\infty$ functions on Δ_d are extended by continuity to the closure of Δ_d.

Definition 2.0.1 *Let $a \geq 0$. Let $\mathcal{S}_d(\Omega \times [a, \infty[^l)$ denote the test function space*

$$\mathcal{S}_d(\Omega \times [a, \infty[^l) = \{h_1 \otimes \cdots \otimes h_l \otimes f(T_1, \ldots, T_d) \; : \quad f \in \mathcal{C}_b^\infty(\Delta_d), \; h_1, \ldots, h_l \in \mathcal{C}_b([a, \infty[)\},$$

with $\mathcal{S}_d(\Omega) = \mathcal{S}_d(\Omega \times \mathbf{R}_+^0)$ for $l = 0$.

We recall that if $f \in L^2(\Delta_d, e^{-t_d} dt_1 \cdots dt_d)$ then

$$E[f(T_1, \ldots, T_d)] = \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_d,$$

which follows e.g. from the fact that $(\tau_n)_{n \geq 1} = (T_n - T_{n-1})_{n \geq 1}$ is a family of independent exponential random variables.

2.1 Intrinsic gradient

The intrinsic gradient $\hat{D}$ on Poisson space is defined on $\mathcal{S}_d(\Omega)$ as

$$\hat{D}_t F = \sum_{k=1}^{k=d} 1_{\{T_k\}}(t) \partial_k f(T_1, \ldots, T_d), \quad dN_t - a.e.,$$

with $F = f(T_1, \ldots, T_d)$, $f \in \mathcal{C}_b^\infty(\Delta_d)$, where $\partial_k f$ represents the partial derivative of f with respect to its k-th variable, $1 \leq k \leq d$.

Lemma 1 *Let $F \in \mathcal{S}_d(\Omega)$ and $h \in \mathcal{C}_b^1(\mathbf{R}_+)$. We have the integration by parts formula*

$$E[\langle \hat{D}F, h \rangle_{L^2(\mathbf{R}_+, dN_t)}] = E\left[FU_{h'}^d\right] = -E\left[F \int_0^\infty h'(t) d(N_t - t)\right]. \tag{2.1.2}$$

where $U_{h'}^d = -\left(\sum_{k=1}^{k=d} h'(T_k) - \int_0^{T_d} h'(t) dt\right) \in \mathcal{S}_d(\Omega)$.

Proof. We have by integration by parts on Δ_d:

$$E[\langle \hat{D}F, h \rangle_{L^2(\mathbf{R}_+, dN_t)}] = \sum_{k=1}^{k=d} \int_0^\infty \int_0^{t_d} \cdots \int_0^{t_2} e^{-t_d} h(t_k) \partial_k f(t_1, \ldots, t_d) dt_1 \cdots dt_d$$

$$= \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_1) \partial_1 f(t_1, \ldots, t_d) dt_1 \cdots dt_d$$

$$+ \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} h(t_k) \frac{\partial}{\partial t_k} \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_d$$

$$- \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} h(t_k) \int_0^{t_d} \cdot \int_0^{t_k} \int_0^{t_{k-2}} \cdot \int_0^{t_2} f(t_1, \ldots, t_{k-2}, t_k, t_k, \ldots, t_d) dt_1 \cdot d\hat{t}_{k-1} \cdot dt_d$$

$$= - \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h'(t_1) f(t_1, \ldots, t_d) dt_1 \cdots dt_d$$

$$+ \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_3} h(t_2) f(t_2, t_2, \ldots, t_d) dt_2 \cdots dt_d$$

$$- \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} h'(t_k) \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_d$$

$$+ \int_0^\infty e^{-t_d} h(t_d) \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_d$$

$$+ \sum_{k=2}^{k=d-1} \int_0^\infty e^{-t_d} h(t_{k+1}) \int_0^{t_d} \cdot \int_0^{t_{k+1}} \int_0^{t_{k-1}} \cdot \int_0^{t_2} f(t_1, ., t_{k-1}, t_{k+1}, t_{k+1}, ., t_d) dt_1 \cdot d\hat{t}_k \cdot dt_d$$

$$- \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} h(t_k) \int_0^{t_d} \cdot \int_0^{t_k} \int_0^{t_{k-2}} \cdot \int_0^{t_2} f(t_1, ., t_{k-2}, t_k, t_k, ., t_d) dt_1 \cdot dt_d$$

$$= - \sum_{k=1}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h'(t_k) f(t_1, \ldots, t_d) dt_1 \cdots dt_d$$

$$+ \int_0^\infty e^{-t_d} h(t_d) \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_d$$

$$= - E\left[F\left(\sum_{k=1}^{k=d} h'(T_k) - \int_0^{T_d} h'(t) dt \right) \right],$$

where $d\hat{t}_k$ denotes the absence of dt_k. Concerning the second part of the equality it suffices to notice that if $k > d$,

$$
\begin{aligned}
E[Fh'(T_k)] &= \int_0^\infty e^{-t_k} h'(t_k) \int_0^{t_k} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_k \\
&= \int_0^\infty e^{-t_k} h(t_k) \int_0^{t_k} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_k \\
&\quad - \int_0^\infty e^{-t_{k-1}} h(t_{k-1}) \int_0^{t_{k-1}} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_{k-1} \\
&= E[F(h(T_k) - h(T_{k-1}))] = E\left[F \int_{T_{k-1}}^{T_k} h'(t) dt \right].
\end{aligned}
$$

$\square$

Relation (2.1.2) implies immediately for $F, G \in \mathcal{S}_d(\Omega)$:

$$
\begin{aligned}
E[\langle \hat{D}F, hG \rangle_{L^2(\mathbf{R}_+, dN_t)}] &= E\left[\langle \hat{D}(FG), h \rangle_{L^2(\mathbf{R}_+, dN_t)} - F\langle \hat{D}G, h \rangle_{L^2(\mathbf{R}_+, dN_t)} \right] \\
&= E\left[F(GU_{h'}^d - \langle h, \hat{D}G \rangle_{L^2(\mathbf{R}_+, dN_t)}) \right] \\
&= -E\left[F\left(G \int_0^\infty h'(t) d(N_t - t) + \langle h, \hat{D}G \rangle_{L^2(\mathbf{R}_+, dN_t)} \right) \right].
\end{aligned}
$$

2.2   Damped gradient

Let $r(s, t) = -s \vee t$ denote the Green function associated to the Laplacian $\mathcal{L}$ on $\mathbb{R}_+$:

$$
\mathcal{L}f = -f'', \quad f \in \mathcal{C}_c^\infty(]0, \infty[),
$$

i.e. we have, with $g = -f''$:

$$
\int_0^\infty r(s, t) g(t) dt = -\int_0^s \int_0^t g(u) du dt, \quad s \in \mathbb{R}_+.
$$

Definition 2.2.2 *Given $F \in \mathcal{S}_d(\Omega)$, $F = f(T_1, \ldots, T_d)$, we let*

$$
r^{(1)}(s, t) = \frac{\partial}{\partial s} r(s, t) = -1_{]-\infty, s]}(t), \quad s, t \in \mathbb{R}_+,
$$

and

$$
\tilde{D}_t F = \int_0^\infty r^{(1)}(s, t) \hat{D}_s F dN_s.
$$

We have

$$
\tilde{D}_t F = \sum_{k=1}^{k=d} r^{(1)}(T_k, t) \partial_k f(T_1, \ldots, T_d) = -\sum_{k=1}^{k=d} 1_{[0, T_k]}(t) \partial_k f(T_1, \ldots, T_d).
$$

In fact $\tilde{D}$ is (up to a minor modification) the gradient introduced in [1]. This presentation of $\tilde{D}$ using the Green function $r(s, t)$ is motivated by [9].

Proposition 2.2.1 *We have for $F \in \mathcal{S}_d(\Omega)$ and $h \in \mathcal{C}_c(\mathbb{R}_+)$:*

$$E[\langle \tilde{D}F, h \rangle_{L^2(\mathbf{R}_+, dt)}] = E\left[FU_h^d\right] = E\left[F \int_0^\infty h(t)d(N_t - t)\right], \qquad (2.2.3)$$

where $U_h^d = \sum_{k=1}^{k=d} h(T_k) - \int_0^{T_d} h(t)dt$.

Proof. We have

$$
\begin{aligned}
E\left[\langle \tilde{D}F, h \rangle_{L^2(\mathbf{R}_+, dt)}\right] &= E\left[\int_0^\infty \int_0^\infty r^{(1)}(s,t)\hat{D}_s F h(t)dN_s dt\right] \\
&= -E\left[\left\langle \hat{D}F, \int_0^{\cdot} h(t)dt \right\rangle_{L^2(\mathbf{R}_+, dN_t)}\right] = E\left[F \int_0^\infty h(t)d(N_t - t).\right]
\end{aligned}
$$

$\square$

Relation (2.2.3) also implies that for $F, G \in \mathcal{S}_d(\Omega)$,

$$
\begin{aligned}
E[\langle \tilde{D}F, hG \rangle_{L^2(\mathbf{R}_+, dt)}] &= E\left[\langle \tilde{D}(FG), h \rangle_{L^2(\mathbf{R}_+, dt)} - F\langle \tilde{D}G, h \rangle_{L^2(\mathbf{R}_+, dt)}\right] \qquad (2.2.4) \\
&= E\left[F(GU_h^d - \langle h, \tilde{D}G \rangle_{L^2(\mathbf{R}_+, dt)})\right] \\
&= E\left[F\left(G \int_0^\infty h(t)d(N_t - t) - \langle h, \tilde{D}G \rangle_{L^2(\mathbf{R}_+, dt)}\right)\right].
\end{aligned}
$$

2.3 Finite difference gradient

For completeness we mention the gradient D which is associated to the Fock space structure, and whose properties have been discussed in the introduction.

3 Distribution-valued gradient

For $n \geq 2$ and $F \in \mathcal{S}_d(\Omega)$ we let $dN_{t_1} \otimes \cdots \otimes dN_{t_n} - a.e.$:

$$\hat{D}_{t_1, \ldots, t_n}^n F = \sum_{1 \leq j_1, \ldots, j_n \leq d} 1_{\{T_{j_1}\}}(t_1) \cdots 1_{\{T_{j_n}\}}(t_n) \partial_{j_1} \cdots \partial_{j_n} f(T_1, \ldots, T_d).$$

This is not the n-th iteration of $\hat{D}$, in fact we have

$$\|\hat{D}^n F\|_{L^2(\mathbf{R}_+, dN_t)^{\otimes n}}^2 = \sum_{1 \leq j_1, \ldots, j_n \leq d} (\partial_{j_1} \cdots \partial_{j_i} f(T_1, \ldots, T_d))^2.$$

Definition 3.0.3 *Let $a \in \mathbb{R}_+$ and $l \in \mathbb{N}$.*

i) We denote by $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$ the space of continuous linear forms (distributions) on $\mathcal{S}_d(\Omega \times [a, \infty[^l)$, i.e. $F \in \mathcal{S}'_d(\Omega \times [a, \infty[^l)$ if there exists $k \geq 0$ and $C > 0$ such that

$$|\langle F, h_1 \otimes \cdots \otimes h_l \otimes G \rangle| \leq C \sum_{i=0}^{i=k} \|h_1 \otimes \cdots \otimes h_l\|_\infty \|\hat{D}^i G\|_{L^\infty(\Omega, L^2(\mathbf{R}_+, dN_t)^{\otimes i})},$$

$G \in \mathcal{S}_d(\Omega \times \mathbf{R}^l_+)$, $h_1, \dots, h_l \in \mathcal{C}_c([a, \infty[)$.

ii) A sequence $(F_n)_{n \in \mathbf{N}} \subset \mathcal{S}'_d(\Omega \times [a, \infty[^l)$ is said to converge in $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$ if the sequence $(\langle F_n, G \rangle)_{n \in \mathbf{N}}$ converges to $\langle F, G \rangle$ for all $G \in \mathcal{S}_d(\Omega \times [a, \infty[^l)$.

The notation $\langle \cdot, \cdot \rangle$ will be used to denote the pairing between $\mathcal{S}_d(\Omega \times [a, \infty[^l)$ and $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$, for all values of $l \in \mathbf{N}$. Every $F \in \mathcal{S}_d(\Omega \times [a, \infty[^l)$ is identified to an element of $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$ by letting

$$\langle F, G \rangle = \int_\Omega \langle F(\omega, \cdot), G(\omega, \cdot) \rangle_{L^2([a, \infty[, dt)^{\otimes l}} P(d\omega), \quad G \in \mathcal{S}_d(\Omega \times [a, \infty[^l).$$

The closability property in L^2 of the operator $\tilde{D}$ is a well-known statement which extends to distributions in $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$.

Proposition 3.0.2 *Let* $(F_n)_{n \in \mathbf{N}} \subset \mathcal{S}_d(\Omega \times [a, \infty[^l)$ *such that*

i) $(F_n)_{n \in \mathbf{N}}$ *converges to 0 in* $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$,

ii) $(\tilde{D}F_n)_{n \in \mathbf{N}}$ *converges in* $\mathcal{S}'_d(\Omega \times [a, \infty[^{l+1})$.

Then $(\tilde{D}F_n)_{n \in \mathbf{N}}$ *converges to 0 in* $\mathcal{S}'_d(\Omega \times [a, \infty[^{l+1})$.

Proof. For $l = 0$ this is a direct consequence of the integration by parts formula (2.2.4), which shows that

$$\langle \tilde{D}F_n, hG \rangle = E[F_n(GU_h^d - \langle \tilde{D}G, h \rangle_{L^2(\mathbf{R}_+, dt)})], \quad h \in \mathcal{C}_c([a, \infty[), \ G \in \mathcal{S}_d(\Omega),$$

with $GU_h^d - \langle \tilde{D}G, h \rangle_{L^2(\mathbf{R}_+, dt)} \in \mathcal{S}_d(\Omega)$. The generalization to $l \geq 1$ is straightforward. $\square$

This proposition justifies the following extension of $\tilde{D}$ to generalized functionals.

Definition 3.0.4 *Let* $a \in \mathbf{R}_+$. *We let* $I\!\!D_l([a, \infty[)$ *denote the subspace of* $F \in \mathcal{S}'_d(\Omega \times [a, \infty[^l)$ *such that*

i) there exists $(F_n)_{n \in \mathbf{N}} \subset \mathcal{S}_d(\Omega \times [a, \infty[^l)$ that converges to F in $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$,

ii) $(\tilde{D}F_n)_{n \in \mathbf{N}}$ *converges in* $\mathcal{S}'_d(\Omega \times [a, \infty[^{l+1})$.

Given F as above we define $\tilde{D}F$ as the limit in $\mathcal{S}'_d(\Omega \times [a, \infty[^{l+1})$

$$\tilde{D}F = \lim_{n \to \infty} \tilde{D}F_n, \quad F \in I\!\!D_l([a, \infty[) \subset \mathcal{S}'_d(\Omega \times [a, \infty[^l).$$

4   Iterated gradient in distribution sense

We let for $n \geq 2$:
$$r^{(n)}(T_k, t) = \partial_1^n r(T_k, t) = -\delta_t^{(n-2)}(T_k),$$

in distribution sense, i.e. $r^{(n)}(T_k, t)$ belongs to $\mathcal{S}'_d(\Omega)$ with for $k = 1, \ldots, d$, and $f \in \mathcal{C}_b^\infty(\Delta_d)$:

$$
\begin{aligned}
&\langle r^{(n)}(T_k, t), f(T_1, \ldots, T_d) \rangle \\
&= \quad (-1)^{n+1} 1_{\{k<d\}} \int_0^\infty e^{-s_d} \int_0^{s_d} \cdots \int_0^{s_{k+2}} \\
&\qquad \left(\frac{\partial^{n-2}}{\partial s_k^{n-2}} \int_0^{s_k} \cdots \int_0^{s_2} f(s_1, \ldots, s_d) ds_1 \cdots ds_{k-1} \right)_{|s_k=t} ds_{k+1} \cdots ds_d \\
&\qquad + (-1)^{n+1} 1_{\{k=d\}} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t \int_0^{s_{d-1}} \cdots \int_0^{s_2} e^{-t} f(s_1, \ldots, s_{d-1}, t) ds_1 \cdots ds_{d-1},
\end{aligned}
$$

and for $n = 1$:

$$\langle r^{(1)}(T_k, t), f(T_1, \ldots, T_d) \rangle = - \int_0^\infty \int_0^{s_d} \cdots \int_0^{s_2} e^{-s_d} 1_{[t, \infty[}(s_k) f(s_1, \ldots, s_d) ds_1 \cdots ds_d.$$

Let $\phi \in \mathcal{C}_c^\infty([-1, 1])$, $\phi \geq 0$, such that $\int_{-1}^1 \phi(t) dt = 1$, and let

$$\phi_\varepsilon(t) = \varepsilon^{-1} \phi(\varepsilon^{-1} t), \quad t \in \mathbb{R}, \quad \varepsilon > 0.$$

Let $\phi_\varepsilon * r^{(n)}(T_k, t)$, $n \geq 1$, denote the convolution of ϕ_ε with $r^{(n)}(T_k, t)$ in the first variable, i.e. for $n = 1$:

$$\phi_\varepsilon * r^{(1)}(T_k, t) = - \int_{-\infty}^\infty \phi_\varepsilon(u) 1_{]-\infty, T_k-u]}(t) du = - \int_{-\infty}^\infty \phi_\varepsilon(T_k-u) 1_{]-\infty, u]}(t) du, \quad s, t \in \mathbb{R}_+,$$

and for $n \geq 2$, $l \in \mathbb{N}$:

$$\phi_\varepsilon^{(l)} * r^{(n)}(T_k, t) = \phi_\varepsilon * r^{(n+l)}(T_k, t) = -\phi_\varepsilon^{(n+l-2)}(T_k - t),$$

which converges in $\mathcal{S}'_d(\Omega)$ to $r^{(n+l)}(T_k, t)$ if $n+l \geq 1$ (i.e. to $-\delta_t^{(n+l-2)}(T_k)$ if $n+l \geq 2$), $k = 1, \ldots, d$. Let $t_1 \vee \cdots \vee t_n$, $t_1, \ldots, t_n \in \mathbb{R}_+$.

Proposition 4.0.3 *Let $k \geq 1$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}_+)$. Then for all $n \geq 1$, $\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k) \in \mathbb{D}_0(\mathbb{R}_+)$ for a.a. $(t_1, \ldots, t_n) \in \mathbb{R}_+^n$, and*

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k) = (-1)^n f^{(n)}(T_k) 1_{[0,T_k]}(t_1 \vee \cdots \vee t_n)$$

$$+ (-1)^n \sum_{j=1}^{n-1} 1_{\{t_1 \vee \cdots \vee t_j < t_{j+1} \vee \cdots \vee t_n\}} f^{(n-j)}(t_{j+1} \vee \cdots \vee t_n) \delta_{t_{j+1} \vee \cdots \vee t_n}^{(j-1)}(T_k). \quad (4.0.5)$$

Proof. For $n = 1$ this is the definition of $\tilde{D}$. We proceed by induction, assuming that $\tilde{D}_{t_3} \cdots \tilde{D}_{t_{n+1}} f(T_k) \in \mathbb{D}_0(\mathbb{R}_+)$ for some $n \geq 1$, and

$$\tilde{D}_{t_2} \cdots \tilde{D}_{t_{n+1}} f(T_k) = (-1)^n f^{(n)}(T_k) 1_{[0,T_k]}(t_2 \vee \cdots \vee t_{n+1})$$

$$+ (-1)^n \sum_{j=1}^{n-1} 1_{\{t_2 \vee \cdots \vee t_{j+1} < t_{j+2} \vee \cdots \vee t_{n+1}\}} f^{(n-j)}(t_{j+2} \vee \cdots \vee t_{n+1}) \delta_{t_{j+2} \vee \cdots \vee t_{n+1}}^{(j-1)}(T_k).$$

Let for $\varepsilon > 0$. We define a smooth approximation of $\tilde{D}_{t_2} \cdots \tilde{D}_{t_{n+1}} f(T_k)$ by letting

$$F_\varepsilon(t_2, \ldots, t_{n+1}) = (-1)^n f^{(n)}(T_k) \phi_\varepsilon * r^{(1)}(T_k, t_2 \vee \cdots \vee t_{n+1})$$

$$+ (-1)^n \sum_{j=1}^{n-1} 1_{\{t_2 \vee \cdots \vee t_{j+1} < t_{j+2} \vee \cdots \vee t_{n+1}\}} f^{(n-j)}(t_{j+2} \vee \cdots \vee t_{n+1}) \phi_\varepsilon * r^{(j+1)}(T_k, t_{j+2} \vee \cdots \vee t_{n+1}).$$

Then $F_\varepsilon(t_2, \ldots, t_{n+1}) \in \mathcal{S}_d(\Omega)$ and

$$\begin{aligned}
\tilde{D}_{t_1} F_\varepsilon(t_2, \ldots, t_{n+1}) &= (-1)^{n+1} 1_{[0,T_k]}(t_1) \phi_\varepsilon * r^{(1)}(T_k, t_2 \vee \cdots \vee t_{n+1}) f^{(n+1)}(T_k) \\
&\quad + (-1)^{n+1} 1_{[0,T_k]}(t_1) f^{(n)}(T_k) \phi_\varepsilon' * r^{(1)}(T_k, t_2 \vee \cdots \vee t_{n+1}) \\
&\quad + (-1)^{n+1} 1_{[0,T_k]}(t_1) \sum_{j=1}^{n-1} 1_{\{t_2 \vee \cdots \vee t_{j+1} < t_{j+2} \vee \cdots \vee t_{n+1}\}}(T_k) f^{(n-j)}(t_{j+2} \vee \cdots \vee t_{n+1}) \\
&\quad \times \phi_\varepsilon' * r^{(j+1)}(T_k, t_{j+2} \vee \cdots \vee t_{n+1}) \\
&= (-1)^{n+1} 1_{[0,T_k]}(t_1) \phi_\varepsilon * r^{(1)}(T_k, t_2 \vee \cdots \vee t_{n+1}) f^{(n+1)}(T_k) \\
&\quad + (-1)^{n+1} 1_{[0,T_k]}(t_1) f^{(n)}(T_k) \phi_\varepsilon * r^{(2)}(T_k, t_2 \vee \cdots \vee t_{n+1}) \\
&\quad + (-1)^{n+1} 1_{[0,T_k]}(t_1) \sum_{j=2}^{n} 1_{\{t_2 \vee \cdots \vee t_j < t_{j+1} \vee \cdots \vee t_{n+1}\}}(T_k) f^{(n+1-j)}(t_{j+1} \vee \cdots \vee t_{n+1}) \\
&\quad \times \phi_\varepsilon * r^{(j+1)}(T_k, t_{j+1} \vee \cdots \vee t_{n+1}) \\
&= (-1)^{n+1} 1_{[0,T_k]}(t_1) \phi_\varepsilon * r^{(1)}(T_k, t_2 \vee \cdots \vee t_{n+1}) f^{(n+1)}(T_k) \\
&\quad + (-1)^{n+1} 1_{[0,T_k]}(t_1) \sum_{j=1}^{n} 1_{\{t_2 \vee \cdots \vee t_j < t_{j+1} \vee \cdots \vee t_{n+1}\}}(T_k) f^{(n+1-j)}(t_{j+1} \vee \cdots \vee t_{n+1}) \\
&\quad \times \phi_\varepsilon * r^{(j+1)}(T_k, t_{j+1} \vee \cdots \vee t_{n+1}),
\end{aligned}$$

where we used the relation $\phi'_\varepsilon * r^{(j+1)} = \phi_\varepsilon * r^{(j+2)}$. As $\varepsilon \to 0$, $F_\varepsilon(t_2, \ldots, t_{n+1})$ converges in $\mathcal{S}'_d(\Omega)$ to

$$(-1)^{n+1} f^{(n+1)}(T_k) 1_{[0,T_k]}(t_1 \vee \cdots \vee t_{n+1})$$

$$+(-1)^{n+1} \sum_{j=1}^n 1_{\{t_1 \vee \cdots \vee t_j < t_{j+1} \vee \cdots \vee t_{n+1}\}} f^{(n+1-j)}(t_{j+1} \vee \cdots \vee t_{n+1}) r^{(j+1)}(T_k, t_{j+1} \vee \cdots \vee t_{n+1}).$$

$$\square$$

In particular, for $n \geq 2$:

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} T_k = (-1)^{n+1} 1_{\{0 \leq t_1, \ldots, t_{n-1} < t_n\}} r^{(n)}(T_k, t_n) = (-1)^n 1_{\{0 \leq t_1, \ldots, t_{n-1} < t_n\}} \delta_{t_n}^{(n-2)}(T_k).$$

We note that since $f \in \mathcal{C}_b^\infty(\mathbb{R}_+)$, there exists $C > 0$ such that

$$|\langle \tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k), h_1 \otimes \cdots \otimes h_l \otimes G \rangle| \leq C \sum_{i=0}^{i=k} \|h_1 \otimes \cdots \otimes h_l\|_\infty \|\hat{D}^i G\|_{L^\infty(\Omega, L^2(\mathbb{R}_+, dN_t)^{\otimes i})},$$

$dt_1 \cdots dt_n - a.e.$, for all $G \in \mathcal{S}_d(\Omega)$ and $h_1, \ldots, h_l \in \mathcal{C}_c(\mathbb{R}_+)$. Hence $\tilde{D}^n f(T_k) \in \mathbb{D}_n(\mathbb{R}_+) \subset \mathcal{S}'_d(\Omega \times \mathbb{R}_+^n)$, and (4.0.5) can be written as

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k) = (-1)^n \sum_{j=0}^{(n-1)\vee 0} 1_{\{t_1 \vee \cdots \vee t_j < t_{j+1} \vee \cdots \vee t_n\}} f^{(n-j)}(T_{k,j}) r^{(j+1)}(T_k, t_{j+1} \vee \cdots \vee t_n)$$

with $t_0 = 0$ and

$$T_{k,j} = \begin{cases} T_k & \text{if } j = 0, \\ t_{j+1} \vee \cdots \vee t_n & \text{if } j \geq 1, \end{cases}$$

i.e. if $0 \leq t_1 < \cdots < t_n$:

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k) = (-1)^n \sum_{j=0}^{(n-1)\vee 0} f^{(n-j)}(T_{k,j}) r^{(j+1)}(T_k, t_n).$$

If $t_1 > \cdots > t_n$ then

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F = (-1)^n \sum_{1 \leq j_1, \ldots, j_n \leq d} 1_{[0,T_{j_1}]}(t_1) \cdots 1_{[0,T_{j_n}]}(t_n) \partial_{j_1} \cdots \partial_{j_n} f(T_1, \ldots, T_d).$$

Let

$$\tilde{D}_t = \sum_{k=1}^{k=d} \tilde{D}_{t,k},$$

where $\tilde{D}_{t,k}$ is the partial gradient with respect to the k-th variable, i.e.

$$\tilde{D}_{t,k} f(T_1, \ldots, T_d) = -1_{[0,T_k]}(t) \partial_k f(T_1, \ldots, T_d), \quad 1 \leq k \leq d.$$

With this notation we have in general

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} = \sum_{1 \le j_1,\dots,j_n \le d} \tilde{D}_{t_1,j_1} \cdots \tilde{D}_{t_n,j_n}, \tag{4.0.6}$$

and for $0 < t_1 < \cdots < t_l$, Prop. 4.0.3 shows that

$$\tilde{D}_{t_1,k} \cdots \tilde{D}_{t_l,k} F = (-1)^{l+1} \sum_{j=0}^{j=(l-1)\vee 0} \partial_k^{l-j} f(T_1,\dots,T_{k-1},T_{k,j},T_{k+1},\dots,T_d) r^{(j+1)}(T_k,t_l), \tag{4.0.7}$$

with $t_0 = 0$. Now,

$$\tilde{D}_{t_1,j_1} \cdots \tilde{D}_{t_n,j_n} F$$

contains distribution terms $\delta_{t_l}^{(i)}(T_k)$, $i \ge 0$, in the variable T_k if and only if the indices equal to k in $(j_1,\dots,j_n)$ are consecutive except may be for the greatest of them, or are in number ≤ 2. If these conditions are no met, such distribution terms are killed by derivation with respect to a variable of index different from k, since $T_{k,j}$ in (4.0.7) is deterministic when $j \ge 1$.

Given $j_1,\dots,j_n \in \{1,\dots,d\}$ and $i \in \{1,\dots,d\}$, let

$$a_i(j_1,\dots,j_n) = \mathrm{Card}\{l \ : \ j_l = i\},$$

$$b_i(j_1,\dots,j_n) = \begin{cases} 0 & \text{if } a_i(j_1,\dots,j_n) = 0, \\ a_i(j_1,\dots,j_n) - 1 & \text{if the first } a_i(j_1,\dots,j_n) - 1 \text{ occurrences} \\ & \quad \text{of } i \text{ in } (j_1,\dots,j_n) \text{ are consecutive,} \\ 1 & \text{otherwise.} \end{cases}$$

$$c_i(j_1,\dots,j_n) = \max\{a : j_a = i\},$$

$$T_{l,i_l} = \begin{cases} T_l & \text{if } i_l = 0, \\ t_{c_l} & \text{if } i_l \ge 1. \end{cases}$$

With this notation, the above can be rephrased into the following formula in which only non-zero terms in (4.0.6) are appearing (the indices $j_1,\dots,j_n$ are omitted in $a_i(j_1,\dots,j_n)$, $b_i(j_1,\dots,j_n)$ and $c_i(j_1,\dots,j_n)$).

Theorem 4.0.1 *Let $F = f(T_1,\dots,T_d)$, with $f \in \mathcal{C}_b^\infty(\Delta_d)$. We have $\tilde{D}^n F \in I\!\!D_n(\mathbb{R}_+)$ for all $n \ge 0$, and:*

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F = (-1)^{n+1} \sum_{1 \le j_1,\dots,j_n \le d} \sum_{0 \le i_1 \le b_1} \cdots \sum_{0 \le i_d \le b_d} \tag{4.0.8}$$

$$\partial_1^{a_1-i_1} \cdots \partial_d^{a_d-i_d} f(T_{1,i_1},\dots,T_{d,i_d}) r^{(1+i_1)}(T_1,t_{c_1}) \cdots r^{(1+i_d)}(T_d,t_{c_d}),$$

$(t_1,\dots,t_n) \in \Delta_n$, $n \ge 2$.

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5 Equality of adapted projections in distribution sense

We recall that the adjoint of D extends the compensated Poisson stochastic integral, cf. [3], Th. 6.9, p. 23, i.e. for all adapted square-integrable process $u \in L^2(\Omega \times \mathbb{R}_+)$ we have

$$E[\langle DF, u \rangle_{L^2(\mathbf{R}_+, dt)}] = E\left[F \int_0^\infty u(t) d(N_t - t)\right]. \tag{5.0.9}$$

Since the adapted projections of $\tilde{D}$ and D coincide, cf. [7], Prop. 20:

$$E[\tilde{D}_t F \mid \mathcal{F}_a] = E[D_t F \mid \mathcal{F}_a], \quad 0 < a < t, \tag{5.0.10}$$

the same property hold for $\tilde{D}$:

$$E[\langle \tilde{D} F, u \rangle_{L^2(\mathbf{R}_+, dt)}] = E\left[F \int_0^\infty u(t) d(N_t - t)\right].$$

We now show that Relation (5.0.10) can be extended in distribution sense to $\tilde{D}^n$ and D^n, $n \geq 2$. The next proposition will be interpreted in terms of generalized conditional expectations as

$$E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F \mid \mathcal{F}_a] = E[D_{t_1} \cdots D_{t_n} F \mid \mathcal{F}_a], \quad (t_1, \ldots, t_n) \in \Delta_n \cap [a, \infty[^n.$$

Proposition 5.0.4 *Let $F \in \mathcal{S}_d(\Omega)$ and $G \in \mathcal{S}_d(\Omega)$ be $\mathcal{F}_a$-measurable. We have*

$$\langle G \tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F, 1 \rangle = E[G D_{t_1} \cdots D_{t_n} F], \quad (t_1, \ldots, t_n) \in \Delta_n \cap [a, \infty[^n. \tag{5.0.11}$$

Proof. The proposition holds for $n = 1$. We assume that it holds for some $n \geq 1$. Let $F_\varepsilon(t_2, \ldots, t_{n+1})$ denote the regularization of $\tilde{D}_{t_2} \cdots \tilde{D}_{t_{n+1}} F$ constructed as in the proof of Prop. 4.0.3:

$$F_\varepsilon(t_2, \ldots, t_{n+1}) = (-1)^{n+1} \sum_{1 \leq j_1, \ldots, j_n \leq d} \sum_{0 \leq i_1 \leq b_1} \cdots \sum_{0 \leq i_d \leq b_d}$$

$$\partial_1^{a_1 - i_1} \cdots \partial_d^{a_d - i_d} f(T_{1, i_1}, \ldots, T_{d, i_d}) \phi^\varepsilon * r^{(1+i_1)}(T_1, t_{1+c_1}) \cdots \phi^\varepsilon * r^{(1+i_d)}(T_d, t_{1+c_d}).$$

Let $f_{n+1} \in \mathcal{C}_c^\infty(\Delta_{n+1} \cap [a, \infty[^{n+1})$. Since G is $\mathcal{F}_a$ measurable we have $\tilde{D}_{t_1} G = 0$, $t_1 > a$, hence from Prop. 2.2.1:

$$E\left[\left(\int_a^{t_2} \tilde{D}_{t_1} F_\varepsilon(t_2, \ldots, t_{n+1}) f_{n+1}(t_1, \ldots, t_{n+1}) dt_1 \right) G\right]$$

$$= E\left[\int_a^{t_2} \tilde{D}_{t_1}(G F_\varepsilon(t_2, \ldots, t_{n+1})) f_{n+1}(t_1, \ldots, t_{n+1}) dt_1\right]$$

$$= E\left[G F_\varepsilon(t_2, \ldots, t_{n+1}) \left(\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \ldots, t_{n+1}) - \int_0^{T_d} f_{n+1}(t, t_2, \ldots, t_{n+1}) dt\right)\right],$$

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and

$$\langle \tilde{D}^{n+1}F, 1_{\Delta_{n+1}}f_{n+1}G\rangle$$

$$= \lim_{\varepsilon \to 0} \left\langle \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_2} \tilde{D}_{t_1}F_\varepsilon(t_2, \dots, t_{n+1})f_{n+1}(t_1, \dots, t_{n+1})dt_1 \cdots dt_{n+1}, G \right\rangle$$

$$= \lim_{\varepsilon \to 0} E \left[\int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_2} \tilde{D}_{t_1}F_\varepsilon(t_2, \dots, t_{n+1})f_{n+1}(t_1, \dots, t_{n+1})dt_1 \cdots dt_{n+1}G \right]$$

$$= \lim_{\varepsilon \to 0} E \left[G \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_3} F_\varepsilon(t_2, \dots, t_{n+1}) \right.$$
$$\left. \left(\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \dots, t_{n+1}) - \int_0^{T_d} f_{n+1}(t, t_2, \dots, t_{n+1})dt \right) dt_2 \cdots dt_{n+1} \right]$$

$$= E \left[G \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_3} \right.$$
$$\left. \left(\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \dots, t_{n+1}) - \int_0^{T_d} f_{n+1}(t_1, t_2, \dots, t_{n+1})dt_1 \right) \tilde{D}_{t_2} \cdots \tilde{D}_{t_{n+1}}F dt_2 \cdots dt_{n+1} \right]$$

$$= E \left[G \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_3} \right.$$
$$\left. \left(\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \dots, t_{n+1}) - \int_0^{T_d} f_{n+1}(t_1, t_2, \dots, t_{n+1})dt_1 \right) D_{t_2} \cdots D_{t_{n+1}}F dt_2 \cdots dt_{n+1} \right],$$

where on the last step we used the induction hypothesis with $a = t_2$. This is possible because the functional $\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \dots, t_{n+1}) - \int_0^{T_d} f_{n+1}(t_1, t_2, \dots, t_{n+1})dt_1$ is $\mathcal{F}_{t_2}$-measurable since it depends on T_k only when $T_k \leq t_2$, $1 \leq k \leq d$, due to the fact that $f_{n+1} \in \mathcal{C}_c^\infty(\Delta_{n+1})$.

The proof of Prop. 2.2.1 also shows that

$$E \left[G \left(\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \dots, t_{n+1}) - \int_0^{T_d} f_{n+1}(t_1, t_2, \dots, t_{n+1})dt_1 \right) D_{t_2} \cdots D_{t_{n+1}}F \right]$$

$$= E \left[G \left(\sum_{k=1}^{k=\infty} f_{n+1}(T_k, t_2, \dots, t_{n+1}) - \int_0^\infty f_{n+1}(t, t_2, \dots, t_{n+1})dt \right) D_{t_2} \cdots D_{t_{n+1}}F \right]$$

$$= E \left[G \int_a^{t_2} f_{n+1}(t_1, t_2, \dots, t_{n+1})d(N_{t_1} - t_1) D_{t_2} \cdots D_{t_{n+1}}F \right]$$

$$= E \left[G \int_a^{t_2} f_{n+1}(t_1, \dots, t_{n+1}) D_{t_1} \cdots D_{t_{n+1}}F dt_1 \right],$$

where on the last line we used the duality (5.0.9) between D and the Poisson compensated integral on the adapted processes. Hence

$$\langle \tilde{D}^{n+1}F, 1_{\Delta_{n+1}}f_{n+1}G\rangle$$
$$= E \left[G \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_2} f_{n+1}(t_1, \dots, t_{n+1}) D_{t_1} \cdots D_{t_{n+1}}F dt_1 \cdots dt_{n+1} \right].$$

This shows the almost-sure equality

$$\langle G\tilde{D}_{t_1}\cdots\tilde{D}_{t_n}F, 1\rangle = E[GD_{t_1}\cdots D_{t_n}F], \quad \text{a.e. } (t_1,\ldots,t_n)\in\Delta_n\cap[a,\infty[^n,$$

which becomes an equality for all $(t_1,\ldots,t_n)\in\Delta_n\cap[a,\infty[^n$ since $(t_1,\ldots,t_n)\mapsto \langle G\tilde{D}_{t_1}\cdots\tilde{D}_{t_n}F, 1\rangle$ and $(t_1,\ldots,t_n)\mapsto E[GD_{t_1}\cdots D_{t_n}F]$ are clearly continuous functions on Δ_n when $F, G\in\mathcal{S}_d(\Omega)$. $\qquad\square$

Note that Relation (5.0.11) does not hold if $(t_1,\ldots,t_n)\notin\Delta_n$, see Relation (6.0.16) below.

6 Chaos expansions of jump times functionals

Our result is stated for smooth functions $f(T_1,\ldots,T_d)$ of a finite number of jump times. We start with the simple case of $f(T_d)$. For $n\in\mathbb{Z}$ and $t\in\mathbb{R}_+$, let

$$p_n(t) = P(N_t = n) = \frac{t^n}{n!}e^{-t} = e^{-t}\int_0^t\int_0^{s_{n-1}}\cdots\int_0^{s_2}ds_1\cdots ds_{n-1},$$

if $n\geq 0$, and $p_n(t) = P(N_t = n) = 0$ if $n < 0$, i.e. $p_{n-1}:\mathbb{R}_+\to\mathbb{R}_+$, $n\geq 1$, is the density function of T_n, and

$$p_n^{(k)}(t) = \frac{\partial^k}{\partial t^k}p_n(t) = (-\Delta)^k p_n(t) = (-1)^k\frac{p_n(t)}{t^k}C_k^t(n),$$

where Δ is the finite difference operator $\Delta f(k) = f(k) - f(k-1)$ and C_k^t is the Charlier polynomial of order $k\in\mathbb{N}$ and parameter $t\in\mathbb{R}_+$.

Proposition 6.0.5 *The decomposition*

$$f(T_d) = \sum_{n=0}^{\infty}I_n(h_n 1_{\Delta_n}),$$

is given for $(t_1,\ldots,t_n)\in\Delta_n$ as:

$$h_n(t_1,\ldots,t_n) = (-1)^n\int_{t_n}^{\infty}f^{(n)}(t)p_{d-1}(t)dt + (-1)^n\sum_{j=1}^{j=n-1}(-1)^{j-1}f^{(n-j)}(t_n)p_{d-1}^{(j-1)}(t_n).$$

Proof. From Relation (5.0.11) of Prop. 5.0.4 and Relation (4.0.5) of Prop. 4.0.3 we have

$$h_n(t_1,\ldots,t_n) = E[D_{t_1}\cdots D_{t_n}f(T_d)] = < \tilde{D}_{t_1}\cdots \tilde{D}_{t_n}f(T_d), 1 >$$

$$= (-1)^n \left\langle f^{(n)}(T_d)1_{[0,T_d]}(t_n) + \sum_{j=1}^{n-1} f^{(n-j)}(t_n)\delta_{t_n}^{(j-1)}(T_d), 1 \right\rangle$$

$$= (-1)^n \int_0^\infty 1_{[t_n,\infty[}(s_d)e^{-s_d}f^{(n)}(s_d)\int_0^{s_d}\cdots\int_0^{s_2} ds_1\cdots ds_d$$

$$+(-1)^n \sum_{j=1}^{n-1} f^{(n-j)}(t_n)\int_0^\infty e^{-s_d}\int_0^{s_d}\cdots\int_0^{s_2} ds_1\cdots ds_{d-1}\delta_{t_n}^{(j-1)}(ds_d)$$

$$= (-1)^n \int_{t_n}^\infty e^{-s_d}f^{(n)}(s_d)\frac{s_d^{d-1}}{(d-1)!}ds_d$$

$$+(-1)^n \sum_{j=1}^{n-1} f^{(n-j)}(t_n)\int_0^\infty e^{-s_d}\frac{s_d^{d-1}}{(d-1)!}\delta_{t_n}^{(j-1)}(ds_d)$$

$$= (-1)^n \int_{t_n}^\infty f^{(n)}(t)p_{d-1}(t)dt + (-1)^n \sum_{j=1}^{j=n-1} (-1)^{j-1}f^{(n-j)}(t_n)p_{d-1}^{(j-1)}(t_n).$$

$\square$

By induction this gives for $l = 0,\ldots,n-1$:

$$h_n(t_1,\ldots,t_n) = (-1)^{n+l}\int_{t_n}^\infty f^{(n-l)}(t)p_{d-1}^{(l)}(t)dt + (-1)^n \sum_{j=l+1}^{j=n-1}(-1)^{j-1}f^{(n-j)}(t_n)p_{d-1}^{(j-1)}(t_n),$$

and in particular for $l = n-1$:

$$h_n(t_1,\ldots,t_n) = -\int_{t_n}^\infty f'(s)p_{d-1}^{(n-1)}(s)ds = f(t_n)p_{d-1}^{(n-1)}(t_n) + \int_{t_n}^\infty f(s)p_{d-1}^{(n)}(s)ds,$$

hence

$$f(T_d) = \sum_{n\geq 0}\frac{1}{n!}I_n\left(f(t_1\vee\cdots\vee t_n)p_{d-1}^{(n-1)}(t_1\vee\cdots\vee t_n) + \int_{t_1\vee\cdots\vee t_n}^\infty f(s)p_{d-1}^{(n)}(s)ds\right),$$

(6.0.12)

with the convention $t_1\vee t_0 = 0$. In order to treat the case of d variables we recall the notation

$$a_i(j_1,\ldots,j_n) = \text{Card}\{l \ : \ j_l = i\},$$

$$b_i(j_1,\ldots,j_n) = \begin{cases} 0 & \text{if } a_i(j_1,\ldots,j_n) = 0, \\ a_i(j_1,\ldots,j_n)-1 & \text{if the first } a_i(j_1,\ldots,j_n)-1 \text{ occurrences} \\ & \text{of } i \text{ in } (j_1,\ldots,j_n) \text{ are consecutive,} \\ 1 & \text{otherwise.} \end{cases}$$

$$c_i(j_1,\ldots,j_n) = \max\{a : j_a = i\}.$$

Proposition 6.0.6 *Let $f \in \mathcal{C}_b^\infty(\Delta_d)$. We have*

$$f(T_1, \ldots, T_d) = \sum_{n=0}^{\infty} I_n(h_n 1_{\Delta_n}),$$

with

$$h(t_1, \ldots, t_n) = (-1)^{n+1} \sum_{1 \le j_1, \ldots, j_n \le d} \sum_{0 \le i_1 \le b_1} \cdots \sum_{0 \le i_d \le b_d} \tag{6.0.13}$$

$$\int_0^\infty \int_0^{s_d} \cdots \int_0^{s_2} e^{-s_d} \partial_1^{a_1 - i_1} \cdots \partial_d^{a_d - i_d} f(s_{1,i_1}, \ldots, s_{d,i_d}) r^{(1+i_1)}(ds_1, t_{c_1}) \cdots r^{(1+i_d)}(ds_d, t_{c_d}),$$

where $r^{(1)}(ds, t) = 1_{[t, \infty[}(s) ds$, and

$$s_{l, i_l} = \begin{cases} s_l & \text{if } i_l = 0, \\ t_{c_l} & \text{if } i_l \ge 1, \end{cases} \quad l = 1, \ldots, d.$$

Proof. We apply Th. 4.0.1 and Relation (5.0.11) of Prop. 5.0.4. $\qquad \square$

This expression can be made more explicit by evaluation of the action of $r^{(1+i_l)}(ds_{j_l}, t_{c_l})$, either as an indicator function or as a derivative in t_{c_l}. However this will not be done here in order to keep formula (6.0.13) to a reasonable size.

In [8], another expression (different from (6.0.13)) has been obtained using elementary orthogonal decompositions in Charlier polynomials. Let $n_1, \ldots, n_l \in \mathbb{N}$ with $1 \le n_1 < \cdots < n_l$, and let $f \in \mathcal{C}_b^d(\Delta_l)$. As a convention, if $k_1 \ge 0, \ldots, k_d \ge 0$ satisfy $k_1 + \cdots + k_d = n$, we let for $(t_1, \ldots, t_n) \in \Delta_n$:

$$(t_1^1, \ldots, t_{k_1}^1, t_1^2, \ldots, t_{k_2}^2, \ldots, t_1^d, \ldots, t_{k_d}^d) = (t_1, \ldots, t_n),$$

with $t_{k_i}^i = 0$ if $k_i = 0$, and $(t_1^i, t_0^i) = ()$. We have

$$f(T_{n_1}, \ldots, T_{n_l}) = \sum_{n=0}^{\infty} I_n(1_{\Delta_n} h_n),$$

where

$$h_n(t_1, \ldots, t_n) = \tag{6.0.14}$$

$$(-1)^l \sum_{k_1 + \cdots + k_l = n} \int_{t_{k_l}^l}^{\infty} \cdots \int_{t_{k_i}^i}^{t_1^{i+1}} \cdots \int_{t_{k_1}^1}^{t_1^2} \partial_1 \cdots \partial_l f(s_1, \ldots, s_l) K_{s_1, \ldots, s_l}^{k_1, \ldots, k_l} ds_1 \cdots ds_l,$$

with, for $0 \le s_1 \le \cdots \le s_l$ and $k_1 \ge 0, \ldots, k_l \ge 0$:

$$K_{s_1, \ldots, s_l}^{k_1, \ldots, k_l} = \sum_{\substack{m_1 \ge n_1, \ldots, m_l \ge n_l \\ m_1 \le \cdots \le m_l}} p_{m_1 - m_0}^{(k_1)}(s_1 - s_0) \cdots p_{m_l - m_{l-1}}^{(k_l)}(s_l - s_{l-1}), \quad m_0 = 0, \ s_0 = 0,$$

cf. Th. 1 of [8]. For $l = 1$, i.e. for $f(T_d)$, $n_1 = d$, we have

$$K_s^k = \sum_{m=d}^{\infty} p_m^{(k)}(s) = \frac{\partial^{k-1}}{\partial s^{k-1}} \sum_{m=d}^{\infty} p_m^{(1)}(s) = \frac{\partial^{k-1}}{\partial s^{k-1}} \sum_{m=d}^{\infty} p_{m-1}(s) - p_m(s) = p_{d-1}^{(k-1)}(s)$$

hence

$$h_n(t_1, \ldots, t_n) = -\int_{t_n}^{\infty} f'(s) p_{d-1}^{(n-1)}(s) ds,$$

which coincides with (6.0.12).

Remarks

i) All expressions obtained above for $f(T_1, \ldots, T_d)$, $f \in \mathcal{C}_b^{\infty}(\Delta_d)$, extend to $f \in L^2(\Delta_d, e^{-s_d} ds_1 \cdots ds_d)$, i.e. to square-integrable $f(T_1, \ldots, T_d)$, by repeated integrations by parts.

ii) Chaotic decompositions on the Poisson space on the compact interval $[0, 1]$ as in [4] or [5] can be obtained by considering the functional $f(1 \wedge T_1, \ldots, 1 \wedge T_d)$ instead of $f(T_1, \ldots, T_d)$.

iii) If $t_1 > \cdots > t_n$ then Relation (5.0.11) does not hold, for example we have

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d) = (-1)^n 1_{[0, T_d]}(t_1) f^{(n)}(T_d),$$

and

$$E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d)] = (-1)^n E[1_{[0, T_d]}(t_1) f^{(n)}(T_d)] = (-1)^n \int_{t_1}^{\infty} f^{(n)}(s) p_{d-1}(s) ds, \tag{6.0.15}$$

which differs (if $n \geq 2$) from

$$E[D_{t_1} \cdots D_{t_n} f(T_d)] = -\int_{t_1}^{\infty} f'(s) p_{d-1}^{(n-1)}(s) ds. \tag{6.0.16}$$

References

[1] E. Carlen and E. Pardoux. Differential calculus and integration by parts on Poisson space. In *Stochastics, Algebra and Analysis in Classical and Quantum Dynamics*, pages 63–73. Kluwer, 1990.

[2] R.J. Elliott and A.H. Tsoi. Integration by parts for Poisson processes. *J. Multivariate Anal.*, 44(2):179–190, 1993.

[3] Y. Ito. Generalized Poisson functionals. *Probab. Theory Related Fields*, 77:1–28, 1988.

[4] J. León, J.L. Solé, and J. Vives. The chaos decomposition of functionals of the jump times for the Poisson process over a compact time interval. In *Stochastic models (Guanajuato, 1998)*, pages 269–282. Soc. Mat. Mexicana, México, 1998.

[5] J. León and C. Tudor. The chaos decomposition of the jump times for the Poisson process over a compact time interval. In L.G. Gorostiza, J.A. León, and J.A. López-Mimbela, editors, *IV Simposio de Probabilidad y Procesos Estocásticos*, volume 12 of *Aportaciones Matemáticas*, pages 137–146, Guanajuato, 1996. Sociedad Matemática Mexicana.

[6] D. Nualart and J. Vives. Anticipative calculus for the Poisson process based on the Fock space. In J. Azéma, P.A. Meyer, and M. Yor, editors, *Séminaire de Probabilités XXIV*, volume 1426 of *Lecture Notes in Mathematics*, pages 154–165. Springer Verlag, 1990.

[7] N. Privault. Chaotic and variational calculus in discrete and continuous time for the Poisson process. *Stochastics and Stochastics Reports*, 51:83–109, 1994.

[8] N. Privault. Multiple stochastic integral expansions of arbitrary Poisson jump times functionals. *Statistics & Probability Letters*, 43:179–188, 1999.

[9] N. Privault. Connection, parallel transport, curvature and energy identities on spaces of configurations. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(10):899–904, 2000.

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