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Nonexistence of solutions to systems of higher-order semilinear inequalities in cone-like domains

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Abstract

In this paper we obtain nonexistence results of global solutions to the following system of higher-order semilinear partial differential inequalities

$$\begin{cases} \frac{\partial^k u_i}{\partial t^k} - \Delta(a_i(x,t)u_i(x,t)) \geq t^{\gamma_{i+1}}|x|^{\sigma_{i+1}}|u_{i+1}(x,t)|^{p_{i+1}}, \quad 1 \leq i \leq n, \\ u_{n+1} = u_1, \end{cases}$$

in cones and cone-like domains in $\mathbb{R}^N$, $t > 0$. Our results concern as well the nonnegative solutions as the solutions which change sign. Moreover, general formula of the critical exponent corresponding to the previous system is given. Our proofs are based on the test function method, developed by E. Mitidieri and S. I. Pohozaev.

1 Introduction

This paper is devoted to nonexistence of global solutions to systems of semilinear higher-order evolution differential inequalities in unbounded cone-like domains. Nonexistence results concerning nonnegative solutions of parabolic equations in cones were obtained by C. Bandle & H. A. Levine [1], H. A. Levine & P. Meier [18]. Recently, new nonexistence results dealing with solutions with arbitrary sign were established by G. G. Laptev [11, 13, 14] and A. El Hamidi & G. G. Laptev [6] when the considered domains are cones or product of cones. On the other hand, for cone-like domains, only nonexistence results of nonnegative solutions to semilinear evolution differential inequalities, were obtained [1, 6, 11, 13, 14]. Very recently, G. G. Laptev [15] obtained a nonexistence result for the semilinear parabolic inequality

$$u_t - \Delta(|u|^{m-1}u) \geq |x|^\sigma |u|^q$$

with $1 \leq m < q$ and $\sigma > -2$, in cone-like domains. It is the first result, to our knowledge, dealing with solutions of evolution problems which are not necessarily nonnegative in cone-like domains.

In this paper, we obtain nonexistence results for systems of semilinear higher-order evolution differential inequalities in unbounded cones and cone-like domains. More precisely, for $n \geq 2$, we study the problem

$$\text{(P)} \begin{cases} \frac{\partial^k u_i}{\partial t^k} - \Delta(a_i u_i) \geq t^{\gamma_{i+1}}|x|^{\sigma_{i+1}}|u_{i+1}|^{p_{i+1}}, \quad 1 \leq i \leq n, \\ u_{n+1} = u_1, \end{cases}$$

where x belongs to a cone (or a *cone-like domain*), $t \in]0, +\infty[$, $k \geq 1$, $p_{n+1} = p_1$, $\gamma_{n+1} = \gamma_1$ and $\sigma_{n+1} = \sigma_1$.

For $n = 1$, we study the problem

$$\text{(I)} \quad \frac{\partial^k u}{\partial t^k} - \Delta(au) \geq t^\gamma |x|^\sigma \, |u|^p,$$

where x belongs to a cone (or a *cone-like domain*) and $t \in]0, +\infty[$. Such systems were studied, in the whole space, by J. Renclawowicz [28], M. Guedda & M. Kirane [7], N. Igbida & M. Kirane [8] and M. Kirane, E. Nabana & S. I. Pohozaev [10].

Our results do not concern only the nonnegative weak solutions but all weak solutions. Moreover, we obtain general formulas of the critical exponents corresponding to the systems considered. These formulas are also valid in the scalar case of one inequality ($n = 1$).

Our approach is based on the *test function* method developed by E. Mitidieri & S. I. Pohozaev [19], S. I. Pohozaev & A. Tesei [25], S. I. Pohozaev & L. Véron [27] and G. G. Laptev [11, 13, 14].

Let $\Omega \subset S^{N-1}$ be a connected submanifold of the unit sphere S^{N-1} in $\mathbb{R}^N$ with smooth boundary $\partial\Omega \subset S^{N-1}$ and having positive $N-2$ dimensional measure. By a *cone* in $\mathbb{R}^N$ with *cross section* Ω with vertex at the origin, we mean the set

$$K = \{(r, \omega) \in \mathbb{R}^N;\ 0 < r < +\infty \text{ and } \omega \in \Omega\},$$

where $r = |x|$, $x \in \mathbb{R}^N$. The boundary of K is

$$\partial K = \{(r, \omega); \quad r = 0 \ \text{ or } \ \omega \in \partial\Omega\}.$$

For $\varepsilon > 0$ fixed, the *cone-like domain* K_ε is defined as follows

$$K_\varepsilon = \{x \in K;\ |x| > \varepsilon\}$$

and its boundary is

$$\partial K_\varepsilon = \{(r, \omega); \quad r = \varepsilon \ \text{ or } \ \omega \in \partial\Omega\}.$$

The outward normal vector to the boundary $\partial\Omega$ (resp. ∂K) will be denoted by ν_ω (resp. ν).

The restriction of the laplacian operator Δ to the unit sphere S^{N-1} will be denoted by Δ_ω, it is the Laplace-Beltrami operator. It is well-known that the laplacian operator in $\mathbb{R}^N$ can be written, in polar coordinates (r, ω), as follows

$$\Delta = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_\omega = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\omega.$$

Throughout the remainder of this paper, we shall let λ denote the first Dirichlet eigenvalue, and Φ the corresponding eigenfunction, for the Laplace-Beltrami operator, namely

$$\begin{cases} -\Delta_\omega \Phi = \lambda \Phi & \text{in} \quad \Omega, \\ \Phi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Recall that $\lambda > 0$ and $\Phi(\omega) > 0$, for any $\omega \in \Omega$. We shall assume Φ is normalized so that

$$0 < \Phi(\omega) \leq 1, \quad \forall \omega \in \Omega.$$

The space of the C^2 functions w.r.t. the first variable and C^j, $j \in \mathbb{N}^*$, w.r.t. the second variable, on $K\times]0,+\infty[$, will be denoted by $\mathcal{C}^{2,j}(K\times]0,+\infty[)$.

This article is organized as follows. In Section 2, we introduce notations and establish estimates which we shall use in the sequel. Section 3 is devoted to nonexistence results to the inequality (I), where the parameter $\gamma = 0$. In Section 4, we generalize the results of the Section 3 for $n \geq 2$, the parameters $\gamma_i = 0$, $i \in \{1, 2, ..., n\}$. Section 5 concerns the general system (P), with $\gamma_i \leq 0$, $i \in \{1, 2, ..., n\}$.

2 Preliminary results

Throughout this paper, the letter C denotes a constant which may vary from line to line but is independent of the terms which will take part in any limit process. For any real number $p > 1$, we define the real p' such that $1/p + 1/p' = 1$.

We define now the weak solutions of the problems that we will consider in the sequel. Let us consider the higher order inequality

$$(\mathrm{I}_k) \quad \frac{\partial^k u}{\partial t^k} - \Delta(au) \geq |x|^\sigma\, |u|^p, \qquad x \in K_\varepsilon, \quad t \in]0,+\infty[,$$

where $p > 1$, $\sigma > -2$, with the initial data

$$\begin{cases} u(x,0) & = & u_0(x), \quad \text{in} \quad K_\varepsilon, \\ \frac{\partial^i u}{\partial t^i}(x,0) & = & u_i(x), \quad i \in \{1,2,...,k-1\}, \qquad \text{in} \quad K_\varepsilon. \end{cases}$$

Definition 1 *Let a be in $L^\infty\left(\overline{K}_\varepsilon\times]0,+\infty[\right)$. A weak solution u of the system (I_k) on $K_\varepsilon\times]0,+\infty[$ is continuous function on $\overline{K}_\varepsilon \times [0,+\infty[$ such that the traces $\frac{\partial^j u}{\partial t^j}(x,0)$, $j \in \{1,..,k-1\}$, are well defined and locally integrable on K_ε and estimate*

$$\int_0^\infty \int_{K_\varepsilon} \left(a\, u\Delta\varphi - u(-1)^k \frac{\partial^k \varphi}{\partial t^k} + |x|^\sigma |u|^p \varphi \right) dx\, dt -$$

$$\int_0^\infty \int_{\partial K_\varepsilon} a\, u \frac{\partial \varphi}{\partial \nu}\, dx\, dt + \sum_{j=0}^{k-1} (-1)^j \int_{K_\varepsilon} \frac{\partial^{k-1-j} u}{\partial t^{k-1-j}}(x,0) \frac{\partial^j \varphi}{\partial t^j}(x,0)\, dx \leq 0, \tag{1}$$

holds true, for any nonnegative test function $\varphi \in \mathcal{C}^{2,k}(K_\varepsilon\times]0,+\infty[)$ with compact support, such that $\varphi|_{\partial K_\varepsilon\times]0,+\infty[} = 0$.

Similarly, we define the weak solutions of the system

$$(\mathrm{S}_k^n) \begin{cases} \frac{\partial^k u_i}{\partial t^k} - \Delta(a_i u_i) \geq |x|^{\sigma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad x \in K_\varepsilon, \quad t \in]0,+\infty[, \quad 1 \leq i \leq n, \\ u_{n+1} = u_1, \end{cases}$$

where $p_i > 1$, $\sigma_i > -2$, for $1 \leq i \leq n$, $p_{n+1} = p_1$, $\sigma_{n+1} = \sigma_1$, and the initial data $(u_i^{(0)}, u_i^{(1)}, ..., u_i^{(k-1)}) \in \left[L^1_{\mathrm{loc}}(K_\varepsilon)\right]^k$, $1 \leq i \leq n$.

Definition 2 *Let a_i, $i \in \{1, 2, ..., n\}$, be n bounded measurable functions in $L^\infty\left(\overline{K}_\varepsilon\times]0,+\infty[\right)$. A weak solution $(u_1, ..., u_n)$ of the system (S_k^n) on $K_\varepsilon\times]0,+\infty[$ is a vector of continuous* *functions* $(u_1, ..., u_n)$ *on* $\overline{K}_\varepsilon \times [0, +\infty[$ *such that the traces* $\frac{\partial^j u_i}{\partial t^j}(x, 0)$, $(i, j) \in \{1, .., n\} \times \{1, .., k-1\}$, *are well defined and locally integrable on* K_ε *and the* n *estimates*

$$\int_0^\infty \int_{K_\varepsilon} \left(a_i\, u_i \Delta\varphi - u_i (-1)^k \frac{\partial^k \varphi}{\partial t^k} + |x|^{\sigma_{i+1}} |u_{i+1}|^{p_{i+1}} \varphi \right) dx\, dt -$$

$$\int_0^\infty \int_{\partial K_\varepsilon} a_i\, u_i \frac{\partial \varphi}{\partial \nu}\, dx\, dt + \sum_{j=0}^{k-1} (-1)^j \int_{K_\varepsilon} \frac{\partial^{k-1-j} u_i}{\partial t^{k-1-j}}(x, 0) \frac{\partial^j \varphi}{\partial t^j}(x, 0)\, dx \le 0, \tag{2}$$

for any $i \in \{1, 2, ..., n-1\}$, *and*

$$\int_0^\infty \int_{K_\varepsilon} \left(a_n\, u_n \Delta\varphi - u_n (-1)^k \frac{\partial^k \varphi}{\partial t^k} + |x|^{\sigma_1} |u_1|^{p_1} \varphi \right) dx\, dt -$$

$$\int_0^\infty \int_{\partial K_\varepsilon} a_n\, u_n \frac{\partial \varphi}{\partial \nu}\, dx\, dt + \sum_{j=0}^{k-1} (-1)^j \int_{K_\varepsilon} \frac{\partial^{k-1-j} u_n}{\partial t^{k-1-j}}(x, 0) \frac{\partial^j \varphi}{\partial t^j}(x, 0)\, dx \le 0, \tag{3}$$

hold true, for any nonnegative test function $\varphi \in \mathcal{C}^{2,k}(K_\varepsilon \times]0, +\infty[)$ *with compact support, such that* $\varphi|_{\partial K_\varepsilon \times]0, +\infty[} = 0$.

We shall construct the test functions which will be used in our proofs. Let $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ be the standard *cut-off function*

$$\zeta(y) = \begin{cases} 1 & \text{if} \quad 0 \le y \le 1, \\ \\ 0 & \text{if} \quad y \ge 2 \end{cases} \qquad \text{and} \quad \forall y \in \mathbb{R}, \;\; 0 \le \zeta(y) \le 1.$$

Let $p_0 \ge k+1$ and η the function defined by

$$\eta(y) = [\zeta(y)]^{p_0}.$$

Explicit computation shows that there is a positive constant $C(\eta) > 0$ such that, for any $y \ge 0$ and any p, $1 < p \le p_0$, the estimate

$$|\eta^{(k)}(y)|^p \le C(\eta) \eta^{p-1}(y), \tag{4}$$

holds true.

We introduce the term of the test function which depends on the variable t. Let the parameter $\rho > \varepsilon$, the exponent $\theta > 0$ and the function $t \mapsto \eta(t/\rho^\theta)$. Remark that

$$\text{supp} \left| \eta(t/\rho^\theta) \right| = \{ t \in \mathbb{R}^+, \;\; 0 \le t \le 2\rho^\theta \}$$

and

$$\text{supp} \left| \frac{d^k \eta}{dt^k}(t/\rho^\theta) \right| = \{ t \in \mathbb{R}^+, \;\; \rho^\theta \le t \le 2\rho^\theta \},$$

where "supp" denotes the support. It follows that

$$\int_{\text{supp} \left| \frac{d^k \eta}{dt^k}(t/\rho^\theta) \right|} \frac{\left| \frac{d^k \eta}{dt^k}(t/\rho^\theta) \right|^p}{\eta^{p-1}(t/\rho^\theta)}\, dt \;\le\; c_\eta \rho^{-\theta(kp-1)}. \tag{5}$$

Now, we construct the part of the test function in the space variable $x \equiv (r, \omega)$. Let $s \neq 0$ a real number, then

$$\Delta\left(r^s \Phi(\omega)\right) = r^{s-2}\Phi(\omega)\left(s^2 + (N-2)s - \lambda\right).$$

Denote by s_+ and s_- the two roots of the equation

$$s^2 + (N-2)s - \lambda = 0.$$

These roots are given by

$$s_+ = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda} \quad \text{and} \quad s_- = -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda}\ .$$

Consider the function ζ defined on K_ε by

$$\zeta(x) \equiv \zeta(r, \omega) = \left(\left(\frac{r}{\varepsilon}\right)^{s_+} - \left(\frac{r}{\varepsilon}\right)^{s_-}\right)\Phi(\omega).$$

It is interesting to note that the function ζ is harmonic in K_ε and vanishes on the boundary ∂K_ε. Moreover,

$$\frac{\partial \zeta}{\partial \nu}|_{\partial K_\varepsilon} \leq 0,$$

where ν is the outer normal to the full surface ∂K_ε. Indeed, thanks to the Hopf lemma, the following inequality

$$\frac{\partial \zeta}{\partial \nu_\omega} = \left(\left(\frac{r}{\varepsilon}\right)^{s_+} - \left(\frac{r}{\varepsilon}\right)^{s_-}\right)\frac{\partial \Phi(\omega)}{\partial \nu_\omega} \leq 0,$$

holds true. Moreover, since $s_+ > 0$ and $s_- < 0$, then

$$-\frac{\partial \zeta}{\partial r}|_{r=\varepsilon} = \frac{s_- - s_+}{\varepsilon}\Phi(\omega) \leq 0.$$

Let us consider now the function of r, for $r \geq \varepsilon$,

$$\xi(r) = \left(\left(\frac{r}{\varepsilon}\right)^{s_+} - \left(\frac{r}{\varepsilon}\right)^{s_-}\right)\eta\left(\frac{r}{\rho}\right).$$

Now, we give estimates of $\partial\xi/\partial r$ and $\partial^2\xi/\partial r^2$. First, we have

$$\frac{\partial \xi}{\partial r} = \left(\frac{s_+}{\varepsilon}\left(\frac{r}{\varepsilon}\right)^{s_+-1} - \frac{s_-}{\varepsilon}\left(\frac{r}{\varepsilon}\right)^{s_--1}\right)\eta\left(\frac{r}{\rho}\right) + \frac{1}{\rho}\left(\left(\frac{r}{\varepsilon}\right)^{s_+} - \left(\frac{r}{\varepsilon}\right)^{s_-}\right)\eta'\left(\frac{r}{\rho}\right).$$

Whence, there is a positive constant C, independent of ρ and r, such that

$$\left|\frac{\partial \xi}{\partial r}\right|^p \leq C\left\{\left[\frac{s_+}{\varepsilon}\left(\frac{r}{\varepsilon}\right)^{s_+-1} - \frac{s_-}{\varepsilon}\left(\frac{r}{\varepsilon}\right)^{s_--1}\right]^p \eta^p\left(\frac{r}{\rho}\right) + \right.$$
$$\left.\frac{1}{\rho^p}\left[\left(\frac{r}{\varepsilon}\right)^{s_+} - \left(\frac{r}{\varepsilon}\right)^{s_-}\right]^p \left|\eta'\left(\frac{r}{\rho}\right)\right|^p\right\}.$$

Consequently, the estimate

$$\left|\frac{\partial \xi}{\partial r}\right|^p \leq C\, r^{p(s_+-1)}\eta^{p-1}\left(\frac{r}{\rho}\right)\left(1 + \frac{r^p}{\rho^p}\right). \tag{6}$$

holds true. Moreover,

$$\begin{aligned}\frac{\partial^2\xi}{\partial r^2} &= \left(\frac{s_+(s_+-1)}{\varepsilon^2}\left(\frac{r}{\varepsilon}\right)^{s_+-2}-\frac{s_-(s_--1)}{\varepsilon^2}\left(\frac{r}{\varepsilon}\right)^{s_--2}\right)\eta\left(\frac{r}{\rho}\right)+\\ &\frac{2}{\rho}\left(\frac{s_+}{\varepsilon}\left(\frac{r}{\varepsilon}\right)^{s_+-1}-\frac{s_-}{\varepsilon}\left(\frac{r}{\varepsilon}\right)^{s_--1}\right)\eta'\left(\frac{r}{\rho}\right)+\\ &\frac{1}{\rho^2}\left(\left(\frac{r}{\varepsilon}\right)^{s_+}-\left(\frac{r}{\varepsilon}\right)^{s_-}\right)\eta''\left(\frac{r}{\rho}\right).\end{aligned}$$

Similarly, there is $C>0$, independent on ρ and r, such that

$$\left|\frac{\partial^2\xi}{\partial r^2}\right|^p \leq C\, r^{p(s_+-2)}\eta^{p-1}\left(\frac{r}{\rho}\right)\left(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}\right). \tag{7}$$

We introduce now the final test function of the space variable

$$\psi_\rho(x)=\zeta(x)\,\eta\left(\frac{|x|}{\rho}\right)=\left(\left(\frac{r}{\varepsilon}\right)^{s_+}-\left(\frac{r}{\varepsilon}\right)^{s_-}\right)\eta\left(\frac{r}{\rho}\right)\Phi(\omega).$$

Then

$$\Delta\psi_\rho(x)=\frac{\partial^2\psi_\rho}{\partial r^2}(x)+\frac{N-1}{r}\frac{\partial\psi_\rho}{\partial r}(x)+\frac{1}{r^2}\Delta_\omega\psi_\rho(x),$$

where

$$\Delta_\omega\psi_\rho=\left(\left(\frac{r}{\varepsilon}\right)^{s_+}-\left(\frac{r}{\varepsilon}\right)^{s_-}\right)\eta\left(\frac{r}{\rho}\right)(-\lambda\Phi)=-\lambda\psi_\rho.$$

Whence

$$\begin{aligned}|\Delta\psi_\rho(x)|^p &= \Phi^p(\omega)\left|\left\{\frac{\partial^2}{\partial r^2}+\frac{N-1}{r}\frac{\partial}{\partial r}-\frac{\lambda}{r^2}\right\}(\xi(r))\right|^p,\\ &\leq C\,\Phi^p(\omega)\eta^{p-1}\left(\frac{r}{\rho}\right)r^{p(s_+-2)}\left(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}\right),\\ &\leq C\,\psi_\rho^{p-1}(x)r^{s_+-2p}\left(\frac{(r/\varepsilon)^{s_+}}{(r/\varepsilon)^{s_+}-(r/\varepsilon)^{s_-}}\right)^{p-1}\left(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}\right),\end{aligned} \tag{8}$$

where C is a positive constant, independent of ρ and r. Let us denote $\mathcal{N}=\operatorname{supp}(\Delta\psi_\rho)$. Since $\eta(r/\rho)=1$ for $r\leq\rho$ and $\eta(r/\rho)=0$ for $r\geq 2\rho$, then $\mathcal{N}\subset\{x\in K_\varepsilon;\ \ \rho\leq r\leq 2\rho\}$. Moreover, since $\rho>\varepsilon$, then the expressions

$$1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}\quad\text{and}\quad\frac{(r/\varepsilon)^{s_+}}{(r/\varepsilon)^{s_+}-(r/\varepsilon)^{s_-}}$$

are bounded on $\{x\in K_\varepsilon;\ \ \rho\leq r\leq 2\rho\}$. We conclude that there is a positive constant C such that

$$\forall\, x\in\mathcal{N};\ \ |\Delta\psi_\rho(x)|^p\leq C\,\psi_\rho^{p-1}(x)\frac{r^{s_+}}{\rho^{2p}}.$$

Finally, for ρ sufficiently large, we have the estimate

$$\begin{aligned}\int_{\mathcal{N}}\frac{|\Delta(\psi_\rho)(x)|^p}{\psi_\rho^{p-1}(x)|x|^{\sigma(p-1)}}\,dx &\leq \frac{C}{\rho^{2p}}\int_\rho^{2\rho}\int_\Omega\frac{r^{s_++N-1}}{r^{\sigma(p-1)}}\,d\theta\,dr,\\ &\leq C\begin{cases}\rho^{s_++N-\sigma(p-1)-2p} & \text{if}\ \ s_++N-\sigma(p-1)>0,\\ \rho^{-2p}\ln(\rho) & \text{if}\ \ s_++N-\sigma(p-1)=0,\\ \rho^{-2p} & \text{if}\ \ s_++N-\sigma(p-1)<0.\end{cases}\end{aligned} \tag{9}$$

Consider the final test function of the varaiables x and t:

$$\varphi_\rho(x,t)=\eta\left(\frac{t}{\rho^\theta}\right)\psi_\rho(x). \tag{10}$$

On one hand, the same arguments used in (9) give the estimate

$$\int_0^{+\infty}\int_{\mathcal{N}}\frac{|\Delta(\varphi_\rho)(x,t)|^p}{\varphi_\rho^{p-1}(x,t)|x|^{\sigma(p-1)}}dxdt\leq\frac{C}{\rho^{2p}}\int_0^{2\rho^\theta}\int_\rho^{2\rho}\int_\Omega\frac{r^{s_++N-1}}{r^{\sigma(p-1)}}\,d\theta\,dr\,dt$$

$$\leq C\begin{cases}\rho^{s_++N-\sigma(p-1)+\theta-2p} & \text{if}\quad s_++N-\sigma(p-1)>0,\\ \rho^{\theta-2p}\ln(\rho) & \text{if}\quad s_++N-\sigma(p-1)=0,\\ \rho^{\theta-2p} & \text{if}\quad s_++N-\sigma(p-1)<0.\end{cases} \tag{11}$$

On the other hand, if we denote $\mathcal{C}(\varepsilon,\rho):=\{x\in K_\varepsilon;\ \ \varepsilon\leq|x|\leq2\rho\}$, then

$$\int\int_{\text{supp}\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right|}\frac{\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right|^p}{\varphi_\rho^{p-1}|x|^{(p-1)\sigma}}\,dx\,dt\leq$$

$$\int_{\mathcal{C}(\varepsilon,\rho)}\frac{\psi_\rho(x)}{|x|^{(p-1)\sigma}}\,dx\int_{\text{supp}\left|\frac{d^k\eta}{dt^k}(t/\rho^\theta)\right|}\frac{\left|\frac{d^k\eta}{dt^k}(t/\rho^\theta)\right|^p}{\eta^{p-1}(t/\rho^\theta)}\,dt. \tag{12}$$

Furthermore,

$$\begin{aligned}\int_{\mathcal{C}(\varepsilon,\rho)}\frac{\tilde{\psi}_\rho(x)}{|x|^{(p-1)\sigma}}\,dx &= \int_\Omega\Phi(\omega)\,d\theta\int_\varepsilon^{2\rho}\left(\left(\frac{r}{\varepsilon}\right)^{s_+}-\left(\frac{r}{\varepsilon}\right)^{s_-}\right)\eta\left(\frac{r}{\rho}\right)\frac{r^{N-1}}{r^{(p-1)\sigma}}\,dr\\ &\leq \frac{|\Omega|}{\varepsilon^{s_+}}\int_\varepsilon^{2\rho}r^{s_++N-1-(p-1)\sigma}\,dr\\ &\leq C\begin{cases}\rho^{s_++N-\sigma(p-1)} & \text{if}\quad s_++N-\sigma(p-1)>0,\\ \ln(\rho) & \text{if}\quad s_++N-\sigma(p-1)=0,\\ 1 & \text{if}\quad s_++N-\sigma(p-1)<0.\end{cases}\end{aligned} \tag{13}$$

Combining the estimates (5), (12) and (13), we obtain

$$\int\int_{\text{supp}\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right|}\frac{\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right|^p}{\varphi_\rho^{p-1}|x|^{(p-1)\sigma}}\,dx\,dt\leq$$

$$C\begin{cases}\rho^{s_++N-\sigma(p-1)-\theta(kp-1)} & \text{if}\quad s_++N-\sigma(p-1)>0,\\ \rho^{-\theta(kp-1)}\ln(\rho) & \text{if}\quad s_++N-\sigma(p-1)=0,\\ \rho^{-\theta(kp-1)} & \text{if}\quad s_++N-\sigma(p-1)<0.\end{cases} \tag{14}$$

In the following section, we consider the case $n=1$.

3 Higher Order Evolution Semilinear Inequalities

In this section, we establish nonexistence results of global solutions to the semilinear problem (I_k). The weak solutions of (I_k) are defined in Definition 1.

Theorem 1 *Assume that*

$$\forall (x,t) \in \partial K_\varepsilon \times [0,+\infty[, \quad a(x,t) \geq 0, \quad u(x,t) \geq 0$$

and

$$\forall x \in K_\varepsilon; \quad \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \geq 0.$$

Let

$$\frac{\sigma+2}{p-1} \geq s_+ + N - 2\left(1 - \frac{1}{k}\right).$$

Then there is no weak nontrivial solution u of the system (I_k).

Proof. Assume that (I_k) admits a nontrivial global weak solution u with

$$\frac{\sigma+2}{p-1} \geq s_+ + N - 2\left(1 - \frac{1}{k}\right).$$

In definition 1, let us choose the test function $\varphi(x,t) = \varphi_\rho(x,t)$ defined in (10). Thanks to the Hopf lemma, we have

$$\int_0^\infty \int_{\partial K_\varepsilon} a\, u \frac{\partial \varphi_\rho}{\partial \nu}\, dx\, dt \leq 0.$$

Moreover, the test function φ_ρ satisfies the equalities

$$\frac{\partial^j \varphi_\rho}{\partial t^j}(x,0) = 0, \quad \text{for} \quad j \in \{1, 2, ..., k-1\}.$$

Finally, we have

$$\int_{K_\varepsilon} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\, dx \geq 0.$$

Then, the inequality (1) implies that

$$\int_0^\infty \int_{K_\varepsilon} |x|^\sigma |u|^p \varphi_\rho\, dx\, dt \leq \int_0^\infty \int_{K_\varepsilon} u \left(-a\Delta + (-1)^k \frac{\partial^k}{\partial t^k}\right) \varphi_\rho dx dt. \tag{15}$$

Let us introduce the notations

$$I(\rho) := \int_0^\infty \int_{K_\varepsilon} |x|^\sigma |u|^p \varphi_\rho\, dx\, dt,$$

$$\mathcal{A}(\rho) = \int_0^\infty \int_{K_\varepsilon} \frac{|\Delta \varphi_\rho|^{p'}}{(|x|^\sigma \varphi_\rho)^{p'-1}}\, dx\, dt$$

and

$$\mathcal{B}(\rho) = \int_0^\infty \int_{K_\varepsilon} \frac{|\frac{\partial^k \varphi_\rho}{\partial t^k}|^{p'}}{(|x|^\sigma \varphi_\rho)^{p'-1}}\, dx\, dt.$$

Applying the Hölder inequality to (15), we obtain

$$I(\rho) \leq \max(||a||_\infty, 1)\, I(\rho)^{\frac{1}{p}} \left(\mathcal{A}(\rho)^{\frac{1}{p'}} + \mathcal{B}(\rho)^{\frac{1}{p'}}\right), \tag{16}$$

or equivalently
$$I(\rho)^{1-\frac{1}{p}} \leq \max\left(||a||_{\infty}, 1\right) \left(\mathcal{A}(\rho)^{\frac{1}{p'}} + \mathcal{B}(\rho)^{\frac{1}{p'}}\right).$$
At this stage, we choose the real parameter $\theta = 2/k$ and obtain
$$\mathcal{A}(\rho) \leq C\,\Theta(\rho) \quad \text{and} \quad \mathcal{B}(\rho) \leq C\,\Theta(\rho),$$
where
$$\Theta(\rho) = \begin{cases} \rho^{s_+ + N - \sigma(p'-1) - 2(p'-1/k)} & \text{if} \quad s_+ + N - \sigma(p'-1) > 0, \\ \rho^{-2(p'-1/k)} \ln(\rho) & \text{if} \quad s_+ + N - \sigma(p'-1) = 0, \\ \rho^{-2(p'-1/k)} & \text{if} \quad s_+ + N - \sigma(p'-1) < 0. \end{cases}$$
If $s_+ + N - \sigma(p'-1) > 0$, then explicit computation gives , for ρ sufficiently large,
$$I^{1-\frac{1}{p}} \leq C\rho^{\alpha},$$
where
$$\alpha = \frac{p-1}{p}\left(s_+ + N - 2\left(1 - \frac{1}{k}\right) - \frac{\sigma+2}{p-1}\right).$$
Now, we require that $\alpha \leq 0$, which is equivalent to
$$\frac{\sigma+2}{p-1} \geq s_+ + N - 2\left(1 - \frac{1}{k}\right).$$
In this case, $I(\rho)$ is bounded uniformly w.r.t. the variable ρ. Moreover, the function $I(\rho)$ is increasing in ρ. Consequently, the monotone convergence theorem implies that the function
$$(x,t) \equiv (r,\omega,t) \longmapsto |u(x,t)|^p |x|^{\sigma} \left(\left(\frac{r}{\varepsilon}\right)^{s_+} - \left(\frac{r}{\varepsilon}\right)^{s_-}\right) \Phi(\omega)$$
is in $L^1(K_\varepsilon \times]0, +\infty[)$. Furthermore, note that
$$\operatorname{supp}(\Delta\varphi_\rho) \subset \{t \in \mathbb{R}^+, \quad 0 \leq t \leq 2\rho^{2/k}\} \times \{x \in K_\varepsilon, \quad \rho \leq |x| \leq 2\rho\}$$
and
$$\operatorname{supp}\left(\frac{\partial^k \varphi_\rho}{\partial t^k}\right) \subset \{t \in \mathbb{R}^+, \quad \rho^{2/k} \leq t \leq 2\rho^{2/k}\} \times \{x \in K_\varepsilon, \quad \varepsilon \leq |x| \leq 2\rho\}.$$
Whence, instead of (16) we have more precisely
$$I(\rho) \leq \max\left(||a||_{\infty}, 1\right) \tilde{I}(\rho)^{\frac{1}{p}} \left(\mathcal{A}(\rho)^{\frac{1}{p'}} + \mathcal{B}(\rho)^{\frac{1}{p'}}\right), \tag{17}$$
where
$$\tilde{I}(\rho) = \int_{\mathcal{C}_\rho} |x|^{\sigma} |u|^p \varphi_\rho \, dx \, dt,$$
where
$$\mathcal{C}_\rho = \operatorname{supp}(\Delta\varphi_\rho) \cup \operatorname{supp}\left(\frac{\partial^k \varphi_\rho}{\partial t^k}\right).$$
Finally, using the dominated convergence theorem, we obtain
$$\lim_{\rho \to +\infty} I(\rho) = 0.$$

This means that $u \equiv 0$, which contradicts the fact that u is assumed to be nontrivial weak solution.
Now, if $s_+ + N - \sigma(p'-1) \leq 0$, then

$$\lim_{\rho\to+\infty} \rho^{-2(p'-1/k)} \ln(\rho) = 0 \quad \text{and} \quad \lim_{\rho\to+\infty} \rho^{-2(p'-1/k)} = 0.$$

Therefore the integral $I(\rho)$ is bounded uniformly w.r.t. the variable ρ. The same arguments used previously complete the proof. $\square$

The previous result is also valid for cones instead of cone-like domains. Indeed, let us consider the higher order inequality

$$(\tilde{\mathrm{I}}_k) \quad \frac{\partial^k u}{\partial t^k} - \Delta(au) \geq |x|^\sigma \, |u|^p, \qquad x \in K, \quad t \in]0,+\infty[,$$

where $p > 1$, $\sigma > -2$, with the initial data

$$\begin{cases} u(x,0) & = \quad u_0(x), \quad \text{in} \quad K, \\ \frac{\partial^i u}{\partial t^i}(x,0) & = \quad u_i(x), \quad i \in \{1,2,...,k-1\}, \qquad \text{in} \quad K. \end{cases}$$

Then we have

Theorem 2 *Assume that*

$$\forall (x,t) \in \partial K \times [0,+\infty[, \quad a(x,t) \geq 0, \quad u(x,t) \geq 0$$

and

$$\forall x \in K; \quad \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \geq 0.$$

Let

$$\frac{\sigma+2}{p-1} \geq s_+ + N - 2\left(1 - \frac{1}{k}\right).$$

Then there is no weak nontrivial solution u of the system $(\tilde{I}_k)$.

Proof. Note that the cone K coincides with K_ε for $\varepsilon = 0$. In this case, the test function φ_ρ given by (10) is not well defined. We choose the new test function

$$\tilde{\varphi}_\rho(x,t) \equiv \tilde{\varphi}_\rho(r,\omega,t) = r^{s_+}\,\Phi(\omega)\,\eta\left(\frac{r}{\rho^\theta}\right)\,\eta\left(\frac{t}{\rho}\right).$$

The function

$$\begin{cases} K & \longrightarrow \quad [0,+\infty[\\ (r,\omega) & \longmapsto \quad r^{s_+}\,\Phi(\omega), \end{cases}$$

is also harmonic. Following the different steps of the last proof with $\tilde{\varphi}_\rho$ (resp. K) instead of φ_ρ (resp. K_ε), we obtain the result. $\square$

4 Higher Order Systems of Evolution Semilinear Inequalities

We establish here nonexistence results of global solutions to the system (S_k^n). The weak solutions of (S_k^n) are defined in Definition 2. In this section, the initial conditions $\frac{\partial^j u_i}{\partial t^j}(x,0)$ will be denoted by $u_i^{(j)}(x,0)$, for $(i,j) \in \{1,..,n\}\times\{0,..,k-1\}$, and the vector $(X_1, X_2, ..., X_n)$ will denote the solution of the linear system

$$\begin{pmatrix} -1 & p_1 & 0 & \dots & 0 \\ 0 & -1 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & p_{n-1} \\ p_n & 0 & \dots & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix} = \begin{pmatrix} \sigma_1 + 2 \\ \sigma_2 + 2 \\ \vdots \\ \sigma_{n-1} + 2 \\ \sigma_n + 2 \end{pmatrix}. \tag{18}$$

We have the following non existence result:

Theorem 3 *Assume that*

$$\forall (x,t) \in \partial K_\varepsilon \times [0,+\infty[, \quad \forall i \in \{1,2,...,n\}; \quad u_i(x,t) \geq 0, \quad a_i(x,t) \geq 0$$

and

$$\forall x \in K_\varepsilon, \quad \forall i \in \{1,2,...,n\}; \quad \frac{\partial^{k-1} u_i}{\partial t^{k-1}}(x,0) \geq 0.$$

Let

$$\max\{X_1, X_2, ..., X_n\} \geq s_+ + N - 2\left(1 - \frac{1}{k}\right).$$

Then the problem (S_k^n) has no nontrivial global weak solution.

Proof. In order to simplify the proof, we treat only the case $n = 3$, the general case can be established in the same manner.
The proof is by contradiction. Assume that (S_k^3) admits a nontrivial global weak solution (u_1, u_2, u_3) with $\max\{X_1, X_2, ..., X_n\} \leq s_+ + N - 2(1 - 1/k)$. In definition 2, let us choose the test function $\varphi(x,t) = \varphi_\rho(x,t)$ defined in (10). Thanks to the Hopf lemma, we have

$$\int_0^\infty \int_{\partial K_\varepsilon} a_i\, u_i \frac{\partial \varphi_\rho}{\partial \nu}\, dx\, dt \leq 0, \quad \text{for} \quad i \in \{1,2,3\}.$$

Moreover, the test function φ_ρ satisfies

$$\frac{\partial^j \varphi_\rho}{\partial t^j}(x,0) = 0, \quad \text{for} \quad j \in \{1,2,...,k-1\}.$$

Finally, we have

$$\int_{K_\varepsilon} \frac{\partial^{k-1} u_i}{\partial t^{k-1}}(x,0) \varphi_\rho(x,0)\, dx \geq 0.$$

Then, the inequalities (2) and (3) imply that

$$\begin{cases} \int_0^\infty \int_{K_\varepsilon} |x|^{\sigma_2} |u_2|^{p_2} \varphi_\rho & \leq & \int_0^\infty \int_{K_\varepsilon} u_1 \left(-a_1 \Delta + (-1)^k \frac{\partial^k}{\partial t^k}\right) \varphi_\rho, \\ \int_0^\infty \int_{K_\varepsilon} |x|^{\sigma_3} |u_3|^{p_3} \varphi_\rho & \leq & \int_0^\infty \int_{K_\varepsilon} u_2 \left(-a_2 \Delta + (-1)^k \frac{\partial^k}{\partial t^k}\right) \varphi_\rho, \\ \int_0^\infty \int_{K_\varepsilon} |x|^{\sigma_1} |u_1|^{p_1} \varphi_\rho & \leq & \int_0^\infty \int_{K_\varepsilon} u_3 \left(-a_3 \Delta + (-1)^k \frac{\partial^k}{\partial t^k}\right) \varphi_\rho. \end{cases} \tag{19}$$

For any $i \in \{1,2,3\}$, we use the notations

$$I_i(\rho) := \int_0^\infty \int_{K_\varepsilon} |x|^{\sigma_i} |u_i|^{p_i} \varphi_\rho \, dx\, dt,$$

$$\mathcal{A}_i(\rho) = \int_0^\infty \int_{K_\varepsilon} \frac{|\Delta \varphi_\rho|^{p_i'}}{(|x|^{\sigma_i}\varphi_\rho)^{p_i'-1}} \, dx\, dt$$

and

$$\mathcal{B}_i(\rho) = \int_0^\infty \int_{K_\varepsilon} \frac{|\frac{\partial^k \varphi_\rho}{\partial t^k}|^{p_i'}}{(|x|^{\sigma_i}\varphi_\rho)^{p_i'-1}} \, dx\, dt.$$

Using the Hölder inequality to the system (19), we obtain

$$\left\{\begin{array}{lcl} I_1(\rho) & \leq & \max(||a_3||_\infty, 1)\, I_3(\rho)^{\frac{1}{p_3}} \left(\mathcal{A}_3(\rho)^{\frac{1}{p_3'}} + \mathcal{B}_3(\rho)^{\frac{1}{p_3'}}\right), \\ I_2(\rho) & \leq & \max(||a_1||_\infty, 1)\, I_1(\rho)^{\frac{1}{p_1}} \left(\mathcal{A}_1(\rho)^{\frac{1}{p_1'}} + \mathcal{B}_1(\rho)^{\frac{1}{p_1'}}\right), \\ I_3(\rho) & \leq & \max(||a_2||_\infty, 1)\, I_2(\rho)^{\frac{1}{p_2}} \left(\mathcal{A}_2(\rho)^{\frac{1}{p_2'}} + \mathcal{B}_2(\rho)^{\frac{1}{p_2'}}\right). \end{array}\right. \tag{20}$$

Whence, there is a positice constant C, independent on ρ and r such that

$$\left\{\begin{array}{lcl} I_1^{1-\frac{1}{p_1p_2p_3}} & \leq & C\left(\mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'}\right)^{\frac{1}{p_2p_3}} \left(\mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'}\right)^{\frac{1}{p_3}} \left(\mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'}\right), \\ I_2^{1-\frac{1}{p_1p_2p_3}} & \leq & C\left(\mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'}\right) \left(\mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'}\right)^{\frac{1}{p_1p_3}} \left(\mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'}\right)^{\frac{1}{p_1}}, \\ I_3^{1-\frac{1}{p_1p_2p_3}} & \leq & C\left(\mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'}\right)^{\frac{1}{p_2}} \left(\mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'}\right) \left(\mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'}\right)^{\frac{1}{p_1p_2}}. \end{array}\right.$$

At this stage, we choose the real parameter $\theta = 2/k$ and obtain

$$\mathcal{A}_i(\rho) \leq C\,\Theta_i(\rho) \quad \text{and} \quad \mathcal{B}_i(\rho) \leq C\,\Theta_i(\rho), \quad \text{for} \quad i \in \{1,2,3\},$$

where

$$\Theta_i(\rho) = \left\{\begin{array}{ll} \rho^{s_+ + N - \sigma_i(p_i'-1) - 2(p_i'-1/k)} & \text{if} \quad s_+ + N - \sigma_i(p_i'-1) > 0, \\ \rho^{-2(p_i'-1/k)} \ln(\rho) & \text{if} \quad s_+ + N - \sigma_i(p_i'-1) = 0, \\ \rho^{-2(p_i'-1/k)} & \text{if} \quad s_+ + N - \sigma_i(p_i'-1) < 0. \end{array}\right.$$

In order to estimate the expressions $I_i(\rho)$, $i \in \{1,2,3\}$, we start with the case

<u>Case 1:</u> $\quad s_+ + N - \sigma_i(p_i'-1) > 0, \quad$ for any $\quad i \in \{1,2,3\}$.

Explicit computation gives , for ρ sufficiently large,

$$I_i^{1-\frac{1}{p_1p_2p_3}} \leq C\rho^{\alpha_i}, \quad 1 \leq i \leq 3,$$

where

$$\begin{cases} \alpha_1 &= \left(1-\frac{1}{p_1p_2p_3}\right)\left(s_+ + N - 2\left(1-\frac{1}{k}\right) - \frac{(\sigma_1+2)+p_1(\sigma_2+2)+p_1p_2(\sigma_3+2)}{p_1p_2p_3-1}\right), \\ \alpha_2 &= \left(1-\frac{1}{p_1p_2p_3}\right)\left(s_+ + N - 2\left(1-\frac{1}{k}\right) - \frac{p_2p_3(\sigma_1+2)+(\sigma_2+2)+p_2(\sigma_3+2)}{p_1p_2p_3-1}\right), \\ \alpha_3 &= \left(1-\frac{1}{p_1p_2p_3}\right)\left(s_+ + N - 2\left(1-\frac{1}{k}\right) - \frac{p_3(\sigma_1+2)+p_1p_3(\sigma_2+2)+(\sigma_3+2)}{p_1p_2p_3-1}\right). \end{cases}$$

Now, we require that, at least, one of α_i, $i \in \{1,2,3\}$, is less than zero, which is equivalent to

$$\max\{X_1, X_2, X_3\} \geq s_+ + N - 2\left(1-\frac{1}{k}\right),$$

where the vector $(X_1, X_2, X_3)^T$ is the solution of

$$\begin{pmatrix} -1 & p_1 & 0 \\ 0 & -1 & p_2 \\ p_3 & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \sigma_1+2 \\ \sigma_2+2 \\ \sigma_3+2 \end{pmatrix}. \tag{21}$$

In this case, there is $i_0 \in \{1,2,3\}$ such that $I_{i_0}(\rho)$ is bounded uniformly w.r.t. the variable ρ. Using the systems (19) and (20), we obtain that both $I_1(\rho)$, $I_2(\rho)$ and $I_3(\rho)$ are bounded uniformly w.r.t. the variable ρ. Moreover, the functions $I_i(\rho)$, $i \in \{1,2,3\}$, are increasing in ρ. Consequently, the monotone convergence theorem implies that the functions

$$(x,t) \equiv (r,\omega,t) \longmapsto |u_i(x,t)|^{p_i}|x|^{\sigma_i}\left(\left(\frac{r}{\varepsilon}\right)^{s_+} - \left(\frac{r}{\varepsilon}\right)^{s_-}\right)\Phi(\omega).$$

is in $L^1(K_\varepsilon \times]0,+\infty[)$. Precise that these functions correspnd to

$$\lim_{\rho\to+\infty} |u_i(x,t)|^{p_i}|x|^{\sigma_i}\varphi_\rho(x,t).$$

Furthermore, note that

$$\operatorname{supp}(\Delta\varphi_\rho) \subset \{t \in \mathbb{R}^+, \quad 0 \leq t \leq 2\rho^{2/k}\} \times \{x \in K_\varepsilon, \quad \rho \leq |x| \leq 2\rho\}$$

and

$$\operatorname{supp}\left(\frac{\partial^k \varphi_\rho}{\partial t^k}\right) \subset \{t \in \mathbb{R}^+, \quad \rho^{2/k} \leq t \leq 2\rho^{2/k}\} \times \{x \in K_\varepsilon, \quad \varepsilon \leq |x| \leq 2\rho\}.$$

Whence, instead of (20) we have more precisely

$$\begin{cases} I_1(\rho) &\leq \max(||a_3||_\infty, 1)\, \tilde{I}_3(\rho)^{\frac{1}{p_3}}\left(\mathcal{A}_3(\rho)^{\frac{1}{p_3'}} + \mathcal{B}_3(\rho)^{\frac{1}{p_3'}}\right), \\ I_2(\rho) &\leq \max(||a_1||_\infty, 1)\, \tilde{I}_1(\rho)^{\frac{1}{p_1}}\left(\mathcal{A}_1(\rho)^{\frac{1}{p_1'}} + \mathcal{B}_1(\rho)^{\frac{1}{p_1'}}\right), \\ I_3(\rho) &\leq \max(||a_2||_\infty, 1)\, \tilde{I}_2(\rho)^{\frac{1}{p_2}}\left(\mathcal{A}_2(\rho)^{\frac{1}{p_2'}} + \mathcal{B}_2(\rho)^{\frac{1}{p_2'}}\right). \end{cases} \tag{22}$$

where

$$\tilde{I}_i(\rho) = \int_{\mathcal{C}_\rho} |x|^{\sigma_i}|u_i|^{p_i}\varphi_\rho \, dx\, dt,$$

where

$$\mathcal{C}_\rho = \mathrm{supp}(\Delta\varphi_\rho) \cup \mathrm{supp}\left(\frac{\partial^k \varphi_\rho}{\partial t^k}\right).$$

Finally, using the dominated convergence theorem, we obtain

$$\lim_{\rho\to+\infty} I_i(\rho) = 0, \quad i \in \{1,2,3\}.$$

This means that $(u_1,u_2,u_3) \equiv (0,0,0)$, which contradicts the fact that (u_1,u_2,u_3) is assumed to be nontrivial weak solution.
We end the proof by treating the case

Case 2: There is $i_0 \in \{1,2,3\}$, such that $s_+ + N - \sigma_{i_0}(p'_{i_0} - 1) \le 0$.

The same arguments used in Case 1 give, for ρ sufficiently large,

$$I_i^{1-\frac{1}{p_1p_2p_3}} \le C\rho^{\tilde{\alpha}_i}, \quad 1 \le i \le 3,$$

where $\tilde{\alpha}_i \le \alpha_i$, for $i \in \{1,2,3\}$. Then, if there is $i_1 \in \{1,2,3\}$ such that $\alpha_{i_1} \le 0$ then $\tilde{\alpha}_{i_1} \le 0$. This implies that $I_{i_1}(\rho)$ is bounded uniformly w.r.t. the variable ρ, which leads to the same conclusion as the Case 1. This achieves the proof. $\square$

If the problem (S_k^n) is posed on the cone K instead of the cone-like domain K_ε, our result is also valid and the proof changes very slithly. Indeed, consider

$$(\tilde{\mathrm{S}}_k^n)\begin{cases} \frac{\partial^k u_i}{\partial t^k} - \Delta(a_iu_i) \ge |x|^{\sigma_{i+1}}|u_{i+1}|^{p_{i+1}}, & x \in K, \ t \in]0,+\infty[, \quad 1 \le i \le n, \\ u_{n+1} = u_1, \end{cases}$$

where $p_i > 1$, $\sigma_i > -2$, for $1 \le i \le n$, $p_{n+1} = p_1$, $\sigma_{n+1} = \sigma_1$, and the initial data $(u_i^{(0)}, u_i^{(1)}, ..., u_i^{(k-1)}) \in \left[L^1_{\mathrm{loc}}(K)\right]^k$, $1 \le i \le n$.

The weak solutions of $(\tilde{\mathrm{S}}_k^n)$ are defined similarly as in Definition 1 with K instead of K_ε. Then we have the following result:

Theorem 4 *Assume that*

$$\forall (x,t) \in \partial K \times [0,+\infty[, \ \ \forall i \in \{1,2,...,n\}; \ \ u_i(x,t) \ge 0 \ \ a_i(x,t) \ge 0$$

and

$$\forall x \in K,, \ \ \forall i \in \{1,2,...,n\}; \ \ \frac{\partial^{k-1} u_i}{\partial t^{k-1}}(x,0) \ge 0.$$

Let

$$\max\{X_1, X_2, ..., X_n\} \ge s_+ + N - 2\left(1 - \frac{1}{k}\right).$$

Then the problem $(\tilde{S}_k^n)$ has no nontrivial global weak solution.

Proof. It suffices to use the same arguments of the last proof with $\tilde{\varphi}_\rho$ (resp K) instead of φ_ρ (resp K_ε) to obtain the result. $\square$

5 General case

$$(\mathrm{P}_k^n)\begin{cases} \frac{\partial^k u_i}{\partial t^k} - \Delta(a_i u_i) \geq t^{\gamma_{i+1}} |x|^{\sigma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad x \in K_\varepsilon, \ t \in]0,+\infty[, \quad 1 \leq i \leq n, \\ \\ u_{n+1} = u_1, \end{cases}$$

where $p_i > 1$, $\sigma_i > -2$, for $1 \leq i \leq n$, $p_{n+1} = p_1$, $\gamma_{n+1} = \gamma_1$, $\sigma_{n+1} = \sigma_1$, and the initial data $(u_i^{(0)}, u_i^{(1)}, ..., u_i^{(k-1)}) \in \left[L^1_{\mathrm{loc}}(K_\varepsilon)\right]^k$, $1 \leq i \leq n$. We will assume that $\gamma_i \leq 0$ for $i \in \{1,2,...,n\}$.

We start by giving the new estimates corresponding to (11) and (14), (for given $p > 1$ and $\gamma \leq 0$)

$$\int_0^{+\infty} \int_{\mathcal{N}} \frac{|\Delta(\varphi_\rho)(x,t)|^p}{(t^\gamma |x|^\sigma \varphi_\rho)^{p-1}} dxdt \leq \frac{C}{\rho^{2p}} \int_0^{2\rho^\theta} t^{-\gamma(p-1)}\, dt \int_\rho^{2\rho} \int_\Omega \frac{r^{s_+ + N-1}}{r^{\sigma(p-1)}}\, d\theta\, dr$$

$$\leq \quad C \begin{cases} \rho^{s_+ + N - \sigma(p-1) + \theta - 2p - \gamma\theta(p-1)} & \text{if} \quad s_+ + N - \sigma(p-1) > 0, \\ \rho^{\theta - 2p - \gamma\theta(p-1)} \ln(\rho) & \text{if} \quad s_+ + N - \sigma(p-1) = 0, \\ \rho^{\theta - 2p - \gamma\theta(p-1)} & \text{if} \quad s_+ + N - \sigma(p-1) < 0. \end{cases}$$

$$\leq \quad C \begin{cases} \rho^{s_+ + N - (\sigma+\gamma\theta)(p-1) + \theta - 2p} & \text{if} \quad s_+ + N - \sigma(p-1) > 0, \\ \rho^{\theta - 2p - \gamma\theta(p-1)} \ln(\rho) & \text{if} \quad s_+ + N - \sigma(p-1) = 0, \\ \rho^{\theta - 2p - \gamma\theta(p-1)} & \text{if} \quad s_+ + N - \sigma(p-1) < 0. \end{cases} \tag{23}$$

and

$$\int \int_{\mathrm{supp}\left|\frac{\partial^k \varphi_\rho}{\partial t^k}\right|} \frac{\left|\frac{\partial^k \varphi_\rho}{\partial t^k}\right|^p}{(t^\gamma |x|^\sigma \varphi_\rho)^{p-1}}\, dx\, dt$$

$$\leq C \begin{cases} \rho^{s_+ + N - \sigma(p-1) - \theta(kp-1) - \gamma\theta(p-1)} & \text{if} \quad s_+ + N - \sigma(p-1) > 0, \\ \rho^{-\theta(kp-1) - \gamma\theta(p-1)} \ln(\rho) & \text{if} \quad s_+ + N - \sigma(p-1) = 0, \\ \rho^{-\theta(kp-1) - \gamma\theta(p-1)} & \text{if} \quad s_+ + N - \sigma(p-1) < 0. \end{cases}$$

$$\leq C \begin{cases} \rho^{s_+ + N - (\sigma+\gamma\theta)(p-1) - \theta(kp-1)} & \text{if} \quad s_+ + N - \sigma(p-1) > 0, \\ \rho^{-\theta(kp-1) - \gamma\theta(p-1)} \ln(\rho) & \text{if} \quad s_+ + N - \sigma(p-1) = 0, \\ \rho^{-\theta(kp-1) - \gamma\theta(p-1)} & \text{if} \quad s_+ + N - \sigma(p-1) < 0. \end{cases} \tag{24}$$

Let the vector $(Y_1, Y_2, ..., Y_n)$ be the solution of the linear system

$$\begin{pmatrix} -1 & p_1 & 0 & \dots & 0 \\ 0 & -1 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & p_{n-1} \\ p_n & 0 & \dots & 0 & -1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{n-1} \\ Y_n \end{pmatrix} = \begin{pmatrix} \sigma_1 + 2 + 2\gamma_1/k \\ \sigma_2 + 2 + 2\gamma_2/k \\ \vdots \\ \sigma_{n-1} + 2 + 2\gamma_{n-1}/k \\ \sigma_n + 2 + 2\gamma_n/k \end{pmatrix}. \tag{25}$$

The we have the following non existence result:

Theorem 5 *Assume that*

$$\forall (x,t) \in \partial K_\varepsilon \times [0,+\infty[, \quad \forall i \in \{1,2,...,n\}; \quad u_i(x,t) \geq 0, \quad a_i(x,t) \geq 0$$

and

$$\forall x \in K_\varepsilon, \quad \forall i \in \{1,2,...,n\}; \quad \frac{\partial^{k-1} u_i}{\partial t^{k-1}}(x,0) \geq 0.$$

Let
$$\max\{Y_1, Y_2, ..., Y_n\} \geq s_+ + N - 2\left(1 - \frac{1}{k}\right).$$
Then the problem (P) has no nontrivial global weak solution.

Proof. We follow the last proof and replace the expressions $I_i(\rho)$, $\mathcal{A}_i(\rho)$ and $\mathcal{B}_i(\rho)$ by
$$\overline{I}_i(\rho) := \int_0^\infty \int_{K_\varepsilon} |x|^{\sigma_i} |u_i|^{p_i} \varphi_\rho \, dx \, dt,$$
$$\overline{\mathcal{A}}_i(\rho) = \int_0^\infty \int_{K_\varepsilon} \frac{|\Delta\varphi_\rho|^{p_i'}}{(t^{\gamma_i}|x|^{\sigma_i}\varphi_\rho)^{p_i'-1}} \, dx \, dt$$
and
$$\overline{\mathcal{B}}_i(\rho) = \int_0^\infty \int_{K_\varepsilon} \frac{|\frac{\partial^k \varphi_\rho}{\partial t^k}|^{p_i'}}{(t^{\gamma_i}|x|^{\sigma_i}\varphi_\rho)^{p_i'-1}} \, dx \, dt,$$
respectively. By setting the parameter $\theta = 2/k$, we conclude that
$$\overline{\mathcal{A}}_i(\rho) \leq C\,\overline{\Theta}_i(\rho) \quad \text{and} \quad \overline{\mathcal{B}}_i(\rho) \leq C\,\overline{\Theta}_i(\rho),$$
where
$$\overline{\Theta}_i(\rho) = \begin{cases} \rho^{s_+ + N - (\sigma_i + 2\gamma_i/k)(p_i'-1) - 2(p_i' - 1/k)} & \text{if} \quad s_+ + N - \sigma_i(p_i' - 1) > 0, \\ \rho^{-2(p_i'-1/k) - 2\gamma_i(p_i'-1)/k} \ln(\rho) & \text{if} \quad s_+ + N - \sigma_i(p_i' - 1) = 0, \\ \rho^{-2(p_i'-1/k) - 2\gamma_i(p_i'-1)/k} & \text{if} \quad s_+ + N - \sigma(p_i' - 1) < 0, \end{cases}$$
for any $i \in \{1, 2, ..., N\}$. As in the last proof, note that the leading exponent in the previous estimate is
$$s_+ + N - (\sigma_i + 2\gamma_i/k)(p_i' - 1) - 2(p_i' - 1/k).$$
In other words, the unique difference with the last proof is that the parameter σ_i must be replaced by $\sigma_i + 2\gamma_i/k$. Which achieves the proof. $\square$

Remark. If we consider the semilinear problem (I) instead of (I_k) with $\gamma \leq 0$, then σ has to be replaced by $\sigma + 2\gamma/k$ in Theorem 1 and Theorem 2.

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