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# Systems of Semilinear higher order evolution inequalities on the Heisenberg group

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# Systems of Semilinear Higher Order Evolution Inequalities on the Heisenberg group

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**Abstract**

This paper is devoted to nonexistence results for solutions to the problem

$$(\mathrm{S}_k^m)\begin{cases} \frac{\partial^k u_i}{\partial t^k} - \Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad \eta \in \mathbb{H}^N, \ \ t \in ]0,+\infty[, \quad 1 \leq i \leq m, \\ \\ u_{m+1} = u_1, \end{cases}$$

where $\Delta_{\mathbb{H}}$ is the laplacian on the $(2N+1)-$dimensional Heisenberg group $\mathbb{H}^N$, $|\eta|_{\mathbb{H}}$ is the distance from $\eta$ in $\mathbb{H}$ to the origin, $m \geq 2$, $k \geq 1$, $p_{m+1} = p_1$, $\gamma_{m+1} = \gamma_1$, and $a_i \in L^\infty(\mathbb{H}^N \times ]0,+\infty[)$, $1 \leq i \leq m$. These nonexistence results hold for $Q \equiv 2N+2$ less than critical exponents which depend on $k$, $p_i$ and $\gamma_i$, $1 \leq i \leq m$. For $k=1$, $k=2$ we retreive the results, obtained by A. El Hamidi & M. Kirane [4], corresponding respectively to the parabolic, hyperbolic systems. In order to show that the obtained exponents are also valid for $m=1$, we study the scalar case

$$(\mathrm{I}_k) \quad \frac{\partial^k u}{\partial t^k} - \Delta_{\mathbb{H}}(au) \geq |\eta|_{\mathbb{H}}^{\gamma} \, |u|^p,$$

where $p > 1$, $\gamma$ are real parameters and $a \in L^\infty(\mathbb{H}^N \times ]0,+\infty[)$.

*Key words:* Critical exponent, higher order evolution inequalities, Heisenberg group

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## 1 Introduction

In this section, we quote some background facts concerning the Heisenberg group. Let $\eta = (x, y, \tau) = (x_1, x_2, ..., x_N, y_1, y_2, ..., y_N, \tau) \in \mathbb{R}^{2N+1}$, with $N \geq 1$. The Heisenberg group $\mathbb{H}^N$, whose points will be denoted by $\eta = (x, y, \tau)$ is the set $\mathbb{R}^{2N+1}$ endowed with the group operation $\circ$ defined by

$$\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(< x \, , \, \tilde{y} > - < \tilde{x} \, , \, y >)), \tag{1}$$

where $<,>$ is the usual inner product in $\mathbb{R}^N$. The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}^N$ is obtained, from the vector fields $X_i = \partial_{x_i} + 2y_i \, \partial_\tau$ and $Y_i = \partial_{y_i} - 2x_i \, \partial_\tau$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} (X_i^2 + Y_i^2). \tag{2}$$

Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right). \tag{3}$$

A natural group of dilatations on $\mathbb{H}^N$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \quad \lambda > 0,$$

whose Jacobian determinant is $\lambda^Q$, where

$$Q = 2N + 2$$

is the homogeneous dimension of $\mathbb{H}^N$.

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}^N$ and homogeneous w.r.t. the dilatations $\delta_\lambda$. More precisely, we have

$$\forall (\eta, \tilde{\eta}) \in \mathbb{H}^N \times \mathbb{H}^N, \quad \Delta_{\mathbb{H}}(u(\eta \circ \tilde{\eta})) = (\Delta_{\mathbb{H}} u)(\eta \circ \tilde{\eta})$$

and

$$\Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_\lambda.$$

It is natural to define a distance from $\eta$ to the origin by

$$|\eta|_{\mathbb{H}} = \left( \tau^2 + \sum_{i=1}^{N} (x_i^2 + y_i^2)^2 \right)^{1/4}. \tag{4}$$

In their paper, Pohozaev & Vron [16] gave another proof of a result of Birindelli, Capuzzo-Dolcetta and Cutri [2] concerning the nonexistence of weak solutions of the differential inequality

$$\Delta_{\mathbb{H}}(au) + |\eta|_{\mathbb{H}}^{\gamma}\, |u|^p \leq 0 \quad \text{in}, \quad \mathbb{H}^N$$

for $\gamma > -2$, $1 < p \leq (Q+\gamma)/(Q-2)$ and $a \in L^\infty(\mathbb{H}^N)$.
They then studied the problem of nonexistence of weak solutions to the system

$$\begin{cases} \Delta_{\mathbb{H}}(a_1 u) + |\eta|_{\mathbb{H}}^{\gamma_1}\, |v|^{p_1} \leq 0, \\ \\ \Delta_{\mathbb{H}}(a_2 v) + |\eta|_{\mathbb{H}}^{\gamma_2}\, |u|^{p_2} \leq 0, \end{cases}$$

for $\gamma_i > -2$, $p_i > 1$ and $a_i \in L^\infty(\mathbb{H}^N)$, $i \in \{1, 2\}$. They showed that this system admits no solution defined in $\mathbb{H}^N$ whenever $\gamma_i > -2$ and

$$Q \leq 2 + \min\left\{\frac{\gamma_1 + 2}{p_1 - 1}\,;\, \frac{\gamma_2 + 2}{p_2 - 1}\right\}.$$

Recently, El Hamidi & Kirane [4] improved this result and gave the Fujita's exponent. Indeed, the authors showed that the system admits no solution defined in $\mathbb{H}^N$ whenever

$$Q \leq 2 + \max\left\{\frac{(\gamma_1 + 2) + p_1(\gamma_2 + 2)}{p_1 p_2 - 1}\,;\, \frac{p_2(\gamma_1 + 2) + (\gamma_2 + 2)}{p_1 p_2 - 1}\right\}$$

and verified that

$$\max\left\{\frac{(\gamma_1 + 2) + p_1(\gamma_2 + 2)}{p_1 p_2 - 1}\,;\, \frac{p_2(\gamma_1 + 2) + (\gamma_2 + 2)}{p_1 p_2 - 1}\right\} \geq \min\left\{\frac{\gamma_1 + 2}{p_1 - 1}\,;\, \frac{\gamma_2 + 2}{p_2 - 1}\right\}.$$

They then studied systems of $m$ hypoelliptic, parabolic and hyperbolic semilinear inequalities.

In this paper, we generalize the results obtained in [4] to higher order evolution systems of $m$ semilinear inequalities. We retreive the critical exponent corresponding to the hypoelliptic case by setting formally $k = +\infty$.

For the convenience of the reader, we start with the case $m = 2$.

## 2 Higher Order Evolution Systems of two Semilinear Inequalities

Let us consider the higher order evolution system of two inequalities

$$(\mathrm{S}_k^2)\begin{cases}\frac{\partial^k u}{\partial t^k}-\Delta_{\mathbb{H}}(a_1u)\geq|\eta|_{\mathbb{H}}^{\gamma_1}\;|v|^{p_1},\\[2ex] \frac{\partial^k v}{\partial t^k}-\Delta_{\mathbb{H}}(a_2v)\geq|\eta|_{\mathbb{H}}^{\gamma_2}\;|u|^{p_2},\end{cases}$$

with the initial data

$$\begin{cases}u(\eta,0)\;=u^{(0)}(\eta),\quad v(\eta,0)\;=v^{(0)}(\eta)\quad\text{in}\quad\mathbb{R}^{2N+1},\\[2ex] \frac{\partial^i u}{\partial t^i}(\eta,0)=u^{(i)}(\eta),\quad\frac{\partial^i v}{\partial t^i}(\eta,0)=v^{(i)}(\eta),\;\; i\in\{1,2,...,k-1\},\qquad\text{in}\quad\mathbb{R}^{2N+1}.\end{cases}$$

The product set $\mathbb{R}^{2N+1}\times\mathbb{R}^{+}$ will be denoted by $\mathbb{R}_{+}^{2N+1,1}$ and the integrals $\int_{\mathbb{R}^{2N+1}}$ and $\int_{\mathbb{R}_{+}^{2N+1,1}}$ by $\int$.

**Definition 1.** *Let $a_1$ and $a_2$ be two bounded measurable functions in $\mathbb{R}_{+}^{2N+1,1}$. A weak solution $(u,v)$ of the system $(S_k^2)$ with initial data $(u^{(i)},v^{(i)})\in L^1_{loc}\left(\mathbb{R}^{2N+1}\right)\times L^1_{loc}\left(\mathbb{R}^{2N+1}\right)$, $i\in\{0,1,...,k-1\}$, is a pair of locally integrable functions $(u,v)$ such that*

$$\begin{cases}u\in L^{p_2}_{loc}\left(\mathbb{R}_{+}^{2N+1,1},|\eta|_{\mathbb{H}}^{\gamma_2}\,d\eta dt\right),\\[2ex] v\in L^{p_1}_{loc}\left(\mathbb{R}_{+}^{2N+1,1},|\eta|_{\mathbb{H}}^{\gamma_1}\,d\eta dt\right),\end{cases}$$

*satisfying*

$$\int_0^\infty\int_{\mathbb{R}^{2N+1}}\left(u\left(a_1\Delta_{\mathbb{H}}\varphi-(-1)^k\frac{\partial^k\varphi}{\partial t^k}\right)+|\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi\right)d\eta\,dt+\sum_{i=0}^{k-1}(-1)^i\int_{\mathbb{R}^{2N+1}}\frac{\partial^{k-1-i}u}{\partial t^{k-1-i}}(x,0)\frac{\partial^i\varphi}{\partial t^i}(x,0)\,d\eta\leq 0\qquad(5)$$

*and*

$$\int_0^\infty\int_{\mathbb{R}^{2N+1}}\left(v\left(a_2\Delta_{\mathbb{H}}\varphi-(-1)^k\frac{\partial^k\varphi}{\partial t^k}\right)+|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi\right)d\eta\,dt+\sum_{i=0}^{k-1}(-1)^i\int_{\mathbb{R}^{2N+1}}\frac{\partial^{k-1-i}v}{\partial t^{k-1-i}}(x,0)\frac{\partial^i\varphi}{\partial t^i}(x,0)\,d\eta\leq 0\qquad(6)$$

*for any nonnegative test function $\varphi\in C_c^{2,k}(\mathbb{R}_{+}^{2N+1,1})$.*

Let the test function

$$\varphi_R(\eta,t) = \Phi^{\lambda}\left(\frac{t^{2k}+\tau^2+|x|^4+|y|^4}{R^4}\right), \tag{7}$$

where $\lambda >> 1$, $R > 0$ and $\Phi \in \mathcal{D}([0,+\infty[)$ is the "standard cut-off function"

$$0 \leq \Phi(r) \leq 1, \quad \Phi(r) = \begin{cases} 1 & \text{if } \ 0 \leq r \leq 1, \\ \\ 0 & \text{if } \ r \geq 2. \end{cases} \tag{8}$$

Note that $\mathrm{supp}(\varphi_R)$ is a subset of

$$\Omega_R = \{(x,y,\tau,t) \in \mathbb{H}^N \times [0,+\infty[;\;\; 0 \leq t^{2k}+\tau^2+|x|^4+|y|^4 \leq 2R^4\},$$

while $\mathrm{supp}(\Delta_{\mathbb{H}}\varphi_R)$ and $\mathrm{supp}\left(\frac{\partial^k \varphi_R}{\partial t^k}\right)$ are subsets of

$$\mathcal{C}_R = \{(x,y,\tau,t) \in \mathbb{H}^N \times [0,+\infty[;\;\; R^4 \leq t^{2k}+\tau^2+|x|^4+|y|^4 \leq 2R^4\} \tag{9}$$

and

$$\frac{\partial^i \varphi_R}{\partial t^i}(\eta,0) = 0, \quad i \in \{1,2,...,k-1\}.$$

Moreover, let

$$\rho = \frac{t^{2k}+\tau^2+|x|^4+|y|^4}{R^4}, \tag{10}$$

then

$$\begin{aligned}
&\Delta_{\mathbb{H}}\varphi_R(\eta,t) = \frac{4(N+4)\Phi'(\rho)}{R^4}\lambda\Phi^{\lambda-1}(\rho)(|x|^2+|y|^2) + \\
&\frac{16\Phi''(\rho)}{R^8}\lambda\Phi^{\lambda-1}(\rho)\left((|x|^6+|y|^6)+\tau^2(|x|^2+|y|^2)+2\tau < x\,,\, y > (|x|^2-|y|^2)\right) + \\
&\frac{16\Phi'^2(\rho)}{R^8}\lambda(\lambda-1)\Phi^{\lambda-2}(\rho)\left((|x|^6+|y|^6)+\frac{\tau^2}{4}(|x|^2+|y|^2)+2\tau < x\,,\, y > (|x|^2-|y|^2)\right).
\end{aligned}$$

It follows that there is a positive constant $C_1 > 0$, independent of $R$, such that

$$\forall (\eta,t) \in \Omega_R, \quad |\Delta_{\mathbb{H}}\varphi_R(\eta,t)| \leq \frac{C_1}{R^2}. \tag{11}$$

Furthermore, there is a constant $C_2 > 0$, independent of $R$, such that

$$\left|\frac{\partial^k \varphi_R(\eta,t)}{\partial t^k}\right| \leq \frac{C_2}{R^2}.$$

Then we have the following nonexistence result:
**Theorem 1.** *Assume that* $u^{(k-1)}$ *and* $v^{(k-1)}$ *belong to* $L^1(\mathbb{R}^{2N+1})$ *with* $\int u^{(k-1)}(\eta)\,d\eta \geq 0$ *and* $\int v^{(k-1)}(\eta)\,d\eta \geq 0$. *If*

$$Q \leq Q_k^* = 2\left(1-\frac{1}{k}\right) + \frac{1}{p_1p_2-1}\max\{(\gamma_1+2)+p_1(\gamma_2+2)\,;p_2(\gamma_1+2)+(\gamma_2+2)\}$$

*then there is no weak nontrivial solution* $(u,v)$ *of the system* $(S_k^2)$.

*Proof.* The proof is by contradiction. Let $(u,v)$ be a nontrivial weak solution of $(\mathrm{S}_k^2)$. Using the Hlder inequality, the relation (6) gives:

$$\begin{aligned}\int \varphi_{_R}|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\;d\eta\;dt + a(R) &\leq \int \left(|v|\left|\frac{\partial^k\varphi_{_R}}{\partial t^k}\right| + ||a_2||_\infty|v||\Delta_{\mathbb{H}}\varphi_{_R}|\right)\,d\eta\,dt\\ &\leq \left(\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_{_R}\right)^{1/p_1}\left(\int\left|\frac{\partial^k\varphi_{_R}}{\partial t^k}\right|^{p_1'}(\varphi_{_R}|\eta|_{\mathbb{H}}^{\gamma_1})^{1-p_1'}\right)^{1/p_1'}\\ &+ ||a_2||_\infty\left(\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_{_R}\right)^{1/p_1}\left(\int |\Delta_{\mathbb{H}}\varphi_{_R}|^{p_1'}(\varphi_{_R}|\eta|_{\mathbb{H}}^{\gamma_1})^{1-p_1'}\right)^{1/p_1'}.\end{aligned}$$

Similarly (5) gives

$$\begin{aligned}\int \varphi_{_R}|\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\;d\eta\;dt + b(R) &\leq \int \left(|u|\left|\frac{\partial^k\varphi_{_R}}{\partial t^k}\right| + ||a_1||_\infty|u||\Delta_{\mathbb{H}}\varphi_{_R}|\right)\,d\eta\,dt\\ &\leq \left(\int |\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_{_R}\right)^{1/p_2}\left(\int\left|\frac{\partial^k\varphi_{_R}}{\partial t^k}\right|^{p_2'}(\varphi_{_R}|\eta|_{\mathbb{H}}^{\gamma_2})^{1-p_2'}\right)^{1/p_2'}\\ &+ ||a_1||_\infty\left(\int |\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_{_R}\right)^{1/p_2}\left(\int |\Delta_{\mathbb{H}}\varphi_{_R}|^{p_2'}(\varphi_{_R}|\eta|_{\mathbb{H}}^{\gamma_2})^{1-p_2'}\right)^{1/p_2'},\end{aligned}$$

where

$$a(R) = \int_{\mathbb{H}^N} v^{(k-1)}(\eta)\,\varphi_{_R}(\eta,0)\,d\eta$$

and

$$b(R) = \int_{\mathbb{H}^N} u^{(k-1)}(\eta)\,\varphi_{_R}(\eta,0)\,d\eta.$$

If we set

$$\begin{cases} I(R) &= \displaystyle\int |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi_R \, d\eta \, dt, \\ J(R) &= \displaystyle\int |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi_R \, d\eta \, dt, \\ \mathcal{A}_{p_i,\gamma_i}(R) &= \displaystyle\int |\Delta_{\mathbb{H}} \varphi_R|^{p_i'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_i}\right)^{1-p_i'} \, d\eta \, dt, \quad i \in \{1,2\}, \\ \mathcal{B}_{p_i,\gamma_i}(R) &= \displaystyle\int \left|\frac{\partial^k \varphi_R}{\partial t^k}\right|^{p_i'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_1}\right)^{1-p_i'} \, d\eta \, dt, \quad i \in \{1,2\}, \end{cases}$$

we then have the following system of inequalities

$$\begin{cases} I(R) + a(R) \leq C\, J^{1/p_1}(R) \left[ (\mathcal{A}_{p_1,\gamma_1}(R))^{1/p_1'} + (\mathcal{B}_{p_1,\gamma_1}(R))^{1/p_1'} \right], \\ J(R) + b(R) \leq C\, I^{1/p_2}(R) \left[ (\mathcal{A}_{p_2,\gamma_2}(R))^{1/p_2'} + (\mathcal{B}_{p_2,\gamma_2}(R))^{1/p_2'} \right], \end{cases} \tag{12}$$

where $C$ is a positive constant independent of $R$.

Note that if $\lambda$ is selected sufficiently large then the integrals $\mathcal{A}_{p_i,\gamma_i}(R)$ and $\mathcal{B}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, are convergent. Indeed, the exponent of $\varphi_R$ in the integrands of $\mathcal{A}_{p_i,\gamma_i}(R)$ and $\mathcal{B}_{p_i,\gamma_i}(R)$ is positive if $\lambda$ is selected large enough.

Moreover, the system (12) implies that neither $u$ nor $v$ is trivial. Indeed, if $v$ is trivial then $J(R) = 0$ and we have $I(R) + a(R) \leq 0$. Since $a(R)$ is uniformly bounded w.r.t. $R$, it follows that $I(R)$ is also uniformly bounded w.r.t. $R$. Using the fact that $I(R)$ is increasing in $R$, the monotone convergence theorem shows that the function $u \in L^{p_2}\left(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma_2}\, d\eta dt\right)$. Whence, we have

$$\lim_{R\to+\infty} (I(R) + a(R)) = \int |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \, d\eta \, dt + \int v^{(k-1)}(\eta) \, d\eta \ \leq 0,$$

and the function $u$ is then trivial, which is impossible.

Now, let $\varepsilon$ be a real number such that $0 < \varepsilon < 1$, there is $R_1 > 0$ such that $I(R_1) > 0$. Since

$$0 \leq \lim_{R\to+\infty} a(R) < +\infty,$$

there exists $R_2 \geq R_1$ such that $-\varepsilon\, I(R_1) \leq a(R)$, for any $R \geq R_2$. Moreover, the function $I(R)$ is nonnegative and increasing of $R$, then for any $R \geq R_2$, the inequalities

$$I(R) + a(R) \geq I(R) - \varepsilon\, I(R_1) \geq (1-\varepsilon) I(R)$$

hold true. The same arguments imply that there is $R_3 \geq R_2$ such that $J(R)+b(R) \geq (1-\varepsilon)J(R)$ for any $R \geq R_3$. Finally, the sytem (12) gives

$$\begin{cases} I(R) \leq \frac{C}{1-\varepsilon} J^{\frac{1}{p_1}}(R) \left[ (\mathcal{A}_{p_1,\gamma_1}(R))^{\frac{1}{p_1'}} + (\mathcal{B}_{p_1,\gamma_1}(R))^{\frac{1}{p_1'}} \right], \\ \\ J(R) \leq \frac{C}{1-\varepsilon} I^{\frac{1}{p_2}}(R) \left[ (\mathcal{A}_{p_2,\gamma_2}(R))^{\frac{1}{p_2'}} + (\mathcal{B}_{p_2,\gamma_2}(R))^{\frac{1}{p_2'}} \right], \end{cases} \tag{13}$$

for any $R \geq R_3$. Then, there is a constant $C > 0$, independent of $R$, such that

$$\begin{cases} I(R)^{1-\frac{1}{p_1p_2}} \leq C \left[ \mathcal{A}_{p_1,\gamma_1}^{\frac{1}{p_1'}} + \mathcal{B}_{p_1,\gamma_1}^{\frac{1}{p_1'}} \right] \left[ \mathcal{A}_{p_2,\gamma_2}^{\frac{1}{p_2'}} + \mathcal{B}_{p_2,\gamma_2}^{\frac{1}{p_2'}} \right]^{\frac{1}{p_1}}, \\ \\ J(R)^{1-\frac{1}{p_1p_2}} \leq C \left[ \mathcal{A}_{p_1,\gamma_1}^{\frac{1}{p_1'}} + \mathcal{B}_{p_1,\gamma_1}^{\frac{1}{p_1'}} \right]^{\frac{1}{p_2}} \left[ \mathcal{A}_{p_2,\gamma_2}^{\frac{1}{p_2'}} + \mathcal{B}_{p_2,\gamma_2}^{\frac{1}{p_2'}} \right]. \end{cases} \tag{14}$$

In order to estimate the integrals $\mathcal{A}_{p_i,\gamma_i}(R)$ and $\mathcal{B}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, we introduce the scaled variables

$$\begin{cases} \tilde{t} = R^{-\frac{2}{k}} t \\ \tilde{\tau} = R^{-2} \tau, \\ \tilde{x} = R^{-1} x, \\ \tilde{y} = R^{-1} y. \end{cases} \tag{15}$$

Using the fact that $\mathrm{supp}\varphi_R \subset \Omega_R$, we conclude that

$$\begin{cases} \mathcal{A}_{p_i,\gamma_i}(R) \leq C\, R^{2N+2+2/k-2p_i'+\gamma_i(1-p_i')}, \quad i \in \{1,2\}, \\ \\ \mathcal{B}_{p_i,\gamma_i}(R) \leq C\, R^{2N+2+2/k-2p_i'+\gamma_i(1-p_i')}, \quad i \in \{1,2\}, \end{cases}$$

which is equivalent to

$$\begin{cases} \mathcal{A}_{p_i,\gamma_i}(R) \leq C\, R^{Q+2/k-2p_i'+\gamma_i(1-p_i')}, \quad i \in \{1,2\}, \\ \\ \mathcal{B}_{p_i,\gamma_i}(R) \leq C\, R^{Q+2/k-2p_i'+\gamma_i(1-p_i')}, \quad i \in \{1,2\}. \end{cases}$$

Consequently, the estimates

$$I(R)^{1-\frac{1}{p_1p_2}} \leq C R^{\sigma_I} \quad \text{and} \quad J(R)^{1-\frac{1}{p_1p_2}} \leq C R^{\sigma_J}$$

hold true, where

$$\sigma_I = \frac{1}{p_1}\left((Q+2/k)\frac{p_2-1}{p_2} - 2 - \frac{\gamma_2}{p_2}\right) + \left((Q+2/k)\frac{p_1-1}{p_1} - 2 - \frac{\gamma_1}{p_1}\right)$$

and

$$\sigma_J = \frac{1}{p_2}\left((Q+2/k)\frac{p_1-1}{p_1} - 2 - \frac{\gamma_1}{p_1}\right) + \left((Q+2/k)\frac{p_2-1}{p_2} - 2 - \frac{\gamma_2}{p_2}\right).$$

Finally, the exponents $\sigma_I$ or $\sigma_J$ are less than zero if, and only if,

$$\begin{aligned} Q \le Q_k^* &= 2 + \frac{1}{p_1p_2-1}\max\{(\gamma_1+2)+p_1(\gamma_2+2)\,;p_2(\gamma_1+2)+(\gamma_2+2)\} - 2/k \\ &= 2(1-1/k) + \frac{1}{p_1p_2-1}\max\{(\gamma_1+2)+p_1(\gamma_2+2)\,;p_2(\gamma_1+2)+(\gamma_2+2)\}. \end{aligned}$$

In this case, the integrals $I(R)$ and $J(R)$, which are increasing in $R$, are bounded uniformly w.r.t. $R$. Using the monotone convergence theorem, we deduce that

$$(u,v) \in L^{p_2}\left(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma_2}\, d\eta dt\right) \times L^{p_1}\left(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma_1}\, d\eta dt\right).$$

Note that instead of (12) we have more precisely

$$\begin{cases} I(R) + a(R) \le C\, \tilde{J}^{1/p_1}(R)\left[(\mathcal{A}_{p_1,\gamma_1}(R))^{1/p_1'} + (\mathcal{B}_{p_1,\gamma_1}(R))^{1/p_1'}\right] \\ \\ J(R) + b(R) \le C\, \tilde{I}^{1/p_2}(R)\left[(\mathcal{A}_{p_2,\gamma_2}(R))^{1/p_2'} + (\mathcal{B}_{p_2,\gamma_2}(R))^{1/p_2'}\right] \end{cases} \qquad (16)$$

where

$$\tilde{I}(R) = \int_{\mathcal{C}_R} |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi_R\, d\eta\, dt$$

and

$$\tilde{J}(R) = \int_{\mathcal{C}_R} |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi_R\, d\eta\, dt,$$

where $\mathcal{C}_R$ is defined in (9). Finally, using the dominated convergence theorem, we obtain

$$\lim_{R\to+\infty} \tilde{I}(R) = \lim_{R\to+\infty} \tilde{J}(R) = 0.$$

Hence,

$$\lim_{R\to+\infty}(I(R)+a(R)) = \int |\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\, d\eta\, dt + \int v^{(k-1)}(\eta)\, d\eta \;= 0$$

and

$$\lim_{R\to+\infty}(J(R)+b(R)) = \int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\, d\eta\, dt + \int u^{(k-1)}(\eta)\, d\eta \;= 0,$$

which implies that $u \equiv v \equiv 0$. This completes the proof. □

**Corollary 1.** *Assume that* $\int u^{(k-1)}(\eta)\, d\eta \geq 0,\ \int v^{(k-1)}(\eta)\, d\eta \geq 0,$ *and*

$$Q \leq Q_k^* = 2(1 - 1/k) + \max\{X_1, X_2\},$$

*where the vector* $(X_1, X_2)^T$ *is the solution of the linear system*

$$\begin{pmatrix} -1 & p_1 \\ p_2 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 + 2 \\ \gamma_2 + 2 \end{pmatrix}. \tag{17}$$

*Then there is no weak nontrivial solution* $(u, v)$ *of the system* $(S_k^2)$.

*Proof.* The vector $(X_1, X_2)^T$ is given by

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -1 & p_1 \\ p_2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 + 2 \\ \gamma_2 + 2 \end{pmatrix} = \frac{1}{p_1 p_2 - 1} \begin{pmatrix} (\gamma_1 + 2) + p_1(\gamma_2 + 2) \\ p_2(\gamma_1 + 2) + (\gamma_2 + 2) \end{pmatrix}.$$

□

**Remark 1.** *To determine the critical exponent* $Q_e^*$ *corresponding to the hypoelliptic system*

$$\begin{cases} -\Delta_{\mathbb{H}}(a_1 u) \geq |\eta|_{\mathbb{H}}^{\gamma_1}\, |v|^{p_1}, \\ \\ -\Delta_{\mathbb{H}}(a_2 v) \geq |\eta|_{\mathbb{H}}^{\gamma_2}\, |u|^{p_2}, \end{cases}$$

*it suffices to tend formally* $k$ *to infinity in the exponent* $Q_k^*$ *and obtain*

$$Q_e^* = 2 + \max\{X_1, X_2\}.$$

Now, we are able to treat the case of systems of $m$ semilinear inequalities.

## 3 Higher-Order Evolution Systems of $m$ Semilinear Inequalities

Let $(X_1, X_2, ..., X_m)$ be the solution of the linear system

$$\begin{pmatrix} -1 & p_1 & 0 & \dots & 0 \\ 0 & -1 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & p_{m-1} \\ p_m & 0 & \dots & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{m-1} \\ X_m \end{pmatrix} = \begin{pmatrix} \gamma_1+2 \\ \gamma_2+2 \\ \vdots \\ \gamma_{m-1}+2 \\ \gamma_m+2 \end{pmatrix}, \tag{18}$$

where $p_i > 1$ and $\gamma_i$ are given real numbers, $i \in \{1, 2, ..., m\}$.

Consider the system

$$(S_k^m) \begin{cases} \frac{\partial^k u_i}{\partial t^k} - \Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad \eta \in \mathbb{R}^{2N+1}, \quad t \in ]0, +\infty[, \quad 1 \leq i \leq m, \\ \\ u_{m+1} = u_1, \end{cases}$$

where $p_{m+1} = p_1$, $\gamma_{m+1} = \gamma_1$, and the initial data $(u_i^{(0)}, u_i^{(1)}, ..., u_i^{(k-1)}) \in \left[L^1_{\text{loc}}\left(\mathbb{R}^{2N+1}\right)\right]^k$, $1 \leq i \leq m$.

**Definition 2.** *Let $a_i$, $i \in \{1, 2, ..., m\}$, be $m$ bounded measurable functions on $\mathbb{R}_+^{2N+1,1}$. A weak solution $(u_1, ..., u_m)$ of the system $(S_k^m)$ on $\mathbb{R}_+^{2N+1,1}$ is a vector of locally integrable functions $(u_1, ..., u_m)$ such that*

$$u_i \in L^{p_i}_{loc}\left(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma_i}\, d\eta dt\right), \quad i \in \{1, 2, ..., m\},$$

*satisfying*

$$\int_0^\infty \int_{\mathbb{R}^{2N+1}} \left( u_i \left( a_i \Delta_{\mathbb{H}} \varphi - (-1)^k \frac{\partial^k \varphi}{\partial t^k} \right) + |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}} \varphi \right) d\eta\, dt +$$

$$\sum_{j=0}^{k-1} (-1)^j \int_{\mathbb{R}^{2N+1}} \frac{\partial^{k-1-j} u_i}{\partial t^{k-1-j}}(\eta, 0) \frac{\partial^j \varphi}{\partial t^j}(\eta, 0)\, d\eta \leq 0, \quad i \in \{1, 2, ..., m-1\}, \tag{19}$$

*and*

$$\int_0^\infty \int_{\mathbb{R}^{2N+1}} \left( u_m \left( a_m \Delta_{\mathbb{H}} \varphi - (-1)^k \frac{\partial^k \varphi}{\partial t^k} \right) + |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} \varphi \right) d\eta\, dt +$$
$$\sum_{j=0}^{k-1} (-1)^j \int_{\mathbb{R}^{2N+1}} \frac{\partial^{k-1-j} u_m}{\partial t^{k-1-j}}(\eta, 0) \frac{\partial^j \varphi}{\partial t^j}(\eta, 0)\, d\eta \leq 0 \tag{20}$$

*for any nonnegative test function* $\varphi \in C_c^{2,k}(\mathbb{R}_+^{2N+1,1})$.

**Theorem 2.** *Assume that* $u_i^{(k-1)} \in L^1(\mathbb{R}^{2N+1})$, $1 \le i \le m$, *and*

$$\int_{\mathbb{R}^{2N+1}} u_i^{(k-1)}(\eta)\, d\eta \ge 0, \quad 1 \le i \le m.$$

*Then,* $Q \le 2(1 - 1/k) + \max\{X_1, X_2, ..., X_m\}$ *implies that the system* $(S_k^m)$ *has no nontrivial solution.*

*Proof.* In order to simplify the proof, we treat only the case $m = 3$, the general case can be established in the same manner.

Let $(u_1, u_2, u_3)$ be a nontrivial weak solution of $(\mathrm{S}_k^m)$. The inequalities (19) and (20), with $\varphi = \varphi_R$ defined by (7), imply that

$$\begin{aligned}\int \varphi_R |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1}\, d\eta\, dt + a(R) &\le \int \left( |u_3| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + ||a_3||_\infty |u_3| |\Delta_{\mathbb{H}} \varphi_R| \right) d\eta\, dt \\ &\le \left( \int |\eta|_{\mathbb{H}}^{\gamma_3} |u_3|^{p_3} \varphi_R \right)^{1/p_3} \left( \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p_3'} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_3})^{1-p_3'} \right)^{1/p_3'} \\ &+ ||a_3||_\infty \left( \int |\eta|_{\mathbb{H}}^{\gamma_3} |u_3|^{p_3} \varphi_R \right)^{1/p_3} \left( \int |\Delta_{\mathbb{H}} \varphi_R|^{p_3'} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_3})^{1-p_3'} \right)^{1/p_3'},\end{aligned}$$

$$\begin{aligned}\int \varphi_R |\eta|_{\mathbb{H}}^{\gamma_2} |u_2|^{p_2}\, d\eta\, dt + b(R) &\le \int \left( |u_1| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + ||a_1||_\infty |u_1| |\Delta_{\mathbb{H}} \varphi_R| \right) d\eta\, dt \\ &\le \left( \int |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} \varphi_R \right)^{1/p_1} \left( \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p_1'} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_1})^{1-p_1'} \right)^{1/p_1'} \\ &+ ||a_1||_\infty \left( \int |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} \varphi_R \right)^{1/p_1} \left( \int |\Delta_{\mathbb{H}} \varphi_R|^{p_1'} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_1})^{1-p_1'} \right)^{1/p_1'}\end{aligned}$$

and

$$\int \varphi_R |\eta|_{\mathbb{H}}^{\gamma_3} |u_3|^{p_3}\, d\eta\, dt + c(R) \le \int \left( |u_2| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + ||a_2||_\infty |u_2| |\Delta_{\mathbb{H}} \varphi_R| \right) d\eta\, dt$$

$$\le \left( \int |\eta|_{\mathbb{H}}^{\gamma_2} |u_2|^{p_2} \varphi_R \right)^{1/p_2} \left( \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p_2'} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_2})^{1-p_2'} \right)^{1/p_2'}$$

$$+ ||a_2||_\infty \left( \int |\eta|_{\mathbb{H}}^{\gamma_2} |u_2|^{p_2} \varphi_R \right)^{1/p_2} \left( \int |\Delta_{\mathbb{H}} \varphi_R|^{p_2'} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_2})^{1-p_2'} \right)^{1/p_2'}.$$

Let

$$\begin{cases} I_i(R) \; = \int |\eta|_{\mathbb{H}}^{\gamma_i} |u_i|^{p_i} \varphi_R \, d\eta, \quad 1 \le i \le 3, \\ \\ \mathcal{A}_i(R) = \int |\Delta_{\mathbb{H}} \varphi_R|^{p_i'} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_i})^{1-p_i'}, \quad 1 \le i \le 3, \\ \\ \mathcal{B}_i(R) \; = \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p_i'} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_i})^{1-p_i'} \, d\eta\, dt, \quad 1 \le i \le 3. \end{cases}$$

Then there is a positive constant $C$ such that

$$\begin{cases} I_1 \le C\, I_3^{1/p_3} \left( \mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'} \right), \\ \\ I_2 \le C\, I_1^{1/p_1} \left( \mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'} \right), \\ \\ I_3 \le C\, I_2^{1/p_2} \left( \mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'} \right). \end{cases}$$

Whence, the estimates

$$\begin{cases} I_1^{1-\frac{1}{p_1p_2p_3}} \le C \left( \mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'} \right)^{\frac{1}{p_2p_3}} \left( \mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'} \right)^{\frac{1}{p_3}} \left( \mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'} \right), \\ \\ I_2^{1-\frac{1}{p_1p_2p_3}} \le C \left( \mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'} \right) \left( \mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'} \right)^{\frac{1}{p_1p_3}} \left( \mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'} \right)^{\frac{1}{p_1}}, \\ \\ I_3^{1-\frac{1}{p_1p_2p_3}} \le C \left( \mathcal{A}_1^{1/p_1'} + \mathcal{B}_1^{1/p_1'} \right)^{\frac{1}{p_2}} \left( \mathcal{A}_2^{1/p_2'} + \mathcal{B}_2^{1/p_2'} \right) \left( \mathcal{A}_3^{1/p_3'} + \mathcal{B}_3^{1/p_3'} \right)^{\frac{1}{p_1p_2}}, \end{cases}$$

hold true.

In order to estimate the expressions $I_i$, $1 \le i \le 3$, we use the scaled variables

(15) and obtain

$$I_i^{1-\frac{1}{p_1p_2p_3}} \leq CR^{\sigma_i}, \quad 1 \leq i \leq 3,$$

where

$$\begin{cases} \sigma_1 = \left(1 - \frac{1}{p_1p_2p_3}\right)\left(Q - 2 + \frac{2}{k} - \frac{(\gamma_1+2)+p_1(\gamma_2+2)+p_1p_2(\gamma_3+2)}{p_1p_2p_3-1}\right), \\ \\ \sigma_2 = \left(1 - \frac{1}{p_1p_2p_3}\right)\left(Q - 2 + \frac{2}{k} - \frac{p_2p_3(\gamma_1+2)+(\gamma_2+2)+p_2(\gamma_3+2)}{p_1p_2p_3-1}\right), \\ \\ \sigma_3 = \left(1 - \frac{1}{p_1p_2p_3}\right)\left(Q - 2 + \frac{2}{k} - \frac{p_3(\gamma_1+2)+p_1p_3(\gamma_2+2)+(\gamma_3+2)}{p_1p_2p_3-1}\right). \end{cases}$$

Now, we require that, at least, one of $\sigma_i$, $1 \leq i \leq 3$, is less than zero, which is equivalent to $Q \leq 2(1-1/k) + \max\{X_1, X_2, X_3\}$, where the vector $(X_1, X_2, X_3)^T$ is the solution of

$$\begin{pmatrix} -1 & p_1 & 0 \\ 0 & -1 & p_2 \\ p_3 & 0 & -1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \gamma_1 + 2 \\ \gamma_2 + 2 \\ \gamma_3 + 2 \end{pmatrix}. \tag{21}$$

Following the arguments used in the proof of Theorem 1, we conclude that $(u_1, u_2, u_3) \equiv (0, 0, 0)$. This ends the proof by contradiction. □

**Remark 2.** *To determine the critical exponent $Q_e^*$ corresponding to the hypoelliptic system*

$$(S_k^m)\begin{cases} -\Delta(a_iu_i) \geq |\eta|_{\mathbb{H}}^{\gamma_{i+1}}|u_{i+1}|^{p_{i+1}}, \quad x \in \mathbb{R}^{2N+1}, \quad 1 \leq i \leq m, \\ \\ u_{m+1} = u_1, \end{cases}$$

*it suffices to tend formally $k$ to infinity in the exponent $Q_k^*$ and obtain*

$$Q_e^* = 2 + \max\{X_1, X_2, ..., X_m\}.$$

In the following section, we show that the result of Theorem 2 is also valid for $m = 1$.

## 4 Higher Order Evolution Semilinear Inequalities

Let us consider the higher inequality ($\mathrm{I}_k$) with the initial data

$$\begin{cases} u(\eta,0) &= u^{(0)}(\eta), \quad \text{in} \quad \mathbb{R}^{2N+1}, \\ \\ \frac{\partial^i u}{\partial t^i}(\eta,0) &= u^{(i)}(\eta), \quad i \in \{1,2,...,k-1\}, \quad\quad \text{in} \quad \mathbb{R}^{2N+1}. \end{cases}$$

**Definition 3.** *Let $a$ a bounded measurable functions in $\mathbb{R}_+^{2N+1,1}$. A weak solution $u$ of the inequality ($I_k$) with initial data $u^{(i)} \in L^1_{loc}\left(\mathbb{R}^{2N+1}\right)$, $i \in \{0,1,...,k-1\}$, is a locally integrable function $u$ such that*

$$u \in L^p_{loc}\left(\mathbb{R}_+^{2N+1,1}, |\eta|_{\mathbb{H}}^{\gamma}\, d\eta dt\right),$$

*satisfying*

$$\int_0^\infty \int_{\mathbb{R}^{2N+1}} \left( u \left( a\Delta_{\mathbb{H}}\varphi - (-1)^k \frac{\partial^k \varphi}{\partial t^k} \right) + |\eta|_{\mathbb{H}}^{\gamma} |u|^p \varphi \right) d\eta\, dt +$$

$$\sum_{i=0}^{k-1} (-1)^i \int_{\mathbb{R}^{2N+1}} \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(\eta,0) \frac{\partial^i \varphi}{\partial t^i}(\eta,0)\, d\eta \leq 0, \tag{22}$$

*for any nonnegative test function $\varphi \in C_c^{2,k}(\mathbb{R}_+^{2N+1,1})$.*
**Theorem 3.** *Assume that $u^{(k-1)} \in L^1(\mathbb{R}^{2N+1})$ and $\int u^{(k-1)}(\eta)\, d\eta \geq 0$. If*

$$Q \leq 2\left(1 - \frac{1}{k}\right) + \frac{\gamma+2}{p-1},$$

*then there is no weak nontrivial solution $u$ of the system ($I_k$).*

*Proof.* Let $u$ be a nontrivial weak solution of ($\mathrm{I}_k$). Using the Hlder inequality, the equation (22) gives:

$$\int \varphi_R |\eta|_{\mathbb{H}}^{\gamma} |u|^p\, d\eta\, dt + \tilde{a}(R) \leq \int \left( |u| \left| \frac{\partial^k \varphi_R}{\partial t^k} \right| + |u||\Delta_{\mathbb{H}} \varphi_R| \right) d\eta\, dt$$

$$\leq \left( \int |\eta|_{\mathbb{H}}^{\gamma} |u|^p \varphi_R\, dx\, dt \right)^{1/p} \left( \int \left| \frac{\partial^k \varphi_R}{\partial t^k} \right|^{p'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma}\right)^{1-p'} d\eta\, dt \right)^{1/p'}$$

$$+ ||a||_\infty \left( \int |\eta|_{\mathbb{H}}^{\gamma} |u|^p \varphi_R\, d\eta\, dt \right)^{1/p} \left( \int |\Delta_{\mathbb{H}} \varphi_R|^{p'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma}\right)^{1-p'} d\eta\, dt \right)^{1/p'},$$

where

$$\tilde{a}(R) = \int_{\mathbb{H}^N} u^{(k-1)}(\eta)\, \varphi_R(\eta,0)\, d\eta.$$

Let us set

$$\tilde{I}(R) = \int \varphi_R |\eta|_{\mathbb{H}}^{\gamma} |u|^p \, d\eta\, dt,$$

$$\tilde{\mathcal{A}}_{p,\gamma}(R) = \int |\Delta_{\mathbb{H}} \varphi_R|^{p'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma}\right)^{1-p'} \, d\eta\, dt,$$

and

$$\tilde{\mathcal{B}}_{p,\gamma}(R) = \int \left|\frac{\partial^k \varphi_R}{\partial t^k}\right|^{p'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma}\right)^{1-p'} \, d\eta\, dt.$$

Following the same method described in the last proof, we obtain

$$I(R) \le C\left(\tilde{\mathcal{A}}_{p,\gamma}(R)^{1/p'} + \tilde{\mathcal{B}}_{p,\gamma}(R)^{1/p'}\right) I(R)^{1/p}, \tag{23}$$

where $C$ is a positive constant independent of $R$. Using the same scaled variables as before, we have the estimate

$$I(R)^{1-1/p} \le C R^{\sigma},$$

where

$$\sigma = -2 - \frac{\gamma}{p} + \left(Q + \frac{2}{k}\right)\frac{p-1}{p}.$$

Now, we require $\sigma \le 0$ which is equivalent to

$$Q \le 2\left(1 - \frac{1}{k}\right) + \frac{\gamma+2}{p-1}. \tag{24}$$

In this case, the integral $I(R)$, increasing in $R$, is bounded uniformly w.r.t. $R$. The monotone convergence theorem implies that $|\eta|_{\mathbb{H}}^{\gamma}|u|^p$ belongs to $L^1(\mathbb{R}_+^{2N+1,1})$. Note that instead of (23) we have more precisely

$$\int |\eta|_{\mathbb{H}}^{\gamma}|u|^p \varphi_R \, d\eta\, dt \le ||a||_{L^\infty} \left(\int_{\mathcal{C}_R} |\eta|_{\mathbb{H}}^{\gamma}|u|^p \varphi_R \, d\eta\, dt\right)^{1/p} \left(\tilde{\mathcal{A}}_{p,\gamma}(R)^{1/p'} + \tilde{\mathcal{B}}_{p,\gamma}(R)^{1/p'}\right)$$
$$\le C \int_{\mathcal{C}_R} |\eta|_{\mathbb{H}}^{\gamma}|u|^p \varphi_R \, d\eta\, dt,$$

where $\mathcal{C}_R$ is defined in (9). Finally, using the dominated convergence theorem, we obtain that

$$\lim_{R\to+\infty} \int_{\mathcal{C}_R} |\eta|_{\mathbb{H}}^{\gamma}|u|^p \varphi_R \, d\eta\, dt = 0.$$

Hence

$$\int |\eta|_{\mathbb{H}}^{\gamma}|u|^p \, d\eta\, dt = 0,$$

which implies that $u \equiv 0$. This contradicts the fact that $u$ is a nontrivial weak solution of ($\mathrm{I}_k$), which achieves the proof. □

**Remark 3.** *To determine the critical exponent for the hypoelliptic inequality*

$$-\Delta_{\mathbb{H}}(au) \geq |\eta|_{\mathbb{H}}^{\gamma}\, |u|^p,$$

*it suffices to tend formally $k$ to infinity and obtain* $2 + \frac{\gamma+2}{p-1}$.

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