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Necessary Conditions for Local and Global Solvability of Nondiagonal Degenerate Systems

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Necessary Conditions for Local and Global Solvability of Nondiagonal Degenerate Systems

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Résumé

Cross-diffusion has been widely considered either in the mechanical description of diffusion or in the stochastic point process description of interacting populations, in the mathematical modelling of spatially structured epidemic or ecological systems and for the geographical diffusion of innovation. In this paper, specific attention is devoted to blowing-up solutions of some systems which may reflect either failures in the modelling or genuine phenomena like aggregation of populations. Furthemore, necessary conditions for local and global existence of solutions to the considered systems are presented.

Keywords : *Nonlinear degenerate reaction-diffusion systems, porous media, necessary conditions for solvability, blowing-up.*

AMS subject classification : 35K45, 35K55, 35K65.

1 Introduction

The role of spatial heterogeneities and dispersal for chemical reacting species or biological interacting populations in the linear or nonlinear regime has been the subject of a sizeable literature (see e.g. the authorative books of Aris [1] and Cussler [10]).
In particular cross-diffusion in modelling interactions among different species has attracted special attentions. Apart the above quoted books, one can cite [16, 18, 32, 33, 34] in physical-chemistry, [7, 23] in epidemics, [19, 29] in ecology and population dynamics, [21] in biology and very recently [8] in economics.
The recent papers [20, 23, 30, 31] on reaction-diffusion systems with ”non diagonal” diffusion matrices are devoted to global existence and large time behaviour.

In this article we consider the system

$$\begin{cases} u_t(x,t) = \Delta\left(|u|^{n-1}u\right) + \alpha\Delta\left(|v|^{m-1}v\right) + f(x,t)\,|v|^p + w_1(x,t) \\ \\ v_t(x,t) = \Delta\left(|v|^{l-1}v\right) + g(x,t)\,|u|^q + w_2(x,t) \end{cases}$$

for $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$, subject to the initial distributions $(u(x,0), v(x,0)) = (u_0(x), v_0(x))$, $x \in \mathbb{R}^N$, the constant α may be positive or negative, f and g are given nonnegative functions, the functions w_1 and w_2 may represent some "noises". The cross-term $\alpha\Delta\left(|v|^{m-1}v\right)$ gives a measure of the flux of one component engendered by the concentration gradient of the second component.
Before announcing our main results, let us dwell for a while on the modelling part [12].
Consider, for example, two substances (species, chemicals, etc.) that are activating or inhibating each other according to some law of reaction and diffusing in a spatial domain by Fick's law but the diffusion of one of the substances is influenced also by the other one and vice versa. The density of the two substances at time t and place x are denoted by $u(x,t)$ and $v(x,t)$ respectively.
On one hand, the substance u flows from places where its density is high towards places where the density is low. On the other hand, v has an attracting or repelling effect on u, so that u flows towards high, resp. low density places of v. In this situation the flow vector of u is given by

$$J_u = -d_{11}(u,v)\nabla u - d_{12}(u,v)\nabla v,$$

where $d_{11}(u,v) > 0$ and $d_{12}(u,v) \le 0$, resp. ≥ 0 according as v attracts, resp repels u.
Similarly, the flow of v is given by

$$J_v = -d_{21}(u,v)\nabla u - d_{22}(u,v)\nabla v,$$

where $d_{22}(u,v) > 0$ and $d_{21}(u,v) \le 0$, resp. ≥ 0 according as u attracts, resp repels v.
Then we obtain the reaction-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \nabla.\left(d_{11}(u,v)\nabla u + d_{12}(u,v)\nabla v\right) + U(u,v) \\ \\ \dfrac{\partial v}{\partial t} &= \nabla.\left(d_{21}(u,v)\nabla u + d_{22}(u,v)\nabla v\right) + V(u,v) \end{cases} \tag{1}$$

where $U(u,v)$ and $V(u,v)$ are the reaction terms. In the particular case :

$$d_{11} = nu^{n-1},\ d_{22} = lv^{l-1},\ d_{12} = d_{21} = 0,\ U = v^p \text{ and } V = u^q,$$

we obtain the system

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \Delta\left(u^n\right) + v^p \\ \\ \dfrac{\partial v}{\partial t} &= \Delta\left(v^l\right) + u^q \end{cases} \tag{2}$$

Note that the system (2) describes the processes of diffusion of heat and burning in two-component continuous media with nonlinear conductivity and volume energy release. The functions u and v can thus be treated as temperatures of interacting components of a combustible mixture [15].

When the cross-diffusion $d_{12}(u,v)$ obeys to a similar law as d_{11} and d_{22}, say

$$d_{12}(u,v) = mv^{m-1},$$

we obtain the system

$$\left\{\begin{array}{lcl} \dfrac{\partial u}{\partial t} & = & \Delta\left(u^n\right) + \Delta\left(v^m\right) + v^p \\ \\ \dfrac{\partial v}{\partial t} & = & \Delta\left(v^l\right) + u^q \end{array}\right. \tag{3}$$

which concerns the present paper. With an aim of giving more general results, we consider the case where the reaction terms also depend on t and x.

Section 2 is motivated by the paper [5] in which Baras and Kersner showed that the problem

$$u_t = \Delta u + h(x)u^p, \quad u(x,0) = u_0(x) \geq 0,$$

has no nonnegative local weak solution if the initial data satisfies

$$\lim_{|x|\to+\infty} u_0^{p-1} h(x) = +\infty,$$

and any possible nonnegative local weak solution blows up at a finite time if

$$\lim_{|x|\to+\infty} u_0^{p-1} h(x)|x|^2 = +\infty.$$

We show similar results for a degenerated nonlinear parabolic system with triangular diffusion matrix.

Section 3 deals with Fujita's type results. Its aim is not only to generalize the results in [9] to triangular diffusion matrix systems but also to weaken the assumptions on the data. Indeed, we require nonnegative integrability of the initial data and of the nonhomogeneous forcing terms while [9] requires their positivity.

2 Necessary conditions for local and global solvability

Consider the system

$$\text{(P)} \left\{\begin{array}{lll} u_t(x,t) = \Delta\left(|u|^{n-1}u\right) \pm \Delta\left(|v|^{m-1}v\right) + f(x,t)\,|v|^p + w_1(x,t) & \text{in} & Q_T = \mathbb{R}^N \times]0,T[\\ \\ v_t(x,t) = \Delta\left(|v|^{l-1}v\right) + g(x,t)\,|u|^q + w_2(x,t) & \text{in} & Q_T \\ \\ u(x,0) = u_0(x) & \text{in} & \mathbb{R}^N \\ \\ v(x,0) = v_0(x) & \text{in} & \mathbb{R}^N \end{array}\right.$$

where $p \geq 1,\ q \geq 1,\ m \geq 1,\ n \geq 1$ and $l \geq 1$, $0 < T \leq +\infty$, with the following hypotheses on the data :

(H1) $(f,g) \in L^{p'}_{loc}(Q_T)\times L^{q'}_{loc}(Q_T)$, $f \geq 0$ and $g \geq 0$, where $(p',q') = (p/(p-1), q/(q-1))$;

(H2) $w_i \in L^1(Q_T)$ and $\displaystyle\int_{Q_T} w_i \, dx\, dt \geq 0$, $i = 1,\ 2$;

(H3) $(u_0, v_0) \in L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$, with $\displaystyle\int_{\mathbb{R}^N} u_0 \, dx \geq 0$ and $\displaystyle\int_{\mathbb{R}^N} v_0 \, dx \geq 0$.

In the sequel, if $T = +\infty$, the domain Q_T will be denoted by Q.

Definition 1. A pair of functions (u,v) is called a weak solution of (P) in Q_T if

(i) $u,\ v\ : Q_T \longrightarrow \mathbb{R}$

(ii) $(u,v) \in L^q_{loc}(Q_T) \times L^p_{loc}(Q_T)$

(iii) for any $\varphi \in \mathcal{D}(\mathbb{R}^N \times [0,T])$ which vanishes at $t = T$ if $T < +\infty$ or for any $\varphi \in \mathcal{D}(Q)$ if $T = +\infty$, one has

$$\int_{Q_T} \left(u\varphi_t + \left(|u|^{n-1}u \pm |v|^{m-1}v\right)\Delta\varphi + (f|v|^p + w_1)\varphi\right) dxdt + \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)dx = 0 \tag{4}$$

and

$$\int_{Q_T} \left(v\varphi_t + |v|^{l-1}v\Delta\varphi + g|u|^q\varphi + w_2\varphi\right) dxdt + \int_{\mathbb{R}^N} v_0(x)\varphi(x,0)dx = 0. \tag{5}$$

We attempt to get insight into the relationship between local and global solvability of (P) on one hand, and the behaviour at infinity of the data f, g, w_1, w_2, u_0 and v_0 on the other. In this section we will confine ourselves to the following case : $f(x,t) = t^\alpha F(x)$, $g(x,t) = t^\beta G(x)$, $w_1(x,t) = t^{\gamma_1} W_1(x)$ and $w_2(x,t) = t^{\gamma_2} W_2(x)$, where F and G are positive and continuous functions, u_0 and v_0 are nonnegative and integrable functions. We add the following assumption

(H4) $p > \max\{l, m, l+\alpha, m+\alpha\}$, $q > \max\{n, n+\beta\}$, and $\min\{m,n,l\} > 1$.

2.1 Necessary conditions for local solvability

Before stating our first result, we need to assume that, for R sufficiently large, the estimates

$$\int_{|x|\leq 2R} G(x)^{-\frac{1}{q-n}}\, dx = o\left(R^{\frac{2q}{q-n}}\right), \tag{6}$$

$$\int_{|x|\leq 2R} F(x)^{-\frac{1}{p-l}}\, dx = o\left(R^{\frac{2p}{p-l}}\right), \tag{7}$$

and

$$\int_{|x|\leq 2R} F(x)^{-\frac{1}{p-m}}\, dx = o\left(R^{\frac{2p}{p-m}}\right) \tag{8}$$

hold.

Theorem 1. *If the problem (P) has a nonnegative local solution defined in Q_T, $T < +\infty$, then the following estimates*

$$\liminf_{|x|\to+\infty} u_0(x)\mathcal{F}(x) \le \frac{K_1}{T^\delta}, \quad \liminf_{|x|\to+\infty} v_0(x)\mathcal{F}(x) \le \frac{K_1}{T^\delta} \tag{9}$$

and

$$\liminf_{|x|\to+\infty} W_1(x)\mathcal{F}(x) \le \frac{K_2}{T^{1+\gamma_1+\delta}}, \quad \liminf_{|x|\to+\infty} W_2(x)\mathcal{F}(x) \le \frac{K_3}{T^{1+\gamma_2+\delta}} \tag{10}$$

hold, where

$$\mathcal{F}(x) = \frac{F^{p'/p}(x)\, G^{q'/q}(x)}{F^{p'/p}(x) + G^{q'/q}(x)},$$

$$T^\delta = \min\left(T^{\frac{\alpha+1}{p-1}},\, T^{\frac{\beta+1}{q-1}}\right),$$

and the constants K_1, K_2, K_3 are specified in the proof.

Démonstration. Let (u,v) be a nonnegative weak solution of (P) in Q_T. For any $\varphi \in \mathcal{D}([0,T]\times\mathbb{R}^N)$ with $\varphi \ge 0$, $\varphi(.,T) \equiv 0$, one has

$$\int_{Q_T} \left(u\varphi_t + (u^n \pm v^m)\,\Delta\varphi + fv^p\varphi + w_1\varphi\right) dxdt + \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)dx = 0,$$

$$\int_{Q_T} \left(v\varphi_t + v^l\Delta\varphi + gu^q\varphi + w_2\varphi\right) dxdt + \int_{\mathbb{R}^N} v_0(x)\varphi(x,0)dx = 0.$$

Then

$$\int_{\mathbb{R}^N} u_0\varphi(0) + \int_{Q_T} fv^p\varphi + \int_{Q_T} w_1\varphi \le \int_{Q_T} \left(u|\varphi_t| + (u^n + v^m)\,(-\Delta\varphi)_+\right),$$

$$\int_{\mathbb{R}^N} v_0\varphi(0) + \int_{Q_T} gu^q\varphi + \int_{Q_T} w_2\varphi \le \int_{Q_T} \left(v|\varphi_t| + v^l(-\Delta\varphi)_+\right),$$

where $\varphi(0) \equiv \varphi(.,0)$ and $(-\Delta\varphi)_+ = \max(0, -\Delta\varphi)$. Furthermore, Young's inequality gives

$$\int_{Q_T} u|\varphi_t| \;\; \le \;\; \frac{1}{2}\int_{Q_T} u^q\,(g\,\varphi) + c_1\int_{Q_T} |\varphi_t|^{q/(q-1)}\,(g\,\varphi)^{-1/(q-1)},$$

and

$$\int_{Q_T} u^n(-\Delta\varphi)_+ \le \frac{1}{2}\int_{Q_T} u^q\,(g\,\varphi) + c_2\int_{Q_T} (-\Delta\varphi)_+^{q/(q-n)}\,(g\,\varphi)^{-1/(q-n)}.$$

Hence

$$\int_{\mathbb{R}^N} u_0\varphi(0) + \int_{Q_T} (fv^p\varphi + w_1\varphi) \le \int_{Q_T} \left(v|\varphi_t| + \left(v^l + v^m\right)(-\Delta\varphi)_+\right) + A(\varphi,g,T),$$

where

$$A(\varphi,g,T) = c_1X_1(\varphi,g,T) + c_2X_2(\varphi,g,T),$$

with

$$\begin{aligned} X_1(\varphi, g, T) &= \int_{Q_T} |\varphi_t|^{q/(q-1)} \, (g\,\varphi)^{-1/(q-1)}, \\ X_2(\varphi, g, T) &= \int_{Q_T} (-\Delta\varphi)_+^{q/(q-n)} \, (g\,\varphi)^{-1/(q-n)}. \end{aligned}$$

Similarly, Young's inequality allows us to obtain

$$\int_{Q_T} \left(v|\varphi_t| + \left(v^l + v^m\right)(-\Delta\varphi)_+ \right) \leq \int_{Q_T} f v^p \varphi + B(\varphi, f, T)$$

where

$$B(\varphi, f, T) = c_3\, X_3(\varphi, f, T) + c_4\, X_4(\varphi, f, T) + c_5\, X_5(\varphi, f, T),$$

with

$$\begin{aligned} X_3(\varphi, f, T) &= \int_{Q_T} |\varphi_t|^{p/(p-1)} \, (f\,\varphi)^{-1/(p-1)}, \\ X_4(\varphi, f, T) &= \int_{Q_T} (-\Delta\varphi)_+^{p/(p-l)} \, (f\,\varphi)^{-l/(p-l)}, \\ X_5(\varphi, f, T) &= \int_{Q_T} (-\Delta\varphi)_+^{p/(p-m)} \, (f\,\varphi)^{-m/(p-m)}. \end{aligned}$$

Finally, we have

$$\int_{\mathbb{R}^N} u_0 \varphi(0) + \int_{Q_T} w_1 \varphi \leq \sum_{i=1}^{5} c_i\, X_i. \tag{11}$$

Following the same idea, we show that there are positive constants d_i, $i = 1, ..5$, so that

$$\int_{\mathbb{R}^N} v_0 \varphi(0) + \int_{Q_T} w_2 \varphi \leq \sum_{i=1}^{5} d_i\, X_i. \tag{12}$$

At this stage, the test function φ is chosen as follows

$$\varphi(x,t) = \left[\eta\left(\frac{t}{T}\right) \right]^s \Phi\left(\frac{x}{R}\right),$$

where

i) $\Phi \in \mathcal{D}(\mathbb{R}^N)$, $0 \leq \Phi \leq 1$, supp $(\Phi) \subset \{1 < |x| < 2\}$ and $-\Delta\Phi \leq \Phi$,

ii) $\eta \in \mathcal{D}(\mathbb{R}^+)$, $0 \leq \eta \leq 1$, and

$$\eta(r) = \begin{cases} 1 \;\text{ if }\; r \leq 1/2, \\ 0 \;\text{ if }\; r \geq 1, \end{cases}$$

iii) $s = \max(p', q')$ and $R > 0$.

The choice of this test function is inspired by the paper of P. Baras and R. Kersner [5]. It allows us to obtain interesting estimations connecting the initial data and the reaction terms.

Consequently, the integrals X_i, $i = 1, ..5$, are convergent, more precisely

$$\begin{aligned} X_1 &\le \frac{s^{q'}||\eta'||_\infty^{q'}}{T^{q'}} \frac{T^{1-\frac{\beta}{q-1}}}{1-\frac{\beta}{q-1}} \int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right) G(x)^{-\frac{q'}{q}}\, dx \\ &= \frac{(q-1)\, s^{q'}||\eta'||_\infty^{q'}}{q-(\beta+1)} T^{-\frac{\beta+1}{q-1}} \int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right) G(x)^{-\frac{q'}{q}}\, dx, \end{aligned}$$

and similarly

$$\begin{aligned} X_2 &\le \frac{q-n}{q-(\beta+n)} T^{\frac{q-(\beta+n)}{q-n}} R^{-\frac{2q}{q-n}} \int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right) G(x)^{-\frac{1}{q-n}}\, dx, \\ X_3 &\le \frac{(p-1)\, s^{p'}||\eta'||_\infty^{p'}}{p-(\alpha+1)} T^{-\frac{\alpha+1}{p-1}} \int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right) F(x)^{-\frac{p'}{p}}\, dx, \\ X_4 &\le \frac{p-l}{p-(\alpha+l)} T^{\frac{p-(\alpha+l)}{p-l}} R^{-\frac{2p}{p-l}} \int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right) F(x)^{-\frac{1}{p-l}}\, dx, \\ X_5 &\le \frac{p-m}{p-(\alpha+m)} T^{\frac{p-(\alpha+m)}{p-m}} R^{-\frac{2p}{p-m}} \int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right) F(x)^{-\frac{1}{p-m}}\, dx. \end{aligned}$$

Let

$$K_1 = \max\left(c_1 \frac{(q-1)\, s^{q'}||\eta'||_\infty^{q'}}{q-(\beta+1)},\ c_2 \frac{(p-1)\, s^{p'}||\eta'||_\infty^{p'}}{p-(\alpha+1)}\right).$$

In view of (11) and the above estimates we see that

$$\int_{\mathbb{R}^N} u_0(x)\Phi\left(\frac{x}{R}\right) \le K_1 \int_{\mathbb{R}^N} \mathcal{G}(x,T)\Phi\left(\frac{x}{R}\right) + X_2 + X_4 + X_5, \qquad (13)$$

and

$$\frac{(T/2)^{1+\gamma_1}}{1+\gamma_1} \int_{\mathbb{R}^N} W_1(x)\Phi\left(\frac{x}{R}\right) \le K_1 \int_{\mathbb{R}^N} \mathcal{G}(x,T)\Phi\left(\frac{x}{R}\right) + X_2 + X_4 + X_5, \qquad (14)$$

where

$$\mathcal{G}(x,T) = F(x)^{-\frac{p'}{p}} T^{-\frac{\alpha+1}{p-1}} + G(x)^{-\frac{q'}{q}} T^{-\frac{\beta+1}{q-1}}.$$

Hence,

$$\left\{\inf_{|x|>R} \frac{u_0(x)}{\mathcal{G}(x,T)}\right\} \int_{\mathbb{R}^N} \mathcal{G}(x,T)\Phi\left(\frac{x}{R}\right) \le K_1 \int_{\mathbb{R}^N} \mathcal{G}(x,T)\Phi\left(\frac{x}{R}\right) + X_2 + X_4 + X_5.$$

Finally, using the estimates (6)-(8), we have

$$\lim_{R\to+\infty} \inf_{|x|>R} \frac{u_0(x)}{\mathcal{G}(x,T)} \le C,$$

or

$$\liminf_{|x|\to+\infty} u_0(x)\mathcal{F}(x) \le \frac{K_1}{\min\left(T^{\frac{\alpha+1}{p-1}},\, T^{\frac{\beta+1}{q-1}}\right)}. \qquad (15)$$

The second estimate in (9) and the estimates in (10) are obtained in the same manner by setting $K_2 = (1+\gamma_1)K_1\, 2^{1+\gamma_1}$ and $K_3 = (1+\gamma_2)K_1\, 2^{1+\gamma_2}$. This completes the proof. □

Consequences :
The previous theorem asserts that if $\max\{\alpha,\beta\} < -1$ and $\liminf_{|x|\to+\infty} u_0(x)\mathcal{F}(x) \neq 0$ then (P) has no nonnegative local solution, as it can be seen from (15) by letting $T \to 0$. Furthermore, if (P) has a nonnegative global solution then, by letting $T \to +\infty$, we have

i) $\min\{\alpha,\beta\} > -1 \quad \Longrightarrow \quad \liminf_{|x|\to+\infty} u_0(x)\mathcal{F}(x) \;=\; \liminf_{|x|\to+\infty} v_0(x)\mathcal{F}(x) \;= 0,$

ii) $\min\{\alpha,\beta,\gamma_1,\gamma_2\} > -1 \quad \Longrightarrow \quad \liminf_{|x|\to+\infty} W_1(x)\mathcal{F}(x) \;=\; \liminf_{|x|\to+\infty} W_2(x)\mathcal{F}(x) \;= 0.$

In the limit case $\min\{\alpha,\beta\} = -1$, if $\liminf_{|x|\to+\infty} u_0(x)\mathcal{F}(x) > K_1$, then there is no nonnegative global solution.

2.2 Necessary conditions for global solvability

Before stating the main results of this subsection, we need to introduce some notations and hypotheses. Let

$$a = \min\left\{\frac{\alpha+1}{p-1}, \frac{\beta+1}{q-1}\right\}$$

and

$$H = \min\{F^{\frac{p'}{p}}, G^{\frac{q'}{q}}, F^{\frac{1}{p-l}}, F^{\frac{1}{p-m}}, G^{\frac{1}{q-n}}\}.$$

We assume the following hypothesis
(H5) $\min\{\alpha,\beta\} > -1.$

Theorem 2. *If the problem (P) has a nonnegative global solution, then there exists $\theta > 0$ for which the following limits*

$$\liminf_{|x|\to+\infty} u_0(x)H(x)|x|^{a\,\theta}, \quad \liminf_{|x|\to+\infty} v_0(x)H(x)|x|^{a\,\theta} \tag{16}$$

are bounded. The real number θ is specified in the proof.

Démonstration. The inequality (13) implies that there is a positive constant C such that

$$\int_{\mathbb{R}^N} u_0(x)\Phi\left(\frac{x}{R}\right) \leq C\ \mathcal{H}(R,T)\int_{\mathbb{R}^N}\frac{1}{H(x)}\Phi\left(\frac{x}{R}\right), \tag{17}$$

where

$$\mathcal{H}(R,T) = \frac{T^{\alpha_1}}{R^{\beta_1}} + \frac{T^{\alpha_2}}{R^{\beta_2}} + \frac{T^{\alpha_3}}{R^{\beta_3}} + \frac{1}{T^{\alpha_4}} + \frac{1}{T^{\alpha_5}},$$

$$\alpha_1 = \frac{p-(\alpha+l)}{p-l}, \quad \beta_1 = \frac{2p}{p-l},$$

$$\alpha_2 = \frac{p-(\alpha+m)}{p-m}, \quad \beta_2 = \frac{2p}{p-m},$$

$$\alpha_3 = \frac{q-(\beta+n)}{q-n}, \quad \beta_3 = \frac{2q}{q-n},$$

$$\alpha_4 = \frac{1+\alpha}{p-1}, \quad \alpha_5 = \frac{1+\beta}{q-1}.$$

Note that under the hypotheses (H4)-(H5), the parameters α_i and β_i are positive. Now, we have to minimize the function $\mathcal{H}$ with respect to T. For this, one has

$$\frac{\partial \mathcal{H}}{\partial T}(R,T) = \frac{1}{T}\mathcal{L}(R,T)$$

where

$$\mathcal{L}(R,T) = \alpha_1 \frac{T^{\alpha_1}}{R^{\beta_1}} + \alpha_2 \frac{T^{\alpha_2}}{R^{\beta_2}} + \alpha_3 \frac{T^{\alpha_3}}{R^{\beta_3}} - \frac{\alpha_4}{T^{\alpha_4}} - \frac{\alpha_5}{T^{\alpha_5}}. \tag{18}$$

Then

$$\frac{\partial \mathcal{H}}{\partial T}(R,T) = 0 \Longleftrightarrow \mathcal{L}(R,T) = 0.$$

Moreover, it is clear that the function $\mathcal{L}$ is strictly increasing in $T > 0$ and

$$\lim_{T\to 0^+} \mathcal{L}(R,T) = -\infty \quad \text{and} \quad \lim_{T\to +\infty} \mathcal{L}(R,T) = +\infty,$$

which imply that for any $R > 0$ there is a unique $T_*(R) > 0$ such that $\mathcal{L}(R, T_*(R)) = 0$. The implicit function theorem asserts that the function T_* is smooth in R and

$$\frac{dT_*}{dR}(R) = -\frac{\partial \mathcal{L}/\partial R}{\partial \mathcal{L}/\partial T}(R, T_*(R)) > 0.$$

Hence T_* is strictly increasing in R and we easily see that

$$\lim_{R\to +\infty} T_*(R) = +\infty.$$

Finally,

$$\begin{aligned} \frac{\partial^2 \mathcal{H}}{\partial T^2}(R, T_*(R)) &= -\frac{1}{T_*^2(R)}\mathcal{L}(R, T_*(R)) + \frac{1}{T_*(R)}\frac{\partial \mathcal{L}}{\partial T}(R, T_*(R)) \\ &= \frac{1}{T_*(R)}\frac{\partial \mathcal{L}}{\partial T}(R, T_*(R)) > 0, \end{aligned}$$

implies that for any fixed $R > 0$, the function $\mathcal{H}$ has a unique minimum at $(R, T_*(R))$. Now, we have to determine the asymptotic behaviour of $T_*(R)$ as R tends to $+\infty$. Recall that $a = \min(\alpha_4, \alpha_5)$ and that the pair $(R, T_*(R))$ verifies the identity

$$\alpha_1 \frac{T_*^{\alpha_1}(R)}{R^{\beta_1}} + \alpha_2 \frac{T_*^{\alpha_2}(R)}{R^{\beta_2}} + \alpha_3 \frac{T_*^{\alpha_3}(R)}{R^{\beta_3}} = \frac{\alpha_4}{T_*^{\alpha_4}(R)} + \frac{\alpha_5}{T_*^{\alpha_5}(R)}.$$

Then, there is $\ell > 0$ such that

$$\lim_{R\to +\infty} \left\{ \alpha_1 \frac{T_*^{\alpha_1 + a}(R)}{R^{\beta_1}} + \alpha_2 \frac{T_*^{\alpha_2 + a}(R)}{R^{\beta_2}} + \alpha_3 \frac{T_*^{\alpha_3 + a}(R)}{R^{\beta_3}} \right\} = \ell.$$

Setting $k_i = \beta_i/(a + \alpha_i)$, $i \in \{1,2,3\}$, $\theta = \min\{k_i;\ i \in \{1,2,3\}\}$ and $\Theta = \{i \in \{1,2,3\};\ k_i = \theta\}$, we have

$$\sum_{i=1}^{3} \alpha_i \left(\frac{T_*(R)}{R^{k_i}}\right)^{a+\alpha_i} = \sum_{i\in\Theta} \alpha_i \left(\frac{T_*(R)}{R^{\theta}}\right)^{a+\alpha_i} + \varepsilon(R,T). \tag{19}$$

Since the functions
$$R \longmapsto \frac{T_*^{\alpha_i+a}(R)}{R^{\beta_i}}, \quad i \in \{1,2,3\},$$
are bounded uniformly with respect to R, we conclude that
$$\lim_{R\to+\infty} \sum_{i\in\Theta} \alpha_i \left(\frac{T_*(R)}{R^\theta}\right)^{a+\alpha_i} = \ell \quad \text{and} \quad \lim_{R\to+\infty} \varepsilon(R,T) = 0.$$
Let the function
$$P(X) = \sum_{i\in\Theta} \alpha_i\, X^{a+\alpha_i}.$$
It is clear that P is strictly increasing and $P(0)=0$. Hence
$$\lim_{R\to+\infty} \frac{T_*(R)}{R^\theta} = \lim_{R\to+\infty} P^{-1}\circ P\left(\frac{T_*(R)}{R^\theta}\right) = P^{-1}(\ell) > 0,$$
because P^{-1} is continuous, and consequently
$$T_*(R) \sim P^{-1}(\ell)\, R^\theta \quad \text{for } R \text{ large enough.}$$
Finally, using (17), there is a constant $K>0$ such that, for R large enough,
$$\int_{\mathbb{R}^N} u_0(x)\Phi\left(\frac{x}{R}\right) \le \frac{K}{R^{a\,\theta}} \int_{\mathbb{R}^N} \frac{1}{H(x)}\Phi\left(\frac{x}{R}\right). \tag{20}$$
Using the fact that $\mathrm{supp}(\Phi) \subset \{x \in \mathbb{R}^N,\ 1 < |x| < 2\}$, we conclude, for R large enough, that
$$\begin{aligned}\left\{\inf_{|x|>R} u_0(x)\, H(x)\, |x|^{a\,\theta}\right\} \int_{\mathbb{R}^N} \frac{|x|^{-a\,\theta}}{H(x)}\Phi\left(\frac{x}{R}\right) &\le \frac{K}{R^{a\,\theta}} \int_{\mathbb{R}^N} \frac{|x|^{a\,\theta}\,|x|^{-a\,\theta}}{H(x)}\Phi\left(\frac{x}{R}\right)\\ &\le \frac{(2R)^{a\,\theta}K}{R^{a\,\theta}} \int_{\mathbb{R}^N} \frac{|x|^{-a\,\theta}}{H(x)}\Phi\left(\frac{x}{R}\right).\end{aligned}$$
Whence the boundedness of $\liminf_{|x|\to+\infty} u_0(x)\, H(x)\, |x|^{a\,\theta}$ is established.
The boundedness of the second limit is similar as above ; this achieves the proof. □

Before stating the last result of this section, we will assume $\gamma_j > -1$, $j \in \{1,2\}$, and distinguish the two hypotheses
(H6) $\min\{\alpha_i,\ \ i \in \{1,2,3\}\} > \gamma_j + 1$,
(H7) $\max\{\alpha_i,\ \ i \in \{1,2,3\}\} < \gamma_j + 1$,
for $j=1$ or $j=2$.

Theorem 3. *Assume (H4)-(H5) and either (H6) or (H7) holds. If the problem (P) has a nonnegative global solution, then there are constants a_j and θ_j such that the following limits*
$$\liminf_{|x|\to+\infty} W_j(x)H(x)|x|^{a_j\theta_j}, \quad j \in \{1,2\}, \tag{21}$$
are bounded. The real numbers a_j and θ_j are specified in the proof.

Démonstration. As before, the inequalities (11) and (12) imply that there is a positive constant C such that for $j \in \{1,2\}$, one has

$$T^{\gamma_j+1}\int_{\mathbb{R}^N} W_j(x)\Phi\left(\frac{x}{R}\right) \leq C\ \mathcal{H}(R,T)\int_{\mathbb{R}^N}\frac{1}{H(x)}\Phi\left(\frac{x}{R}\right). \tag{22}$$

– If (H4), (H5) and (H_6) hold. Consider the function

$$\tilde{\mathcal{H}}(R,T) = \frac{\mathcal{H}(R,T)}{T^{\gamma_j+1}}.$$

The situation then is similar to that of the last theorem with $\tilde{\mathcal{H}}$ instead of $\mathcal{H}$. Following the last proof we obtain

$$\theta_j = \theta \quad \text{and} \quad a_j = a + 1 + \gamma_j,$$

where a and θ are defined in the last proof.

– If (H4), (H5) and (H7) hold, let θ_j be the unique positive real number defined by

$$\min(\beta_i + \theta_j(\gamma_j + 1 - \alpha_i),\ \ i \in \{1,2,3\}) = (\gamma_j + 1 + a)\theta_j.$$

It follows from (22) that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} W_j(x)\Phi\left(\frac{x}{R}\right) \leq \frac{C}{R^{(a+\gamma_j+1)\theta_{4,j}}}\int_{\mathbb{R}^N}\frac{1}{H(x)}\Phi\left(\frac{x}{R}\right).$$

Proceeding as in the end of the last proof, we show that

$$\liminf_{|x|\to+\infty} W_j(x)H(x)|x|^{(a+\gamma_j+1)\theta_j} \tag{23}$$

is bounded. This achieves the proof. □

The next section deals with the nonexistence of global (nontrivial) solutions to the problem (P) from a different angle : We will present results of Fujita's type. These results will take into account the dimension N instead of the behaviour at infinity of the data and of the nonhomogeneous terms. We refer the interested reader to the valuable surveys by Levine [25], Bandle and Brunner [4] and Deng and Levine [11] for some background.

3 Necessary conditions for global solvability : Fujita's type results

The hypotheses considered in this section are (H1)-(H3). In order to simplify the presentation, we initially set $f \equiv g \equiv 1$. We start with the following result.

Lemma 1. *Let (u,v) be a weak solution of (P) in Q. Then, if $u \equiv 0$ or $v \equiv 0$, one has $u \equiv v \equiv 0$.*

We show this lemma in a general way in order to use some contained results in the sequel.

Démonstration. Let $\varphi_{_R} \in \mathcal{D}(\mathbb{R}^N \times [0,+\infty[)$ be a nonnegative function such that

$$\varphi_{_R}(x,t) = \Phi^\lambda\left(\frac{t+|x|^2}{R}\right), \quad \lambda > 1,$$

where $R > 0$ and $\Phi \in \mathcal{D}([0,+\infty[)$ is the "standard cut-off function"

$$0 \le \Phi(r) \le 1, \quad \Phi(r) = \begin{cases} 1 & \text{if} \quad 0 \le r \le 1, \\ 0 & \text{if} \quad r \ge 2. \end{cases} \tag{24}$$

Then the equation (5) gives :

$$\begin{aligned} \int_Q \varphi_{_R}|u|^q\,dx\,dt + a(R) &\le \int_Q \left(|v||\varphi_{_{Rt}}| + |v|^l|\Delta\varphi_{_R}|\right)\,dx\,dt \\ &\le \left(\int_Q |v|^p\varphi_{_R}\,dx\,dt\right)^{1/p}\left(\int_Q |\varphi_{_{Rt}}|^{p/(p-1)}\varphi_{_R}^{-1/(p-1)}\,dx\,dt\right)^{(p-1)/p} \\ &+ \left(\int_Q |v|^p\varphi_{_R}\,dx\,dt\right)^{l/p}\left(\int_Q |\Delta\varphi_{_R}|^{p/(p-l)}\varphi_{_R}^{-l/(p-l)}\,dx\,dt\right)^{(p-l)/p}, \end{aligned}$$

and (4) gives also :

$$\begin{aligned} \int_Q \varphi_{_R}|v|^p\,dx\,dt + b(R) &\le \int_Q \left(|u||\varphi_{_{Rt}}| + |u|^n|\Delta\varphi_{_R}| + |v|^m|\Delta\varphi_{_R}|\right)\,dx\,dt \\ &\le \left(\int_Q |u|^q\varphi_{_R}\,dx\,dt\right)^{1/q}\left(\int_Q |\varphi_{_{Rt}}|^{q/(q-1)}\varphi_{_R}^{-1/(q-1)}\,dx\,dt\right)^{(q-1)/q} \\ &+ \left(\int_Q |u|^q\varphi_{_R}\,dx\,dt\right)^{n/q}\left(\int_Q |\Delta\varphi_{_R}|^{q/(q-n)}\varphi_{_R}^{-n/(q-n)}\,dx\,dt\right)^{(q-n)/q} \\ &+ \left(\int_Q |v|^p\varphi_{_R}\,dx\,dt\right)^{m/p}\left(\int_Q |\Delta\varphi_{_R}|^{p/(p-m)}\varphi_{_R}^{-m/(p-m)}\,dx\,dt\right)^{(p-m)/p}, \end{aligned}$$

where

$$a(R) = \int_{\mathbb{R}^N} v_0(x)\,\varphi_{_R}(x,0)\,dx + \int_Q w_2(x,t)\,\varphi_{_R}(x,t)\,dx\,dt,$$

and

$$b(R) = \int_{\mathbb{R}^N} u_0(x)\,\varphi_{_R}(x,0)\,dx + \int_Q w_1(x,t)\,\varphi_{_R}(x,t)\,dx\,dt.$$

If we set $X(R) = \left(\int_Q |u|^q\varphi_{_R}\,dx\,dt\right)^{1/q}$, $Y(R) = \left(\int_Q |v|^p\varphi_{_R}\,dx\,dt\right)^{1/p}$,

$$A = \left(\int_Q |\varphi_{_{Rt}}|^{p/(p-1)}\varphi_{_R}^{-1/(p-1)}\,dx\,dt\right)^{(p-1)/p}, \quad B = \left(\int_Q |\Delta\varphi_{_R}|^{p/(p-l)}\varphi_{_R}^{-l/(p-l)}\,dx\,dt\right)^{(p-l)/p},$$

$$C = \left(\int_Q |\varphi_{_{Rt}}|^{q/(q-1)}\varphi_{_R}^{-1/(q-1)}\,dx\,dt\right)^{(q-1)/q}, \quad D = \left(\int_Q |\Delta\varphi_{_R}|^{q/(q-n)}\varphi_{_R}^{-n/(q-n)}\,dx\,dt\right)^{(q-n)/q},$$

$$E = \left(\int_Q |\Delta\varphi_{_R}|^{p/(p-m)}\varphi_{_R}^{-m/(p-m)}\,dx\,dt\right)^{(p-m)/p},$$

we then have the following system of inequalities :

$$\left\{ \begin{array}{lcl} X^q(R) + a(R) & \leq & A\, Y(R) + B\, Y^l(R), \\ \\ Y^p(R) + b(R) & \leq & C\, X(R) + D\, X^n(R) + E\, Y^m(R). \end{array} \right. \tag{25}$$

It is easy to see that if λ is selected sufficiently large then the integrals A, B, C, D and E are convergent.
On one hand, if $v \equiv 0$ then $X^q(R)$ is bounded and increasing function of R. Using the monotone convergence theorem, we deduce that $|u|^q$ is in $L^1(Q)$ and

$$\lim_{R\to+\infty} (X^q(R) + a(R)) = \int_Q |u|^q \, dx\, dt + \int_{\mathbb{R}^N} v_0(x)\, dx + \int_Q w_2(x,t)\, dx\, dt = 0.$$

Then, we have necessarily $u \equiv 0$, and consequently $(u, v) \equiv (0, 0)$.
On the other hand, if $u \equiv 0$ then there is a positive constant C_0 such that $Y^p(R) \leq C_0 + EY^m(R)$. Since $m < p$, there is a positive constant C_1 such that $Y(R) \leq C_1$. Similarly, the function $|v|^p$ is in $L^1(Q)$.

Note that instead of (25) we have more precisely

$$\left\{ \begin{array}{lcl} X^q(R) + a(R) & \leq & A\, \tilde{Y}(R) + B\, \tilde{Y}^l(R) \\ \\ Y^p(R) + b(R) & \leq & C\, \tilde{X}(R) + D\, \tilde{X}^n(R) + E\, \tilde{Y}^m(R), \end{array} \right. \tag{26}$$

where

$$\tilde{X}(R) = \left(\int_{\Omega_R} |u|^q \varphi_{_R}\, dx\, dt \right)^{1/q}, \quad \tilde{Y}(R) = \left(\int_{\Omega_R} |v|^p \varphi_{_R}\, dx\, dt \right)^{1/p},$$

and $\Omega_R = \{(x,t) \in Q;\ R \leq t + |x|^2 \leq 2\, R\}$. Indeed, as before, the equation (5) gives :

$$\begin{array}{rcl} \displaystyle\int_Q \varphi_{_R} |u|^q \, dx\, dt + a(R) & \leq & \displaystyle\int_Q \left(|v||\varphi_{_R t}| + |v|^l |\Delta \varphi_{_R}| \right)\, dx\, dt \\ & = & \displaystyle\int_{\Omega_R} \left(|v||\varphi_{_R t}| + |v|^l |\Delta \varphi_{_R}| \right)\, dx\, dt \\ & \leq & \displaystyle\left(\int_{\Omega_R} |v|^p \varphi_{_R}\, dx\, dt \right)^{1/p} \left(\int_{\Omega_R} |\varphi_{_R t}|^{p/(p-1)} \varphi_{_R}^{-1/(p-1)}\, dx\, dt \right)^{(p-1)/p} \\ & + & \displaystyle\left(\int_{\Omega_R} |v|^p \varphi_{_R}\, dx\, dt \right)^{l/p} \left(\int_{\Omega_R} |\Delta\varphi_{_R}|^{p/(p-l)} \varphi_{_R}^{-l/(p-l)}\, dx\, dt \right)^{(p-l)/p}. \end{array}$$

This implies that

$$X^q(R) + a(R) \leq A\, \tilde{Y}(R) + B\, \tilde{Y}^l(R).$$

Similarly, we obtain the second inequality

$$Y^p(R) + b(R) \leq C\, \tilde{X}(R) + D\, \tilde{X}^n(R) + E\, \tilde{Y}^m(R).$$

Now we return to the system (26). Using the dominated convergence theorem, we obtain $\lim_{R\to+\infty} \tilde{Y}(R) = 0$. Hence,

$$\lim_{R\to+\infty}(Y^p(R) + b(R)) = \int_Q |v|^p \, dx\, dt + \int_{\mathbb{R}^N} u_0(x)\, dx + \int_Q w_1(x,t)\, dx\, dt = 0,$$

which implies that $v \equiv 0$. This completes the proof. □

The following lemma gives a generalization of lemma 1.

Lemma 2. *We use the same notations as before. If $X(R)$ or $Y(R)$ is bounded then $u \equiv v \equiv 0$.*

Démonstration. If $X(R)$ or $Y(R)$ is bounded, it follows via (25) that $X(R)$ and $Y(R)$ are bounded. Then $(|v|^q, |v|^p)$ is in $L^1(Q) \times L^1(Q)$. Finally, using (26) and the dominated convergence theorem, we obtain the result. □

Theorem 4. *Assume that $p > \max\{m, l\}$ and $q > n$, with $1 < \min\{m, n, l\}$. If one of the following conditions is satisfied :*

a) $\dfrac{p+1}{q+1} < \min\left(\dfrac{l-1}{n-1},\ \dfrac{p}{n}, \dfrac{m}{n}\right), \quad \dfrac{N}{2} \le \dfrac{1+p}{p-n+pq-pn}$

b) $\dfrac{m}{n} < \dfrac{p+1}{q+1} < \min\left(\dfrac{l-1}{n-1}, \dfrac{p}{n}\right), \quad \dfrac{N}{2} \le \dfrac{p}{p-m} - \dfrac{pq-1}{p-n+pq-pn}$

c) $\left\{\dfrac{l-1}{n-1} < \dfrac{p+1}{q+1} < \dfrac{p}{n}\right\}, \ \left\{\dfrac{p+1}{q+1} \ne \dfrac{m}{n},\ \ \dfrac{p+l}{q+n} > \dfrac{m}{n}\right\}, \ \dfrac{N}{2} \le \dfrac{p}{p-m} - \dfrac{pq-1}{p-n+pq-pn}$

d) $\left\{\dfrac{l-1}{n-1} < \dfrac{p+1}{q+1} < \dfrac{p}{n}\right\}, \ \left\{\dfrac{p+1}{q+1} \ne \dfrac{m}{n},\ \ \dfrac{p+l}{q+n} < \dfrac{m}{n}\right\}, \ \dfrac{N}{2} \le \dfrac{p(n+q)}{pq-nl} - \dfrac{pq-1}{p-n+pq-pn}$

then, the problem (P) has no nontrivial global weak solution.

Démonstration. Let (u, v) be a nontrivial weak solution of (P) and $\varphi_R \in \mathcal{D}(\mathbb{R}^N \times [0, +\infty[)$ be a nonnegative function such that

$$\varphi_R(x,t) = \Phi^\lambda\left(\frac{t + |x|^\delta}{R^\delta}\right), \quad \lambda > 1,$$

where $\delta > 0$, $R > 0$ and $\Phi \in \mathcal{D}([0, +\infty[)$ is the cut-off function defined before. Following the same method described in the previous proof, we deduce the two systems (25) and (26). We precise that the different terms appearing in those two systems depend on δ.
Using the fact that $\lim_{R\to+\infty} a(R) \ge 0$ and $\lim_{R\to+\infty} b(R) \ge 0$ and applying Young's inequality in the system (25) one has, for some ε, $0 < \varepsilon < 1$,

$$\begin{aligned}(1-\varepsilon)\, Y^{pq} \ &\le\ c_{1,\varepsilon}\,(A\,C^q)^{pq/(pq-1)} + c_{2,\varepsilon}\,(B\,C^q)^{pq/(pq-l)} + c_{3,\varepsilon}\,(E^q)^{p/(p-m)} \\ &+\ c_{4,\varepsilon}\,(A^n\,D^q)^{pq/(pq-n)} + c_{5,\varepsilon}\,(B^n\,D^q)^{pq/(pq-nl)},\end{aligned}$$

$$\begin{aligned}(1-\varepsilon)\, X^{pq} \ &\le\ c'_{1,\varepsilon}\,(A^p\,C)^{pq/(pq-1)} + c'_{2,\varepsilon}\,(A^p\,D)^{pq/(pq-n)} + c'_{3,\varepsilon}\left(B^p\,C^l\right)^{pq/(pq-l)} \\ &+\ c'_{4,\varepsilon}\left(B^p\,D^l\right)^{pq/(pq-nl)} + c'_{5,\varepsilon} A^p\,E^{p/(p-m)} + c'_{6,\varepsilon} B^p\,E^{lp/(p-m)}.\end{aligned}$$

At this stage, we introduce the scaled variables $\tau = R^{-\delta}\, t$ and $y = R^{-1}\, x$. It is easy to check that for R large enough

$$A \leq c_1\, R^{\alpha_1}, \;\; B \leq c_2\, R^{\alpha_2}, \;\; C \leq c_3\, R^{\alpha_3}, \;\; D \leq c_4\, R^{\alpha_4} \;\; \text{and} \;\; E \leq c_5\, R^{\alpha_5},$$

where

$$\begin{aligned}
\alpha_1 &= (N + \delta - \delta p/(p-1))(p-1)/p, \\
\alpha_2 &= (N + \delta - 2p/(p-l))(p-l)/p, \\
\alpha_3 &= (N + \delta - \delta q/(q-1))(q-1)/q, \\
\alpha_4 &= (N + \delta - 2q/(q-n))(q-n)/q, \\
\alpha_5 &= (N + \delta - 2p/(p-m))(p-m)/p.
\end{aligned}$$

Finally, we have

$$\left\{ \begin{array}{l} (1-\varepsilon) Y(R)^{pq} \leq c_\varepsilon \left\{ R^{r_1(\delta)} + R^{r_2(\delta)} + R^{r_3(\delta)} + R^{r_4(\delta)} + R^{r_5(\delta)} \right\}, \\ \\ (1-\varepsilon) X^{pq} \leq c'_\varepsilon \left\{ R^{s_1(\delta)} + R^{s_2(\delta)} + R^{s_3(\delta)} + R^{s_4(\delta)} + R^{s_5(\delta)} + R^{s_6(\delta)} \right\}, \end{array} \right. \tag{27}$$

where

$$\begin{aligned}
r_1(\delta) &= Nq + \frac{q\,(-\delta - \delta\, p)}{pq - 1}, \\
r_2(\delta) &= Nq + \frac{q\,(-l\delta - 2\,p)}{pq - l}, \\
r_3(\delta) &= Nq + \frac{q\,(-\delta\, n + \delta\, pq - \delta\, pn - 2\,pq)}{pq - n}, \\
r_4(\delta) &= Nq + \frac{q\,(\delta\, pq - \delta\, nl - 2\,pn - 2\,pq)}{pq - nl}, \\
r_5(\delta) &= Nq + \frac{q\,(\delta\, p - \delta\, m - 2\,p)}{p - m}, \\
s_1(\delta) &= Np + \frac{p\,(-\delta\, q - \delta)}{pq - 1}, \\
s_2(\delta) &= Np + \frac{p\,(-\delta\, n - 2\,q)}{pq - n}, \\
s_3(\delta) &= Np + \frac{p\,(\delta\, pq - lq\delta - l\delta - 2\,pq)}{pq - l}, \\
s_4(\delta) &= Np + \frac{p\,(\delta\, pq - \delta\, nl - 2\,pq - 2\,lq)}{pq - nl}, \\
s_5(\delta) &= Np - 2\,\frac{p}{p - m}, \\
s_6(\delta) &= Np + \frac{p\,(\delta\, p - \delta\, m - 2\,l - 2\,p + 2\,m)}{p - m}.
\end{aligned}$$

The parameter δ is fixed now such that $r_1(\delta) = r_3(\delta)$, *i.e.*

$$\delta = \delta_1 = 2\frac{pq - 1}{p(q+1) - n(p+1)},$$

with $p(q+1)-n(p+1)>0$. Therefore,

$$\begin{aligned} r_1(\delta_1)-r_2(\delta_1) &= 2\,\frac{(n+pn+q-lq-l-p)\,pq}{(pq-l)\,(-p+n-pq+pn)}, \\ r_1(\delta_1)-r_4(\delta_1) &= -2\,\frac{(n+pn+q-lq-l-p)\,npq}{(-p+n-pq+pn)\,(-pq+nl)}, \\ r_1(\delta_1)-r_5(\delta_1) &= 2\,\frac{(n+pn-m-qm)\,pq}{(-p+n-pq+pn)\,(p-m)}. \end{aligned}$$

Note that if $(p+1)/(q+1) < \min((l-1)/(n-1), p/n, m/n)$ (case **(a)** in Theorem 1) then $r_1(\delta_1) = \max\limits_{1\le i\le 5} r_i(\delta_1)$. In this case, if $N/2 \le (1+p)(p-n+pq-pn)$ then $r_1(\delta_1)\le 0$ and there is a constant C such that

$$Y(R)^{pq} = \left(\int_Q |v|^p \varphi_{_R}\,dx\,dt\right)^q = \left(\int_{Q_R} |v|^p \varphi_{_R}\,dx\,dt\right)^q \le C$$

where $Q_R = \{(x,t)\in Q;\ 0\le t+|x|^{\delta_1}\le 2\,R^{\delta_1}\}$. According to Lemma 1, we deduce that $u\equiv v\equiv 0$. This contradicts our assumption.

Assertions **b)**, **c)** and **d)** can be showed in the same manner. □

Theorem 5. *Assume that* $p>\max\{m,l\}$ *and* $q>n$*, with* $1<\min\{m,n,l\}$*. If one of the following conditions is satisfied :*

a) $\left\{(m+1)-l > \dfrac{p+1}{q+1} > \max\left(\dfrac{l-1}{n-1}, \dfrac{l}{q}\right)\right\},\quad \dfrac{N}{2} \le \dfrac{q+1}{pq+q-ql-l}$

b) $\left\{\dfrac{p+1}{q+1} > \max\left(\dfrac{l-1}{n-1}, \dfrac{l}{q}, (m+1)-l\right)\right\},\quad \dfrac{N}{2} \le \dfrac{pq-1}{-pq-q+ql+l} + \dfrac{l+p-m}{p-m}$

c) $\left\{\dfrac{l}{q} < \dfrac{p+1}{q+1} < \min\left(\dfrac{l-1}{n-1}, (m+1)-l\right)\right\},\quad \dfrac{N}{2} \le \dfrac{pq-1}{-pq-q+ql+l} + \dfrac{q(p+l)}{pq-nl}$

d) $\left\{\dfrac{l-1}{n-1} > \dfrac{p+1}{q+1} > \max\left(\dfrac{l}{q}, (m+1)-l\right)\right\},\quad \dfrac{N}{2} \le \min\{N_1,\ N_2\}$

where

$$N_1 = \frac{pq-1}{-pq-q+ql+l} + \frac{q(p+l)}{pq-nl} \quad \textit{and} \quad N_2 = \frac{pq-1}{-pq-q+ql+l} + \frac{l+p-m}{p-m},$$

then, the problem (P) has no nontrivial global weak solution.

Démonstration. We follow the proof of Theorem 4 and choose the parameter δ such that $s_1(\delta)=s_3(\delta)$, *i.e.* $\delta=\delta_2 = 2(pq-1)/(q(p+1)-l(q+1))$ with $q(p+1)-l(q+1)>0$. If $(m+1)-l > (p+1)(q+1) > \max\{(l-1)(n-1), l/q\}$ (case **(a)** in Theorem 2) then $s_1(\delta_2) = \max\limits_{1\le i\le 6} s_i(\delta_2)$. In this case, if $N/2 \le (q+1)(pq+q-ql-l)$ then $s_1(\delta_2)\le 0$ and there is a constant C such that

$$X^{pq}(R) = \left(\int_Q |u|^q \varphi_{_R}\,dx\,dt\right)^p = \left(\int_{Q_R} |u|^q \varphi_{_R}\,dx\,dt\right)^p \le C.$$

The result of Lemma 1 completes the proof.

Assertions **b), c)** and **d)** can be proved similarly. □

Comments

1) Note that in Section 3, the positivity of the solutions of (P) is not guaranteed even if the data are positive. It is then natural that the initial data $u_0(x)$ and $v_0(x)$ may change signs as well as the nonhomogeneous terms $w_1(x,t)$ and $w_2(x,t)$. Now, our hypotheses are weaker than those in the literature, *i.e.* the data may change signs but must have nonnegative integral. This difficulty was first solved in the scalar case in [22].

2) We are now able to treat the case where $f \geq 0$, $g \geq 0$, $f \sim t^{\gamma_1}|x|^{\theta_1}$ and $g \sim t^{\gamma_2}|x|^{\theta_2}$ for t and $|x|$ large enough. A slight change in the proof (Theorem 4 and Theorem 5) shows that we have the systems (25)-(26) with

$$\begin{aligned}
X(R) &= \left(\int_Q |u|^q \varphi_R g\, dx\, dt\right)^{1/q}, \quad Y(R) = \left(\int_Q |v|^p \varphi_R f\, dx\, dt\right)^{1/p} \\
A &= \left(\int_Q |\varphi_{Rt}|^{p/(p-1)} (\varphi_R f)^{-1/(p-1)}\, dx\, dt\right)^{(p-1)/p} \\
B &= \left(\int_Q |\Delta\varphi_R|^{p/(p-l)} (\varphi_R f)^{-l/(p-l)}\, dx\, dt\right)^{(p-l)/p} \\
C &= \left(\int_Q |\varphi_{Rt}|^{q/(q-1)} (\varphi_R g)^{-1/(q-1)}\, dx\, dt\right)^{(q-1)/q} \\
D &= \left(\int_Q |\Delta\varphi_R|^{q/(q-n)} (\varphi_R g)^{-n/(q-n)}\, dx\, dt\right)^{(q-n)/q} \\
E &= \left(\int_Q |\Delta\varphi_R|^{p/(p-m)} (\varphi_R f)^{-m/(p-m)}\, dx\, dt\right)^{(p-m)/p}.
\end{aligned}$$

This naturally changes the α_i, $i = 1, ..., 5$ into $\alpha_1 = (N + \delta - \delta p/(p-1) + \beta_1)(p-1)/p$, $\alpha_2 = (N + \delta - 2p/(p-l) + \beta_2)(p-l)/p$, $\alpha_3 = (N + \delta - \delta q/(q-1) + \beta_3)(q-1)/q$, $\alpha_4 = (N + \delta - 2q/(q-n) + \beta_4)(q-n)/q$, $\alpha_5 = (N + \delta - 2p/(p-m) + \beta_5)(p-m)/p$, where
$\beta_1 = -(\theta_1 + \gamma_1\delta)/(p-1)$, $\beta_2 = -l(\theta_1 + \gamma_1\delta)/(p-l)$, $\beta_3 = -(\theta_2 + \gamma_2\delta)/(q-1)$, $\beta_4 = -n(\theta_2 + \gamma_2\delta)/(q-n)$, $\beta_5 = -m(\theta_1 + \gamma_1\delta)/(p-m)$.

3) This work can be easily generalized to higher order systems with triangular diffusion matrices under the same type of hypotheses.

4) The method described above can also be used for the more general system :

$$\begin{cases}
u_t(x,t) = |x|^\alpha \left\{(-\Delta)^{\alpha_1/2}(\varphi(u)) + (-\Delta)^{\alpha_2/2}(\psi(v))\right\} + f(x,t)k(u) + w_1(x,t) \\
v_t(x,t) = |x|^\beta (-\Delta)^{\alpha_3/2}(\chi(v)) + g(x,t)l(v) + w_2(x,t)
\end{cases}$$

where $-(-\Delta)^{\alpha_i/2}$ is the fractional power of the laplacian. A suitable choice of the functions φ, ψ and χ are required.

5) If the parabolic problem (P) is replaced by the hyperbolic one, *i.e.*, (u_t, v_t) is replaced by (u_{tt}, v_{tt}), our study remains valid. The nonnegativity assumptions on (u_0, v_0) are set on (u_{0_t}, v_{0_t}) and the test function changes slightly ; for example

$$\varphi_{_R}(x,t) = \Phi^\lambda\left(\frac{t^2 + |x|^2}{R}\right), \quad \lambda >> 1.$$

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