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On a conjecture of Wolfgang Wasow concerning the nature of turning points

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Abstract

For linear systems of singularly perturbed ordinary differential equations, the formal fundamental solutions are expressed as a formal power series in a rational power of the parameter ε with coefficient functions in x, times a diagonal exponential polynomial in negative powers of ε and also with coefficient functions in x. We prove that if all the coefficient functions in x of the formal fundamental solution are holomorphic in x in some region $\mathcal{D}$, then there exists a fundamental holomorphic solution for x in $\mathcal{D}$ and for ε in some sector. This settles a conjecture of W. Wasow concerning the nature of the turning points.

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1 Introduction.

Consider a system of n singularly perturbed ordinary differential equations

$$\varepsilon^h \frac{dy}{dx} = A(x,\varepsilon)y \tag{1}$$

where x is a complex variable, ε is a small complex parameter, h is a positive integer and A is a holomorphic function in a neighbourhood of $(x_0, 0)$ of $\mathbb{C} \times \mathbb{C}$. Under suitable assumptions at the point $x = x_0$ on the leading matrix $A_0(x) = A(x, 0)$ and the leading matrices $\tilde{A}_0(x)$ which appear by transformations of the system (1), M. Hukuhara, Y. Sibuya, H.L. Turrittin and W. Wasow [Was85] have shown the existence of a formal fundamental solution of (1) of the form

$$\hat{Y}(x,\varepsilon) = \left(\sum_{r=0}^{+\infty} Y_r(x)\varepsilon^{r/m}\right) \exp\left(\sum_{r=0}^{mh-1} Q_r(x)\varepsilon^{(r-mh)/m}\right), \tag{2}$$

where the matrix-valued functions $\hat{Y}_r(x)$ and $Q_r(x)$ are holomorphic in a set $\mathcal{D}$ to be specified later. In general, the series (2) is divergent, and classically, we investigate the existence of a basis of actual solutions of the above system that is asymptotic to $\hat{Y}(x,\varepsilon)$

when $\varepsilon \longrightarrow 0$ in some sectorial region. The problem was partially solved by Y. Sibuya [Was85], under the condition that the point $x = x_0$ is asymptotically simple, i.e. the functions $Q_r(x)$ satisfy certain restrictive conditions. According to the following definition of W. Wasow of a turning point: *a point $x = x_0$ is called a turning point for the system (1), if none of the formal fundamental solutions (2) is an asymptotic representation of a fundamental solution in a neighbourhood of $x = x_0$*, we deduce that an asymptotically simple point is not a turning point.

In this paper, we study the nature of the points $x = x_0$ for the system (1). As will be shown in section 2, there are systems for which there exists a formal fundamental solution of the form (2) with coefficients that are holomorphic functions in x in a neighbourhood of $x = x_0$, eventhough $x = x_0$ is not asymptotically simple. Our purpose is to prove that for such systems, $x = x_0$ is also not a turning point. Our main concern here will be to redefine a turning point: "*Only the points where some of the coefficients of the formal*" fundamental solution (2) "*have singularities are turning points*". This definition is W. Wasow conjecture [Was85]. Therefore we claim that *a point $x = x_0$ is a turning point for the system (1) if and only if a coefficient of the formal fundamental solution (2) has a singularity at this point.*

If there exists a coefficient with a singularity at $x = x_0$, then by definition x_0 is a turning point for the system (1). Therefore it remains to prove the converse. We suppose that the system (1) admits a formal fundamental solution (2) where all the coefficients $\hat{Y}_r(x)$ and $Q_r(x)$ are holomorphic functions in a neighbourhood of $x = x_0$, and we want to study the existence of a fundamental solution Y such that

$$Y(x, \varepsilon) \sim \hat{Y}(x, \varepsilon) \tag{3}$$

as $\varepsilon \longrightarrow 0$ in a given sectorial region, uniformly in x in some neighbourhood of x_0.

Our approach is based on Gevrey asymptotic theory and is as follows. In section 3.2, we define the set $\mathcal{V}$ of the valuations in terms of $\varepsilon^{1/m}$

$$\mathcal{V} = \left\{ \mathrm{val}_{\varepsilon^{1/m}} \left(\varepsilon^h \frac{dq_{jk}}{dx}(x, \varepsilon) \right) \big| (j, k) \in \{1, \ldots, n\}^2 \right\}$$

where the polynomials $\varepsilon^{1/m} \left(\varepsilon^h \frac{dq_{jk}}{dx}(x, \varepsilon) \right)$ are those which appear previously in the definition of asymptotically simple points of Y. Sibuya and which arise from the diagonal matrix

$$Q(x, \varepsilon) = \sum_{r=0}^{mh-1} Q_r(x) \varepsilon^{(r-mh)/m}$$

of (2). With this definition, we have

$$\mathcal{V} = \begin{cases} \{+\infty\} \\ \text{or} \\ \{s_1, \ldots, s_\ell\} \cup \{+\infty\}, \end{cases}$$

where the s_i are integers. If the set $\mathcal{V}$ reduces to $\{+\infty\}$, we call (1) an essentially scalar system, otherwise the system (1) is said to have ℓ levels.

Our method to prove the existence of an actual fundamental solution (3) is to split the initial system (1), which we suppose to have ℓ levels, into a set of independent systems having less than ℓ levels. In other words, we proceed by a induction of the number of levels; but first in section 3.4, we use a theorem of simplification to connect the new coefficient matrix A with the diagonal matrix Q, and so the set $\mathcal{V}$ which remains unchanged by the transformation. Then, in section 3.6 to initialize our inductive process we prove the theorem for essentially scalar systems.

A system of this kind is equivalent to a system where the parameter disappears on the left hand side. Therefore it is not difficult to prove the Gevrey character of the formal fundamental solution. The existence of an actual fundamental solution follows by using formal Borel transforms and truncated Laplace transforms which provide quasi-solution and to conclude we use Gronwall's lemma as in the paper of M. Canalis-Durand et al. [CDRSS00]. In our article, we write the proof for any Gevrey character whereas other papers deal with the Gevrey order one.

In section 3.7, we begin the induction with systems having one level. The splitting is done by sorting, in a precise way, the coefficients of the diagonal matrix $Q(x,\varepsilon)$ into two blocks and by using a holomorphic transformation to yield two independent systems of the size of the two blocks. To prove the existence of this holomorphic transformation, we first show that there exists a unique formal transformation satisfying all our properties. The key of the formal existence rests in the way to split the diagonal matrix $Q(x,\varepsilon)$ and the use of the main results of the paper of M. Canalis-Durand et al. [CDRSS00]. In order to show the existence of a holomorphic transformation, we again write our problem into a solved problem of the article [CDRSS00]. Thus, we obtain either two independent essentially scalar systems or systems of one level but of smaller size. Repeating this splitting process we obtain a finite set of independent essentially scalar systems.

For systems with several levels the same idea works, the critical point is the splitting of the systems in the right order. This will be explained in the section 3.8.

2 Preliminaries.

Consider a system of n singularly perturbed ordinary differential equations

$$\varepsilon^h \frac{dy}{dx} = A(x,\varepsilon)y \tag{4}$$

where x is a complex variable, ε is a small complex parameter and h is a positive integer. The function A is a holomorphic function of the variables x and ε in the open neighbourhood
$\mathcal{D}_\tau(x_0) \times \mathcal{D}_{\varepsilon_0}(0) \subseteq \mathbb{C} \times \mathbb{C}$ of $(x_0, 0)$ where τ, ε_0 are positive reals.
Here the notation $\mathcal{D}_\alpha(\beta)$ means the open disk of radius α and center β.
As a first consequence, A admits a convergent power series expansion

$$A(x,\varepsilon) = \sum_{r=0}^{+\infty} A_r(x)\varepsilon^r, \qquad x \in \mathcal{D}_\tau(x_0),\ \varepsilon \in \mathcal{D}_{\varepsilon_0}(0), \tag{5}$$

where the coefficients $A_r(x)$ are holomorphic in $\mathcal{D}_\tau(x_0)$. We assume that

$$A_0(x) \not\equiv 0,$$

since, otherwise, equation (4) could be divided by ε.
The structure of the leading matrix $A_0(x)$ is of great importance in the construction of the algorithm for the formal solution of the system (4). The reader interested will find all the details in [Was85] (ch.II). Here we only recall the theorem of the existence of formal solutions, which is the result of the algorithm.

Theorem (M. Hukuhara, Y. Sibuya, H.L. Turrittin, W. Wasow) *Let $\sum_{r=0}^{+\infty} A_r(x)\varepsilon^r$ be a formal series in which the A_r are $n \times n$-dimensional matrix-valued functions of the complex variable x, holomorphic in a region $\mathcal{D} \subseteq \mathbb{C}$, and ε is a complex parameter. Let h be a positive integer. Then the formal differential equation*

$$\varepsilon^h \frac{dy}{dx} = \left(\sum_{r=0}^{+\infty} A_r(x)\varepsilon^r \right) y$$

can be satisfied by replacing y with a formal expression of the form

$$\left(\sum_{r=0}^{+\infty} Y_r(x)\varepsilon^{r/m} \right) \exp\left(\sum_{r=0}^{mh-1} Q_r(x)\varepsilon^{(r-mh)/m} \right). \tag{6}$$

The symbols in (6) are defined as follows

1. *The coefficients Y_r are $n \times n$ matrix-valued functions locally holomorphic in $\mathcal{D}\backslash\mathcal{I}$, where $\mathcal{I}$ is a set of isolated points, the same for all r, where they have branch points or poles. The coefficients Q_r are diagonal matrices and have the same properties as the Y_r, and m is some positive integer.*
2. *The formal series for* $\det\left(\sum_{r=0}^{+\infty} Y_r(x)\varepsilon^{r/m}\right)$ *in powers of $\varepsilon^{1/m}$ is not identically zero.*

The set $\mathcal{I}$ is the set of points of $\mathcal{D}$ where A_0 or the analogous leading coefficient matrices in the later stages of the computation of formal solution fail to satisfy the following hypothesis

Hypothesis *The Jordan matrix for A_0 is holomorphic in $\mathcal{V}$, where $\mathcal{V}$ is some subset of $\mathcal{D}$, and the multiplicity of the eigenvalues of $A_0(x)$ is constant in $\mathcal{V}$.*

The singularities of Y_r and Q_r are points of $\mathcal{I}$. A theorem of linear algebra for holomorphic matrices is used to prove that there are no other singularities which appears in $\mathcal{D}$, see [Was85] (ch XII). Thus, in a region in which the coefficients A_r of (5) are holomorphic the only singularities of Y_r and Q_r in (6) are points of $\mathcal{I}$. It is known that the set $\mathcal{I}$ has no accumulation points in $\mathcal{D}$ and that the Y_r and Q_r are analytically continuable along every curve lying in $\mathcal{D}\backslash\mathcal{I}$ and also on an appropriate Riemann domain (cf. [Was85], [SV90]).
Now, by the theorem above, we have the existence of a formal fundamental solution of (4). We want to study the existence of <u>bases of true solutions</u>, i.e. fundamental solutions which admit as asymptotic representation a formal fundamental solution (6). A true

solution of (4), is a holomorphic function of the variables x in an open neighbourhood $\mathcal{D}_\tau(x_0) \subseteq \mathbb{C}$ of x_0 where τ is positive and holomorphic for ε in an open sector $\mathcal{S}$ of the complex plane whose vertex is at the origin.

The expression asymptotic representation means here that if $Y(x,\varepsilon)$ is a fundamental solution of (4), then we have

$$Y(x,\varepsilon)\exp\left(\sum_{r=0}^{mh-1} Q_r(x)\varepsilon^{(r-mh)/m}\right) \sim \sum_{r=0}^{+\infty} Y_r(x)\varepsilon^{r/m}$$

as $\varepsilon \longrightarrow 0$ in $\mathcal{S}$, uniformly for x in a neighbourhood of x_0. Remark that the fundamental solution is not unique, because multiplication of the solution on the right hand side by a power series in ε, with the series 1 as asymptotic, yields an other solution of our problem. The problem of the existence of a fundamental solution which admits as asymptotic representation the formal fundamental solution given by the theorem above, was partially solved by Y. Sibuya [Was85].

The sufficient condition he gives, is that the point x_0 be asymptotically simple. Before we state the definition of an asymptotically simple point, we introduce some notations. We denote by q_j the j-th diagonal element of the diagonal matrix Q

$$Q(x,\varepsilon) = \sum_{r=0}^{mh-1} Q_r(x)\varepsilon^{(r-mh)/m} = \mathrm{diag}_{j=1}^{n}\left(q_j(x,\varepsilon)\right).$$

Moreover

$$q_{jk}(x,\varepsilon) = q_j(x,\varepsilon) - q_k(x,\varepsilon),$$

and

$$\varepsilon^h \frac{dq_{jk}}{dx}(x,\varepsilon) = \lambda_{jk}(x)\varepsilon^{s/m} + c_{s+1}^{jk}(x)\varepsilon^{(s+1)/m} + \ldots + c_{mh-1}^{jk}(x)\varepsilon^{(mh-1)/m}$$

where s is a non-negative integer and $\lambda_{jk}(x)$ is not identically zero in case $s < mh-1$.

Definition 1. *A point x_0 will be called asymptotically simple with respect to q_k if those of the functions $\lambda_{jk}(x)$, $j = 1, 2 \ldots, n$, which are not identically zero are holomorphic and different from zero at $x = x_0$. And a point x_0 will be called asymptotically simple if for all k the point x_0 is asymptotically simple with respect to q_k.*

Or in other words, x_0 is an asymptotically simple point with respect to q_k, if the valuations (in terms of $\varepsilon^{1/m}$) of $\varepsilon^h \dfrac{dq_{jk}}{dx}(x,\varepsilon)$, $j = 1, 2, \ldots, n$ are constant in a neighbourhood of x_0. For such asymptotically simple points, Y. Sibuya proved the following main result on asymptotic validity.

Theorem (Y. Sibuya) *If the point x_0 is asymptotically simple, then the formal expression (6) represents asymptotically a fundamental solution of the system of differential equations (4) in a neighbourhood of x_0.*

Thus, if we consider the definition given by W. Wasow for a turning point, we deduce from the above theorem that an asymptotically simple point for all q_k is not a turning point for the system (4).

Definition 2. *A point x_0 is a* <u>*turning point*</u> *for the system (4), if* **none** *of the formal fundamental solutions (6) is an* <u>*asymptotic representation*</u> *of a fundamental solution in a neighbourhood of x_0.*

Our problem of the existence of fundamental solutions in a neighbourhood of a point x_0 is equivalent to the problem of the nature of the point x_0, is it a turning point for the system (4) or not. There are essentially two questions

- the hypothesis of asymptotic simplicity is it a necessary condition?
- a point in the set $\mathcal{I}$, of isolated points defined by the theorem of the existence of formal fundamental solutions, is it a turning point for the system (4) or not?

Consider, for instance, the system

$$\varepsilon \frac{dy}{dx} = \left[\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x^{2p+1} \\ 0 & 0 \end{pmatrix} \varepsilon \right] y$$

where p is a positive integer. Here the two eigenvalues of the leading matrix

$$A_0(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

coalesce at the point $x = 0$. At $x = 0$, A_0 does not satisfy the above hypothesis, so there may or may not be a turning point there. The formal resolution, which involves divisions by x, introduces in general poles at $x = 0$ into the coefficients of the formal fundamental solution (6). In our example, however a fundamental solution is given by

$$Y(x,\varepsilon) = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & x^{2p} \\ 0 & 0 \end{pmatrix} \varepsilon + \begin{pmatrix} 0 & 2px^{2p-2} \\ 0 & 0 \end{pmatrix} \varepsilon^2 + \ldots + \begin{pmatrix} 0 & 2^p p! \\ 0 & 0 \end{pmatrix} \varepsilon^{p+1} \right]$$
$$\exp\left\{ \begin{pmatrix} x^2/2 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\varepsilon} \right\}.$$

In this example, the point $x = 0$ is neither asymptotically simple, because the holomorphic function

$$\varepsilon \frac{dq_{12}}{dx}(x,\varepsilon) = x$$

is not constant in a neighbourhood of 0, nor a turning point.

Consider, on the other hand the system

$$\varepsilon \frac{dy}{dx} = \left[\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \varepsilon \right] y.$$

The leading matrix A_0 is the same as in the previous case. The formal computation gives us the following formal fundamental expression

$$\hat{Y}(x,\varepsilon) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1/x \\ 0 & 0 \end{pmatrix} \varepsilon + \begin{pmatrix} 0 & -1/x^3 \\ 0 & 0 \end{pmatrix} \varepsilon^2 + \ldots \right] \exp\left\{ \begin{pmatrix} x^2/2 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\varepsilon} \right\}.$$

The dots indicate a series of positive powers of ε. Here the coefficients of the formal fundamental solution have a singularity at $x = 0$, so it is obvious that this point is a turning point for our system, because there exists no holomorphic solution $Y(x,\varepsilon)$ satisfying

$$Y(x,\varepsilon)\exp\left\{-\begin{pmatrix} x^2/2 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{\varepsilon}\right\} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1/x \\ 0 & 0 \end{pmatrix}\varepsilon + \begin{pmatrix} 0 & -1/x^3 \\ 0 & 0 \end{pmatrix}\varepsilon^2 + \dots$$

as $\varepsilon \longrightarrow 0$ in a sector of the ε-plane and for x in a neighbourhood of 0.
If we look at the construction of the set $\mathcal{I}$ of isolated points which appears in the theorem of the existence of a formal fundamental solution of (4), for all two cases the point $x = 0$ belongs to $\mathcal{I}$. Therefore based on the two examples above we remark that, it is not because a point x_0 is in the set $\mathcal{I}$, that this point is or not a turning point for the system that we consider. More, a point that is not asymptotically simple is not necessarily a turning point.
It is W. Wasow's conjecture [Was85], pp46, which gives us a way to answer the problem of the existence of a basis of actual solutions. He writes that " It is a tempting conjecture that *only* the points where some of the coefficients of the formal solutions described in " the theorem of the existence of a formal fundamental solution " have singularities are turning points."

3 Proof of W. Wasow's conjecture.

We recall the system

$$\varepsilon^h \frac{dy}{dx} = A(x,\varepsilon)y \tag{7}$$

and the form of the formal fundamental solution

$$\hat{Y}(x,\varepsilon) = \left(\sum_{r=0}^{+\infty} \hat{Y}_r(x)\varepsilon^{r/m}\right)\exp\left(\sum_{r=0}^{mh-1} Q_r(x)\varepsilon^{(r-mh)/m}\right). \tag{8}$$

Theorem 1. *A point of the x-plane is a turning point for the system (7) <u>if and only if</u> there exists a coefficient $\hat{Y}_r(x)$ or $Q_r(x)$ with a singularity at this point.*

<u>Proof</u>: It is obvious that if a coefficient in the formal fundamental solution $\hat{Y}$ has a singularity at a point of the x-plane, then this point is a turning point for the system. Indeed as we said it above, there exists no holomorphic solution $Y(x,\varepsilon)$ such that

$$Y(x,\varepsilon)\exp\left(-\sum_{r=0}^{mh-1} Q_r(x)\varepsilon^{(r-mh)/m}\right) \sim \sum_{r=0}^{+\infty} \hat{Y}_r(x)\varepsilon^{r/m}$$

as $\varepsilon \longrightarrow 0$ in a sector of the ε-plane and for x in a neighbourhood of this point of the x-plane.
To complete the proof, it remains to check the reciprocal. For this, we consider the contrapositive sentence. Our hypothesis is: We have a formal fundamental solution (8) of the system (7) and any coefficient $\hat{Y}_r(x)$ or $Q_r(x)$ has not a singularity at x_0 a point

of the x-plane. It is convenient to make the following change of variable, we replace x by $x_0 + x$, thus the new initial value is $x_0 = 0$.
In order to prove the theorem 1, we will show that there exists, for each sufficiently small sector $\mathcal{S}$ (of sufficiently small opening), a fundamental solution Y where

$$Y \sim \hat{Y}$$

as $\varepsilon \longrightarrow 0$ in $\mathcal{S}$, uniformly for x in a neighbourhood of 0.

Our method to prove this statement is based on Gevrey theory.

3.1 Gevrey asymptotic expansions.

We recall two definitions about Gevrey expansions.

Definition 3. *The formal series $\sum_{r=0}^{+\infty} \alpha_r(x)\varepsilon^r$ is of Gevrey order s uniformly for x in $\mathcal{D}$, if there exists two positive constants M and N such that*

$$|\alpha_r(x)| \leq MN^r\Gamma(rs+1)$$

for all r and for all x in $\mathcal{D}$.

Here Γ represents the usual Gamma function.

Definition 4. *The function $f(x,\varepsilon)$ is asymptotic of Gevrey order s to the formal series $\sum_{r=0}^{+\infty} \alpha_r(x)\varepsilon^r$ as $\varepsilon \longrightarrow 0$ in a sector $\mathcal{S}$ uniformly for x in $\mathcal{D}$, if there exists two positive constants M and N such that*

$$\left| f(x,\varepsilon) - \sum_{r=0}^{p-1} \alpha_r(x)\varepsilon^r \right| \leq MN^p\Gamma(ps+1)\,|\varepsilon|^p$$

for all ε in $\mathcal{S}$, all x in $\mathcal{D}$ and all positive integer p.

3.2 Notations and definitions.

We define the following set $\mathcal{V}$.

Definition 5.

$$\mathcal{V} = \left\{ \mathrm{val}_{\varepsilon^{1/m}} \left(\varepsilon^h \frac{dq_{jk}}{dx}(x,\varepsilon) \right) \big| (j,k) \in \{1,\ldots,n\}^2 \right\}.$$

Here and in the sequel, $\mathrm{val}_{\varepsilon^{1/m}}$ means the valuation in terms of $\varepsilon^{1/m}$ defined as follows. Consider a polynomial

$$p(x,\varepsilon) = p_0(x) + p_1(x)\varepsilon^{1/m} + \ldots + p_{v-1}(x)\varepsilon^{(v-1)/m} + p_v(x)\varepsilon^{v/m} + \ldots + p_d(x)\varepsilon^{d/m}$$

where d is a non-negative integer. We put

$$\mathrm{val}_{\varepsilon^{1/m}}\left(p(x,\varepsilon)\right) = v$$

if and only if

$$\begin{cases} p_i(x) \equiv 0, & i = 0, \ldots, v-1 \\ p_v(x) \not\equiv 0, \end{cases}$$

with the convention that

$$\mathrm{val}_{\varepsilon^{1/m}}(0) = +\infty.$$

The set $\mathcal{V}$ depends in fact on the variable x. If we consider the definition of an asymptotically simple point, then in a neighbourhood of this point the set $\mathcal{V}$ is constant. The assumptions of our theorem imply that $\mathcal{V}$ is constant in a punctured neighbourhood of the origin, and $\mathcal{V}$ is allowed to change at $x = 0$.
With the definition of the set $\mathcal{V}$, there are two possibilities for the form of $\mathcal{V}$. We have

$$\mathcal{V} = \begin{cases} \{+\infty\} \\ \text{or} \\ \{s_1, \ldots, s_\ell\} \cup \{+\infty\}, \end{cases}$$

where ℓ is a positive integer and where the s_i are integers such that $0 \leq s_1 < \ldots < s_\ell \leq mh - 1$.

Definition 6. *An essentially scalar system is a system where $\mathcal{V}$ is reduced to the singleton $\{+\infty\}$.*

In the case of an essentially scalar system, the matrix Q has the form

$$Q(x, \varepsilon) = q(x, \varepsilon) I,$$

where $q(x, \varepsilon)$ is a polynomial in negative powers of $\varepsilon^{1/m}$ with holomorphic coefficients.

Definition 7. *A system of ℓ levels is a system where $\mathcal{V} = \{s_1, \ldots, s_\ell\} \cup \{+\infty\}$.*

It is convenient to make the following change of parameter

$$\varepsilon \longleftarrow \varepsilon^{1/m}.$$

This substitution leads to a problem with integral powers of ε, therefore without loss of generality in the sequel we suppose that $m = 1$.

3.3 Outlines of the proof.

Our approach to prove our main result is to decompose the initial system (7), which we suppose of ℓ levels, into a set of independent systems of less than ℓ levels. First we start the proof of W. Wasow's conjecture with an essentially scalar system, then we go on to a system of one level, and end with the generalization for systems of several levels. First of all, however we need a fundamental step: the simplification of the problem.

3.4 Theorem of simplification.

The proof is based on the nature of the set $\mathcal{V}$. This means, by definition of $\mathcal{V}$, on the diagonal matrix Q which appears in the formal fundamental solution (8). Subsequently, informations about the coefficient matrix A of the system (7) will be needed, but at this stage we only know that this matrix is a matrix-valued function holomorphic of the variables x and ε.

Theorem 2. *Under the conditions of theorem 1., there exists a positive integer N, sufficiently large, such that the transformation*
$y = \tilde{Y}(x,\varepsilon)w$ with

$$\tilde{Y}(x,\varepsilon) = \sum_{r=0}^{N} \hat{Y}_r(x)\varepsilon^r$$

reduces the system (7) into the following problem

1. *the new system is*

$$\varepsilon^h \frac{dw}{dx} = \tilde{A}(x,\varepsilon)w, \qquad \tilde{A}(x,\varepsilon) = \sum_{r=0}^{+\infty} \tilde{A}_r(x)\varepsilon^r, \tag{9}$$

   for all x in $\mathcal{D}_x(0)$ and for all ε in $\mathcal{D}_\varepsilon(0)$,

2. *the formal fundamental solution $\hat{W}$ of (9) has the form*

$$\hat{W}(x,\varepsilon) = \left(I + \varepsilon^h \sum_{r=0}^{+\infty} \hat{W}_r(x)\varepsilon^r\right) \exp\left(\sum_{r=0}^{h-1} Q_r(x)\varepsilon^{r-h}\right) \tag{10}$$

3. *the coefficients of the formal solution $\hat{W}$ satisfy*
   - *(a) all the coefficients $\hat{W}_r(x)$, $\tilde{A}_r(x)$ and $Q_r(x)$ have no singularity at $x = 0$,*
   - *(b) the matrix Q and the set $\mathcal{V}$ are unchanged,*
   - *(c) for all integer r in $\{0,\ldots,h-1\}$, $\dfrac{dQ_r}{dx}(x) = \tilde{A}_r(x)$.*

Here, $\mathcal{D}_\alpha(0)$ denotes an open disk in the α-plane of center the origin.

Proof : We define $\hat{H}(x,\varepsilon) = \sum_{r=0}^{+\infty} \hat{Y}_r(x)\varepsilon^r$. Then $\det\left(\hat{H}(x,\varepsilon)\right)$ is of the form

$$\det\left(\hat{H}(x,\varepsilon)\right) = \sum_{r=L}^{+\infty} h_r(x)\varepsilon^r \tag{11}$$

where L is a non-negative integer such that $h_L(x)$ does not identically vanish. First we prove that

$$h_L(x) \neq 0 \qquad\qquad \text{for all } x \text{ in } \mathcal{D}_x(0).$$

We consider the Wronskian $w(x,\varepsilon) = \det\left(\hat{Y}(x,\varepsilon)\right)$, which satisfies the differential equation

$$\frac{dw}{dx} = \varepsilon^{-h}\mathrm{tr}\left(A(x,\varepsilon)\right) w. \tag{12}$$

The general formal solution of (12) is given by the expression

$$w(x,\varepsilon) = c(\varepsilon)\exp\left(\varepsilon^{-h}\int_0^x \mathrm{tr}\left(A(t,\varepsilon)\right) dt\right) \tag{13}$$

where $c(\varepsilon)$ is a formal series in non-negative powers of ε. The definition of $w(x,\varepsilon)$ gives us that

$$w(x,\varepsilon) = \det\left(\hat{H}(x,\varepsilon)\right)\det\left(\exp Q(x,\varepsilon)\right) = \det\left(\hat{H}(x,\varepsilon)\right)\exp\left(\mathrm{tr}\ Q(x,\varepsilon)\right). \tag{14}$$

The equations (13) and (14) imply that

$$\det\left(\hat{H}(x,\varepsilon)\right) = c(\varepsilon)\exp\left[\varepsilon^{-h}\mathrm{tr}\left\{\int_0^x A(t,\varepsilon)dt - \varepsilon^h Q(x,\varepsilon)\right\}\right].$$

As $\det\left(\hat{H}(x,\varepsilon)\right)$ is a formal series in non-negative powers of ε, we have

$$\mathrm{tr}\left\{\sum_{r=0}^{h-1}\left(\int_0^x A_r(t)dt\right)\varepsilon^r - \sum_{r=0}^{h-1} Q_r(x)\varepsilon^r\right\} \equiv 0.$$

Using (11), we deduce that $c(\varepsilon)\varepsilon^{-L} = \tilde{c}(\varepsilon)$, where $\tilde{c}(\varepsilon)$ is a formal series in non-negative powers of ε. Therefore

$$\tilde{c}(\varepsilon)\exp\left[\mathrm{tr}\left\{\sum_{r=0}^{+\infty}\left(\int_0^x A_{r+h}(t)dt\right)\varepsilon^r\right\}\right] = \sum_{r=0}^{+\infty} h_{r+L}(x)\varepsilon^r. \tag{15}$$

Let $\varepsilon = 0$ in (15). We obtain

$$\tilde{c}(0)\exp\left[\mathrm{tr}\left\{\int_0^x A_h(t)dt\right\}\right] = h_L(x).$$

As the coefficient $h_L(x)$ is not identically null, $\tilde{c}(0) \neq 0$. This proves that $h_L(x) \neq 0$ for all x in $\mathcal{D}_x(0)$.

Consider now $\tilde{H}(x,\varepsilon) = \sum_{r=0}^{N}\hat{Y}_r(x)\varepsilon^r$, where N is a positive integer that we will choose later. We can write $\hat{H}$ in the form

$$\hat{H}(x,\varepsilon) = \tilde{H}(x,\varepsilon) + \varepsilon^{N+1}R(x,\varepsilon)$$

where $R(x,\varepsilon) = \sum_{r=0}^{+\infty}\hat{Y}_{r+N+1}(x)\varepsilon^r$. Then, the determinant of this expression has the form

$$\det\left(\hat{H}(x,\varepsilon)\right) = \det\left(\tilde{H}(x,\varepsilon)\right) + \varepsilon^{N+1}r(x,\varepsilon)$$

where r is a formal power series in ε with holomorphic coefficients in $\mathcal{D}_x(0)$. Suppose that $N \geq L$, then

$$\det\left(\tilde{H}(x,\varepsilon)\right) = \varepsilon^L \left(h_L(x) + \sum_{r=1}^{M} \tilde{h}_{r+L}(x)\varepsilon^r\right)$$

where M is a non-negative integer. Applying Cramer's formula for the inversion of matrices, we find that $\varepsilon^{-L}\tilde{H}(x,\varepsilon)$ admits as inverse matrix $\mathrm{Inv}(x,\varepsilon)$, which is a convergent series in powers of ε, for ε sufficiently small. We use the property that $1/\det \tilde{H}$ has a sense, because

$$\frac{1}{\det \tilde{H}} = \varepsilon^{-L}\left(\frac{1}{h_L(x) + \ldots}\right)$$

where $h_L(x) \neq 0$ for all x in $\mathcal{D}_x(0)$.

The transformation $y = \tilde{H}(x,\varepsilon)w$ of (7) yields the new system

$$\varepsilon^{h+L}\frac{dw}{dx} = \left(\varepsilon^{-L}\tilde{H}(x,\varepsilon)\right)^{-1}\left(A(x,\varepsilon)\tilde{H}(x,\varepsilon) - \varepsilon^h\frac{d\tilde{H}}{dx}(x,\varepsilon)\right)w. \qquad (16)$$

The coefficient matrix of the system (16)

$$B(x,\varepsilon) := \mathrm{Inv}(x,\varepsilon)\left(A(x,\varepsilon)\tilde{H}(x,\varepsilon) - \varepsilon^h\frac{d\tilde{H}}{dx}(x,\varepsilon)\right) =: \sum_{r=0}^{+\infty} B_r(x)\varepsilon^r$$

is a convergent series in powers of ε for x in $\mathcal{D}_x(0)$ and ε in $\mathcal{D}_\varepsilon(0)$.
If $\hat{W}$ is the formal fundamental solution of (16), then we have $\hat{Y}(x,\varepsilon) = \tilde{H}(x,\varepsilon)\hat{W}(x,\varepsilon)$ and $\hat{W}$ is of the form

$$\hat{W}(x,\varepsilon) = \left(I + \varepsilon^{N+1-L}\sum_{r=0}^{+\infty}\hat{W}_r(x)\varepsilon^r\right)\exp\left(Q(x,\varepsilon)\right),$$

where $\mathrm{Inv}(x,\varepsilon)R(x,\varepsilon) = \sum_{r=0}^{+\infty}\hat{W}_r(x)\varepsilon^r$. Consider now $\hat{w}(x,\varepsilon) = \hat{W}(x,\varepsilon)\exp\left(-Q(x,\varepsilon)\right)$, which is a formal solution of the system

$$\varepsilon^{h+L}\frac{dw}{dx} = B(x,\varepsilon)w - \varepsilon^{h+L}w\frac{dQ}{dx}(x,\varepsilon).$$

If we rewrite this equation using the series and if we choose N such that

$$N + 1 - L > h - 1 + L,$$

then we have necessarily

$$\sum_{r=L}^{h-1+L}\frac{dQ_{r-L}}{dx}(x)\varepsilon^r = \sum_{r=0}^{h-1+L} B_r(x)\varepsilon^r.$$

Finally, by identification of the coefficients of ε^r we have the theorem. ♣

This theorem can be generalized to a system (7) where the coefficient matrix is asymptotic of Gevrey order $1/h$ to a formal series. The proof of this general case is analogous.

3.5 Our hypothesis.

By the considerations of subsection 3.4, we can assume that we have the system of ordinary differential equations

$$\varepsilon^h \frac{dy}{dx} = A(x,\varepsilon)y \tag{17}$$

where the coefficient matrix $A(x,\varepsilon)$ is a holomorphic function of the variables x and ε, for x in a neighbourhood $\mathcal{D}_x(0)$ and ε in a sector $\mathcal{S}$. Furthermore $A(x,\varepsilon)$ is asymptotic of Gevrey order $1/h$ to the formal series

$$\hat{A}(x,\varepsilon) = \sum_{r=0}^{+\infty} \hat{A}_r(x)\varepsilon^r$$

as $\varepsilon \longrightarrow 0$ in $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$. The system (17) admits a formal fundamental solution $\hat{Y}$ of the form

$$\hat{Y}(x,\varepsilon) = \left(I + \sum_{r=h}^{+\infty} \hat{Y}_r(x)\varepsilon^r\right) \exp\left(\sum_{r=0}^{h-1} Q_r(x)\varepsilon^{r-h}\right), \tag{18}$$

where the coefficients $\hat{A}_r(x)$, $\hat{Y}_r(x)$ and $Q_r(x)$ satisfy the following properties

1. all the coefficients $\hat{A}_r(x)$, $\hat{Y}_r(x)$ and $Q_r(x)$ are analytic at $x = 0$,
2. the set $\mathcal{V}$ has the form
$$\mathcal{V} = \begin{cases} \{+\infty\} \\ \text{or} \\ \{s_1, \ldots, s_\ell\} \cup \{+\infty\} \end{cases}$$
where $0 \leq s_1 < \ldots < s_\ell \leq h-1$,
3. for all integer r in $\{0, \ldots, h-1\}$, $\dfrac{dQ_r}{dx}(x) = \hat{A}_r(x)$.

At this stage, if $\mathcal{V}$ does not reduce to $\{+\infty\}$ it is convenient to make the following simplification. For a system (17) of ℓ levels, using a transformation

$$y = w \exp\left(q_i(x,\varepsilon)I_n\right)$$

for any i in $\{1, \ldots, n\}$, the new system

$$\varepsilon^{h-s_1} \frac{dw}{dx} = B(x,\varepsilon)w$$

is always a system of ℓ levels, but the new set $\mathcal{V}$ is

$$\{0, s_2 - s_1, \ldots, s_\ell - s_1\} \cup \{+\infty\}.$$

Therefore we can suppose that for the system (17), $s_1 = 0$.

3.6 Essentially scalar systems.

In order to show W. Wasow's conjecture by a inductive process, we start with an essentially scalar system. In this case, the system (17) satisfies the hypotheses enumerated in section 3.5 and the set $\mathcal{V}$ is reduced to $\{+\infty\}$. Therefore the formal fundamental solution $\hat{Y}$ of system (17) has the form

$$\hat{Y}(x,\varepsilon) = \left(I + \sum_{r=h}^{+\infty} \hat{Y}_r(x)\varepsilon^r \right) \exp\left(q(x,\varepsilon)I\right), \tag{19}$$

where $q(x,\varepsilon)$ is a polynomial in negative powers of ε and with holomorphic coefficients in $\mathcal{D}_x(0)$.
By the transformation

$$y = w \exp\left(q(x,\varepsilon)I\right), \tag{20}$$

the initial system (17) is equivalent to the system

$$\frac{dw}{dx} = B(x,\varepsilon)w \tag{21}$$

with the coefficient matrix

$$B(x,\varepsilon) = \varepsilon^{-h}\left(A(x,\varepsilon) - \varepsilon^h \frac{dq}{dx}(x,\varepsilon) I \right), \tag{22}$$

which is as $A(x,\varepsilon)$ asymptotic of Gevrey order $1/h$ to the formal series

$$\hat{B}(x,\varepsilon) = \sum_{r=0}^{+\infty} \hat{A}_{r+h}(x)\varepsilon^r$$

as $\varepsilon \longrightarrow 0$ in $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$.

Theorem 3. *Let the system of formal differential equations*

$$\frac{dw}{dx} = \hat{B}(x,\varepsilon)w \tag{23}$$

associated with the system (21).
Then, under the conditions above, there exists a formal fundamental solution $\hat{W}$ of the system (23)

$$\hat{W}(x,\varepsilon) = I + \sum_{r=h}^{+\infty} \hat{W}_r(x)\varepsilon^r.$$

of Gevrey order $1/h$ uniformly for x in $\mathcal{D}_x(0)$.

Proof : According to (19) and the transformation (20), necessarily

$$I + \sum_{r=h}^{+\infty} \hat{Y}_r(x)\varepsilon^r$$

is a formal fundamental solution of (23). If we use the formal series writing for (23)

$$\hat{B}(x,\varepsilon)=\sum_{r=0}^{+\infty}\hat{B}_r(x)\varepsilon^r$$

and identify the coefficients, we yield

$$\begin{cases} \hat{B}_r(x)\equiv 0 & \text{for } r=0,\ldots,h-1, \\ \dfrac{d\hat{Y}_r}{dx}(x)=\hat{B}_r(x) & \text{for } r=h,\ldots,2h-1, \\ \\ \dfrac{d\hat{Y}_r}{dx}(x)=\hat{B}_r(x)+\displaystyle\sum_{l=h}^{r-h}\hat{B}_{r-l}(x)\hat{Y}_l(x) & \text{for } r=2h,2h+1,\ldots. \end{cases} \tag{24}$$

To prove our theorem 3, we show that there exists one formal series $\hat{W}$ solution of the system (24) and of Gevrey order $1/h$ uniformly for x in $\mathcal{D}_x(0)$. Given a real $\tilde{M}$ bigger than 1 and a positive real $\tilde{N}$ (we will define their later precisely), therefore we have

$$|I|\leq\tilde{M}.$$

Let

$$\hat{W}(x,\varepsilon)=I+\sum_{r=h}^{+\infty}\hat{W}_r(x)\varepsilon^r$$

the solution of (24) such that $\hat{W}(0,\varepsilon)=I$, then by integration of $\dfrac{d\hat{W}_r}{dx}(x)$ we obtain the following inequalities

$$\left|\hat{W}_r(x)\right|\leq|x|\int_0^1\left|\frac{d\hat{W}_r}{dx}(xt)\right|dt\leq x_0\int_0^1\left|\frac{d\hat{W}_r}{dx}(xt)\right|dt$$

for all x in $\mathcal{D}_x(0)$ and for all integer $r\geq h$, here x_0 represents the radius of the open disk $\mathcal{D}_x(0)$. By equation (24) and using the Gevrey property of $\hat{B}(x,\varepsilon)$, there exists two constants M and N such that for $r=h,\ldots,2h-1$,

$$\left|\hat{W}_r(x)\right|\leq x_0MN^r\Gamma\left(1+\frac{r}{h}\right)$$

for all x in $\mathcal{D}_x(0)$. We deduce that

$$\left|\hat{W}_r(x)\right|\leq\tilde{M}\tilde{N}^r\Gamma\left(1+\frac{r}{h}\right)$$

for all x in $\mathcal{D}_x(0)$, where $\tilde{M}$ and $\tilde{N}$ satisfy $N<\tilde{N}$ and $x_0M<\tilde{M}$.
By induction, we suppose that for $l=0$ to r we have

$$\left|\hat{W}_l(x)\right|\leq\tilde{M}\tilde{N}^l\Gamma\left(1+\frac{l}{h}\right)$$

for all x in $\mathcal{D}_x(0)$. According to the inequalities before, we can suppose that $r \geq 2h$. Then using equation (24), we yield

$$\left|\hat{W}_{r+1}(x)\right| \leq x_0 M\tilde{M}\tilde{N}^{r+1}\sum_{l=0}^{r+1}\left(\frac{N}{\tilde{N}}\right)^l \Gamma\left(1+\frac{r+1-l}{h}\right)\Gamma\left(1+\frac{l}{h}\right)$$

for all x in $\mathcal{D}_x(0)$. Using the Bêta integral [Bal94], we show that

$$\begin{aligned}\left|\hat{W}_{r+1}(x)\right| \leq\; & x_0 M\tilde{M}\tilde{N}^{r+1}\left(1+\frac{r+1}{h}\right)\Gamma\left(1+\frac{r+1}{h}\right)\\ & \times\left\{\sum_{l=0}^{r+1}\left(\frac{N}{\tilde{N}}\right)^l\int_0^1 (1-x)^{l/h}\, x^{(r+1-l)/h}dx\right\}\end{aligned}$$

for all x in $\mathcal{D}_x(0)$. Finally, using the inequality

$$\int_0^1 (1-x)^{l/h}\, x^{(r+1-l)/h}dx \leq \frac{2h}{r+1+2h} \qquad \forall l=0,\ldots,r+1$$

and rectifying the reals $\tilde{M}$, $\tilde{N}$ and x_0 if necessary, we obtain

$$\left|\hat{W}_{r+1}(x)\right| \leq \tilde{M}\tilde{N}^{r+1}\Gamma\left(1+\frac{r+1}{h}\right)$$

for all x in $\mathcal{D}_x(0)$. This proves our theorem. ♣

Knowing that the formal solution $\hat{W}$ of the formal system (23) is of Gevrey order $1/h$ uniformly for x in $\mathcal{D}_x(0)$, we prove the following theorem.

Theorem 4. *There exists a fundamental solution W of the system (21), holomorphic in x and ε for x in a neighbourhood $\mathcal{D}_x(0)$ of 0, ε in a sector $\mathcal{S}$ and which is asymptotic of Gevrey order $1/h$ to the formal fundamental solution $\hat{W}$ of theorem 3 as $\varepsilon \longrightarrow 0$ in $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$.*

<u>Proof</u>: The two formal series $\hat{W}$ and $\hat{B}$ of theorem 3, are of Gevrey order $1/h$ uniformly for x in $\mathcal{D}_x(0)$. Thus we take their <u>formal Borel transforms of Gervey order $1/h$</u>

$$\left(\mathcal{B}\hat{W}\right)(x,t) := \sum_{r=0}^{+\infty}\frac{\hat{W}_r(x)}{\Gamma(1+r/h)}t^r$$

and

$$\left(\mathcal{B}\hat{B}\right)(x,t) := \sum_{r=0}^{+\infty}\frac{\hat{B}_r(x)}{\Gamma(1+r/h)}t^r$$

for all x in $\mathcal{D}_x(0)$. The first series is analytic for $|t| < 1/\tilde{N}$, as the second is analytic for $|t| < 1/\tilde{N}_B$. We choose a real T such as $0 < T < \min\left(1/\tilde{N}, 1/\tilde{N}_B\right)$ and consider for $\mathcal{B}\hat{W}$ and $\mathcal{B}\hat{B}$ their <u>truncated Laplace transforms of Gevrey order $1/h$</u>

$$\left(\mathcal{L}\left(\mathcal{B}\hat{W}\right)\right)(x,\varepsilon) := \varepsilon^{-h}\int_0^{\left(e^{i\theta}/T\right)^h}\left(\mathcal{B}\hat{W}\right)\left(x,u^{1/h}\right)\exp\left\{-\frac{u}{\varepsilon^h}\right\}du$$

$$+\left(\mathcal{B}\hat{W}\right)\left(x,\frac{e^{i\theta}}{T}\right)\exp\left\{-\left(\frac{e^{i\theta}}{T\varepsilon}\right)^h\right\}$$

and

$$\left(\mathcal{L}\left(\mathcal{B}\hat{B}\right)\right)(x,\varepsilon):=\varepsilon^{-h}\int_0^{\left(e^{i\theta}/T\right)^h}\left(\mathcal{B}\hat{B}\right)(x,u^{1/h})\exp\left\{-\frac{u}{\varepsilon^h}\right\}du$$
$$+\left(\mathcal{B}\hat{B}\right)\left(x,\frac{e^{i\theta}}{T}\right)\exp\left\{-\left(\frac{e^{i\theta}}{T\varepsilon}\right)^h\right\}$$

for x in $\mathcal{D}_x(0)$ and for ε in $\mathcal{S}$ satisfying

$$\operatorname{Re}\left\{\left(\frac{e^{i\theta}}{T\varepsilon}\right)^h\right\}>0 \tag{25}$$

where θ is suitably chosen. We suppose that the inequality (25) is true for all ε in $\mathcal{S}$, otherwise we reduce the sector $\mathcal{S}$. By Watson's lemma we have

$$\left(\mathcal{L}\left(\mathcal{B}\hat{W}\right)\right)(x,\varepsilon)\sim\hat{W}(x,\varepsilon)=\sum_{r=0}^{+\infty}\hat{W}_r(x)\varepsilon^r$$

and

$$\left(\mathcal{L}\left(\mathcal{B}\hat{B}\right)\right)(x,\varepsilon)\sim\hat{B}(x,\varepsilon)=\sum_{r=0}^{+\infty}\hat{B}_r(x)\varepsilon^r,$$

as $\varepsilon\longrightarrow 0$ in $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$. We define the <u>remainder term</u>

$$R(x,\varepsilon):=-\frac{d\mathcal{L}\left(\mathcal{B}\hat{W}\right)}{dx}(x,\varepsilon)+B(x,\varepsilon)\mathcal{L}\left(\mathcal{B}\hat{W}\right)(x,\varepsilon)$$

for x in $\mathcal{D}_x(0)$ and ε in $\mathcal{S}$. It is known (cf. [Bal94]), using Cauchy's formula, that

$$\frac{d\mathcal{L}\left(\mathcal{B}\hat{W}\right)}{dx}(x,\varepsilon)\sim\frac{d\hat{W}}{dx}(x,\varepsilon)$$

as $\varepsilon\longrightarrow 0$ in $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$. Therefore, we check that $R(x,\varepsilon)$ is <u>flat of Gevrey order $1/h$</u>, this means that R is <u>exponentially small</u>. Thus there exists two positive constants K and L such that

$$|R(x,\varepsilon)|\le K\exp\left(-L\,|\varepsilon|^{-h}\right) \tag{26}$$

for x in $\mathcal{D}_x(0)$ and ε in $\mathcal{S}$. Finally, $\mathcal{L}\left(\mathcal{B}\hat{W}\right)$ is a <u>quasi-solution</u> of the system (21).

To find a fundamental solution W of the system (21) near the quasi-solution $\mathcal{L}\left(\mathcal{B}\hat{W}\right)$, we define

$$\Delta(x,\varepsilon):=W(x,\varepsilon)-\left(\mathcal{L}\left(\mathcal{B}\hat{W}\right)\right)(x,\varepsilon).$$

Therefore the system (21) becomes

$$\frac{d\Delta}{dx} = B(x,\varepsilon)\Delta + R(x,\varepsilon). \tag{27}$$

Now we will show that there exists a solution Δ of (27) for x in $\mathcal{D}_x(0)$ and ε in $\mathcal{S}$ which is exponentially small. We fix ε in $\mathcal{S}$, then the theorem of local existence and uniqueness for linear systems of ordinary differential equations proves the existence of a unique solution $\Delta_\varepsilon(x) := \Delta(x,\varepsilon)$ of the system (27) satisfying the initial value $\Delta_\varepsilon(0) = 0$, in $\mathcal{D}_x(0)$. The following stage consists in verifying that the solution $\Delta(x,\varepsilon)$ is exponentially small.
As B is holomorphic there exists a positive real γ such that

$$|B(x,\varepsilon)| \leq \gamma$$

for x in $\mathcal{D}_x(0)$ and ε in $\mathcal{S}$. Using our estimation (26), we apply Gronwall's lemma to

$$\left|\frac{d\Delta}{dx}(x,\varepsilon)\right| \leq \gamma\,|\Delta(x,\varepsilon)| + K\exp\left(-L\,|\varepsilon|^{-h}\right),$$

and obtain that

$$|\Delta(x,\varepsilon)| \leq \tilde{K}\exp\left(-L\,|\varepsilon|^{-h}\right)$$

for some positive real $\tilde{K}$, for x in $\mathcal{D}_x(0)$ and ε in $\mathcal{S}$. For more details, we can also see the paper [CDRSS00] or [Bal94]. This completes the proof. ♣

3.7 Systems of one level.

We still have the hypotheses enumerated in section 3.5., with a set $\mathcal{V}$ of the form $\{0\}\cup\{+\infty\}$. To prove W. Wasow's conjecture in this case, we will split the initial system into a set of independent systems which are essentially scalar.

We define two sets

$$N_1 := \{i \in \{1,\ldots,n\} \mid q_i(x,\varepsilon) \equiv q_1(x,\varepsilon)\} \tag{28}$$

and

$$N_2 := \{1,\ldots,n\}\setminus N_1.$$

We denote by n_1, respectively n_2, the number of their elements. Without loss of generality we can assume that the diagonal matrix Q has the form

$$Q(x,\varepsilon) = \begin{pmatrix} Q_1(x,\varepsilon) & 0 \\ 0 & Q_2(x,\varepsilon) \end{pmatrix} \tag{29}$$

where $Q_i(x,\varepsilon) = \mathrm{diag}_{j\in N_i}\left(q_j(x,\varepsilon)\right)$ for $i = 1,2$.
With the following two theorems we prove that there exists a holomorphic transformation $y = P(x,\varepsilon)w$, that splits (17) according to the size of the matrices Q_1 and Q_2. Remark that in the case of a system of one level the matrix Q_1 can be written as $q_1(x,\varepsilon)I_{n_1}$

3.7.1 Formal splitting theorem.

Theorem 5. *Under the assumptions of theorem 1., there exists a unique formal series $\hat{P}$ of the form $\hat{P}(x,\varepsilon) = I_n + \sum_{r=h}^{+\infty} \hat{P}_r(x)\varepsilon^r$, where the matrix $\hat{P}(x,\varepsilon)$ is a block-matrix*

$$\hat{P}(x,\varepsilon) = \begin{pmatrix} I_{n_1} & \hat{P}^{12}(x,\varepsilon) \\ \hat{P}^{21}(x,\varepsilon) & I_{n_2} \end{pmatrix},$$

and such that the formal transformation $y = \hat{P}(x,\varepsilon)w$ changes the formal system

$$\varepsilon^h \frac{dy}{dx} = \hat{A}(x,\varepsilon)y \tag{30}$$

into the new formal system

$$\varepsilon^h \frac{dw}{dx} = \hat{B}(x,\varepsilon)w, \tag{31}$$

where

$$\hat{B}(x,\varepsilon) := \sum_{r=0}^{+\infty} \hat{B}_r(x)\varepsilon^r,$$

with the following properties

(i) all the coefficients $\hat{B}_r(x)$ are block-diagonal matrices

$$\hat{B}_r(x) = \begin{pmatrix} \hat{B}_r^{11}(x) & 0 \\ 0 & \hat{B}_r^{22}(x) \end{pmatrix}$$

and for all integer r in $\{0,\dots,h-1\}$, $\hat{B}_r(x) \equiv \hat{A}_r(x)$,

(ii) all the coefficients $\hat{B}_r(x)$ and $\hat{P}_r(x)$ have no singularity at $x = 0$,

(iii) the formal series $\hat{P}$ and $\hat{B}$ are of Gevrey order $1/h$.

Proof : If $\hat{P}$ is the formal solution of the system of ordinary differential equations

$$\varepsilon^h \frac{d\hat{P}}{dx} = \hat{A}(x,\varepsilon)\hat{P} - \hat{P}\hat{B}(x,\varepsilon), \tag{32}$$

then the formal transformation $y = \hat{P}(x,\varepsilon)w$ changes the formal system (30) into the formal system (31). Insertion of the formal series $\sum_{r=0}^{+\infty} \hat{P}_r(x)\varepsilon^r$ into (32) and formal identification of like powers of ε produce the formulas

$$\begin{cases} \displaystyle\sum_{l=0}^{r} \Big(\hat{A}_{r-l}(x)\hat{P}_l(x) - \hat{P}_l(x)\hat{B}_{r-l}(x)\Big) \equiv 0 & \text{for } r = 0,\dots,h-1, \\ \displaystyle\frac{d\hat{P}_{r-h}}{dx}(x) = \sum_{l=0}^{r} \Big(\hat{A}_{r-l}(x)\hat{P}_l(x) - \hat{P}_l(x)\hat{B}_{r-l}(x)\Big) & \text{for } r \geq h. \end{cases} \tag{33}$$

With $\hat{P}_0(x) := I_n$ and $\hat{B}_0(x) := \hat{A}_0(x)$, equation (33) is satisfied for $r = 0$. To prove our theorem 5, we must choose

$$\begin{cases} \hat{P}_r(x) :\equiv 0 & \text{for } r = 1, \ldots, h-1, \\ \hat{B}_r(x) :\equiv \hat{A}_r(x) & \text{for } r = 1, \ldots, h-1. \end{cases}$$

For $r = h$, formula (33) reduces to

$$\hat{A}_0(x)\hat{P}_h(x) - \hat{P}_h(x)\hat{A}_0(x) = \hat{B}_h(x) - \hat{A}_h(x).$$

Using the following block-matrix notations

$$\hat{A}_r(x) = \begin{pmatrix} \hat{A}_r^{11}(x) & 0 \\ 0 & \hat{A}_r^{22}(x) \end{pmatrix} \quad \text{for } r = 0, \ldots, h-1,$$

and

$$\hat{A}_r(x) = \begin{pmatrix} \hat{A}_r^{11}(x) & \hat{A}_r^{12}(x) \\ \hat{A}_r^{21}(x) & \hat{A}_r^{22}(x) \end{pmatrix}, \quad \hat{P}_r(x) = \begin{pmatrix} 0 & \hat{P}_r^{12}(x) \\ \hat{P}_r^{21}(x) & 0 \end{pmatrix},$$

$$\hat{B}_r(x) = \begin{pmatrix} \hat{B}_r^{11}(x) & 0 \\ 0 & \hat{B}_r^{22}(x) \end{pmatrix} \quad \text{for } r \geq h,$$

we obtain the system

$$\begin{cases} \hat{B}_h^{11}(x) = \hat{A}_h^{11}(x) \\ \\ \hat{B}_h^{22}(x) = \hat{A}_h^{22}(x) \\ \\ \hat{P}_h^{12}(x)\hat{A}_0^{22}(x) - \hat{A}_0^{11}(x)\hat{P}_h^{12}(x) = \hat{A}_h^{12}(x) \\ \\ \hat{P}_h^{21}(x)\hat{A}_0^{11}(x) - \hat{A}_0^{22}(x)\hat{P}_h^{21}(x) = \hat{A}_h^{21}(x). \end{cases} \tag{34}$$

With the notations

$$\hat{P}_h^{12}(x) = \left(\hat{P}_h^{12}(x)_{ij}\right)_{\substack{i = 1, \ldots, n_1 \\ j = 1, \ldots, n_2}}, \quad \hat{A}_h^{12}(x) = \left(\hat{A}_h^{12}(x)_{ij}\right)_{\substack{i = 1, \ldots, n_1 \\ j = 1, \ldots, n_2}}$$

and

$$\hat{A}_0^{11}(x) = \operatorname{diag}_{i=1}^{n_1}\left(\hat{A}_0^{11}(x)_{ii}\right), \quad \hat{A}_0^{22}(x) = \operatorname{diag}_{i=1}^{n_2}\left(\hat{A}_0^{22}(x)_{ii}\right),$$

the solution $\hat{P}_h^{12}$ of the system (34) is given by

$$\left(\hat{A}_0^{22}(x)_{jj} - \hat{A}_0^{11}(x)_{ii}\right)\hat{P}_h^{12}(x)_{ij} = \hat{A}_h^{12}(x)_{ij}$$

for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. We have similarly an expression for the solution $\hat{P}_h^{21}$ of the system (34).
Let us remind that

$$\operatorname{val}_\varepsilon\left(\varepsilon^h \frac{dq_{1i}}{dx}(x, \varepsilon)\right) > 0 \qquad \text{for } i = 1, \ldots, n_1 \tag{35}$$

and

$$\operatorname{val}_\varepsilon \left(\varepsilon^h \frac{dq_{1j}}{dx}(x,\varepsilon) \right) = 0 \qquad \text{for } j = n_1+1,\dots,n, \tag{36}$$

therefore the difference $\hat{A}_0^{22}(x)_{jj} - \hat{A}_0^{11}(x)_{ii}$ is not identically null for all i in $\{1,\dots,n_1\}$ and j in $\{1,\dots,n_2\}$. Observe that we have not supposed that $x=0$ is an asymptotically simple point, consequently $\hat{P}_h(x)$ might have a singularity at $x=0$.
Suppose that we know $\hat{P}_l$ and $\hat{B}_l$ for $l = h,\dots,r-1$ and that they have the block-matrix form of theorem 5. The system of equations for $\hat{P}_r$ and $\hat{B}_r$ is

$$\hat{A}_0(x)\hat{P}_r(x) - \hat{P}_r(x)\hat{A}_0(x) = \hat{B}_r(x) - \hat{F}_r(x)$$

where

$$\hat{F}_r(x) := \hat{A}_r(x) - \frac{d\hat{P}_{r-h}}{dx}(x) + \sum_{l=h}^{r-1} \hat{A}_{r-l}(x)\hat{P}_l(x) - \hat{P}_l(x)\hat{B}_{r-l}(x).$$

The term $\hat{F}_r(x)$ contains only known coefficients. By using the block-matrix notation, we obtain a system like (34) and an expression for $\hat{P}_r(x)$ of the same form as for $\hat{P}_h(x)$. So by induction, $\hat{P}$ and $\hat{B}$ are uniquely determined but they might have singularities at $x=0$. Using the hypotheses of our theorem, we show that the coefficicents of $\hat{P}$ and $\hat{B}$ have no singularities at $x=0$.

We denote by $\hat{Y}(x,\varepsilon) =: \hat{H}(x,\varepsilon)\exp\left(Q(x,\varepsilon)\right)$ the formal fundamental solution (18) of the formal system (30). The formal system (31) which is a block-diagonal system, admits a formal fundamental solution $\hat{W}$ of the same form

$$\hat{W}(x,\varepsilon) = \begin{pmatrix} \hat{W}^{11}(x,\varepsilon) & 0 \\ 0 & \hat{W}^{22}(x,\varepsilon) \end{pmatrix}.$$

Therefore we obtain

$$\hat{H}(x,\varepsilon) = \hat{P}(x,\varepsilon)\hat{W}(x,\varepsilon)\exp\left(-Q(x,\varepsilon)\right).$$

Writing $\hat{H}$ in block-matrix notation

$$\hat{H}(x,\varepsilon) = \begin{pmatrix} \hat{H}^{11}(x,\varepsilon) & \hat{H}^{12}(x\varepsilon) \\ \hat{H}^{21}(x,\varepsilon) & \hat{H}^{22}(x,\varepsilon) \end{pmatrix},$$

according to (18), we have

$$\begin{cases} \hat{H}^{11}(x,0) = I_{n_1} \\ \hat{H}^{22}(x,0) = I_{n_2}. \end{cases}$$

Hence the formal matrices $\hat{H}^{11}$ and $\hat{H}^{22}$ admit formal matrix series as inverse. Now, if we write $\hat{H}$ as a product of two block-matrices of the form

$$\begin{pmatrix} I_{n_1} & \star \\ \star & I_{n_2} \end{pmatrix} \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix},$$

then we find that

$$\hat{H}(x,\varepsilon)=\begin{pmatrix} I_{n_1} & \hat{H}^{12}(x,\varepsilon)\left(\hat{H}^{22}(x,\varepsilon)\right)^{-1} \\ \hat{H}^{21}(x,\varepsilon)\left(\hat{H}^{11}(x,\varepsilon)\right)^{-1} & I_{n_2} \end{pmatrix}\begin{pmatrix} \hat{H}^{11}(x,\varepsilon) & 0 \\ 0 & \hat{H}^{22}(x,\varepsilon) \end{pmatrix}$$

and so by identification

$$\begin{cases} \hat{P}^{12}(x,\varepsilon)=\hat{H}^{12}(x,\varepsilon)\left(\hat{H}^{22}(x,\varepsilon)\right)^{-1} \\ \hat{P}^{21}(x,\varepsilon)=\hat{H}^{21}(x,\varepsilon)\left(\hat{H}^{11}(x,\varepsilon)\right)^{-1}. \end{cases}$$

Therefore, the coefficients of the formal series $\hat{P}$ but also of $\hat{B}$ have no singularity at $x=0$.

Using block-matrix notation for equation (32), we obtain the following system

$$\begin{cases} \hat{B}^{11}=\hat{A}^{11}+\hat{A}^{12}\hat{P}^{21} \\ \hat{B}^{22}=\hat{A}^{22}+\hat{A}^{21}\hat{P}^{12} \\ \varepsilon^h\dfrac{d\hat{P}^{12}}{dx}=\hat{A}^{11}\hat{P}^{12}-\hat{P}^{12}\hat{A}^{22}-\hat{P}^{12}\hat{A}^{21}\hat{P}^{12}+\hat{A}^{12} \\ \varepsilon^h\dfrac{d\hat{P}^{21}}{dx}=\hat{A}^{22}\hat{P}^{21}-\hat{P}^{21}\hat{A}^{11}-\hat{P}^{21}\hat{A}^{12}\hat{P}^{21}+\hat{A}^{21}. \end{cases} \tag{37}$$

We consider the third system of differential equations (we can do the same for the last system). Therefore we have the system

$$\varepsilon^h\frac{d\hat{T}}{dx}=\hat{A}^{11}\hat{T}-\hat{T}\hat{A}^{22}-\hat{T}\hat{A}^{21}\hat{T}+\hat{A}^{12}. \tag{38}$$

We denote by $\hat{\mathcal{T}}$ the vectorial form of the matrix $\hat{T}$

$$\hat{\mathcal{T}}(x,\varepsilon):=\left(\hat{\mathcal{T}}(x,\varepsilon)_i\right)_{i=1,\dots,n_1n_2}:=\begin{pmatrix} \hat{T}(x,\varepsilon)_{1,1} \\ \dots \\ \hat{T}(x,\varepsilon)_{1,n_2} \\ \hat{T}(x,\varepsilon)_{2,1} \\ \dots \\ \hat{T}(x,\varepsilon)_{n_1,n_2} \end{pmatrix}$$

where the $\hat{T}(x,\varepsilon)_{i,j}$ are the elements of $\hat{T}$ and also we note by $\hat{\mathcal{A}}^{12}$ the vectorial form of $\hat{A}^{12}$. Thus the system (38) is equivalent to the system

$$\varepsilon^h\frac{d\hat{\mathcal{T}}}{dx}=\hat{\mathcal{A}}(x,\varepsilon)\hat{\mathcal{T}}+\sum_{|\vec{v}|=2}f_{\vec{v}}(x,\varepsilon)\hat{\mathcal{T}}^{\vec{v}}+\hat{\mathcal{A}}^{12}(x,\varepsilon) \tag{39}$$

where $\hat{\mathcal{A}}$ is the matrix of the application

$$\begin{array}{lcl} \mathcal{M}_{n_1,n_2} & \longrightarrow & \mathcal{M}_{n_1,n_2} \\ \hat{T} & \longmapsto & \hat{A}^{11}(x,\varepsilon)\hat{T} - \hat{T}\hat{A}^{22}(x,\varepsilon). \end{array}$$

Here $\mathcal{M}_{n_1,n_2}$ represents the space of the matrices of n_1 rows and n_2 columns whose entries are formal power series of ε with coefficients that are holomorphic in $\mathcal{D}_x(0)$. We have used the following notations

$$\vec{v} = (v_1, \ldots, v_{n_1 n_2}), \qquad |\vec{v}| = \sum_{i=1}^{n_1 n_2} v_i$$

where the v_i are non-negative integer and

$$\hat{\mathcal{T}}^{\vec{v}} = \hat{\mathcal{T}}^{v_1}\hat{\mathcal{T}}^{v_2}\ldots\hat{\mathcal{T}}^{v_{n_1 n_2}}.$$

For the next step the structure of the leading matrix $\hat{\mathcal{A}}(x,0) = \hat{\mathcal{A}}_0(x)$ is of great importance. The leading matrix is the diagonal matrix defined by

$$\hat{\mathcal{A}}_0(x) := \mathrm{diag}_{i=1}^{n_1}\left(\mathrm{diag}_{j=1}^{n_2}\left(\hat{A}_0^{11}(x)_{ii} - \hat{A}_0^{22}(x)_{jj}\right)\right). \tag{40}$$

Now, to prove the Gevrey property of $\hat{\mathcal{T}}$ we are going to use results of the article of M. Canalis-Durand et al. [CDRSS00]. They consider systems of singularly perturbed ordinary differential equations with a vector $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_m)$ of m additional parameters

$$\varepsilon^h \frac{d\hat{\mathcal{T}}}{dx} = \hat{\mathcal{A}}(x,\varepsilon)\hat{\mathcal{T}} + \mathcal{B}_0(x)\hat{a} + \sum_{|\vec{v}|=2} \hat{f}_{\vec{v}}(x,\varepsilon)\hat{\mathcal{T}}^{\vec{v}} + \hat{\mathcal{A}}^{12}(x,\varepsilon) \tag{41}$$

where $\mathcal{B}_0$ is a matrix of the space $\mathcal{M}_{n_1 n_2,m}$. To determine in our case the integer m we consider the determinant of the leading matrix $\hat{\mathcal{A}}_0(x)$

$$\det\left(\hat{\mathcal{A}}_0(x)\right) = \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\left(\hat{A}_0^{11}(x)_{ii} - \hat{A}_0^{22}(x)_{jj}\right)$$

and write this determinant as

$$\det\left(\hat{\mathcal{A}}_0(x)\right) = x^m k(x) \tag{42}$$

where m is a non-negative integer and $k(x)$ a holomorphic function in $\mathcal{D}_x(0)$ with $k(0) \neq 0$. According to [CDRSS00], the integer m of (42) will be the number of additional parameters in equation (41). Remark that m is positive because we have a system of one level. Next, we take the <u>Smith normal form</u> of $\hat{\mathcal{A}}_0(x)$

$$\hat{\mathcal{A}}_0(x) =: \hat{\mathcal{C}}(x)x^M\hat{\mathcal{D}}(x)$$

where $\hat{\mathcal{C}}(0)$, $\hat{\mathcal{D}}(0)$ are invertible matrices and M is a diagonal matrix such that

$$\begin{cases} M = \mathrm{diag}_{j=1}^{n_1 n_2}(m_i) \\ 0 \leq m_1 \leq \ldots \leq m_{n_1 n_2} \\ m_1 + \ldots + m_{n_1 n_2} = m. \end{cases}$$

Expression (40) can be rewritten as

$$\hat{\mathcal{A}}_0(x) = \mathrm{diag}_{i=1}^{n_1}\left(\mathrm{diag}_{j=1}^{n_2}\left(x^{m_{ij}}\hat{\mathcal{D}}(x)_{ij}\right)\right)$$

where for all i in $\{1,\ldots,n_1\}$ and j in $\{1,\ldots,n_2\}$ we have $x^{m_{ij}}\hat{\mathcal{D}}(x)_{ij} := \hat{A}_0^{11}(x)_{ii} - \hat{A}_0^{22}(x)_{jj}$ with $\hat{\mathcal{D}}(0)_{ij} \neq 0$. Therefore, with

$$M := \mathrm{diag}_{i=1}^{n_1}\left(\mathrm{diag}_{j=1}^{n_2} m_{ij}\right), \quad \hat{\mathcal{D}}(x) := \mathrm{diag}_{i=1}^{n_1}\left(\mathrm{diag}_{j=1}^{n_2}\hat{\mathcal{D}}(x)_{ij}\right), \quad \hat{\mathcal{C}}(x) := I_{n_1 n_2},$$

we yield a Smith normal form of $\hat{\mathcal{A}}_0(x)$.

For our method of proof, we choose a matrix $\mathcal{B}_0$ which satisfies the following hypothesis of transversality : *the* $m \times m$ *matrix with the rows* $\left[\left(S^k\mathcal{B}_0\right)(0)\right]_l$, *for* $l = 1,\ldots,n_1n_2$ *if* $m_l \neq 0$ *and*
$k = 0,\ldots,m_l - 1$, *is invertible.*
Here S is the shift operator defined by

$$\begin{aligned} Sf(x) &= \frac{f(x)-f(0)}{x} \quad \text{if } x \neq 0 \\ Sf(0) &= \frac{df}{dx}(0) \end{aligned}$$

for f holomorphic in $\mathcal{D}_x(0)$ and $[\,]_l$ denotes the l^{th} row of a matrix.
Let l_0 the non-negative integer such that for all l in $\{1,\ldots,l_0\}$ we have $m_l = 0$ but $m_{l_0+1} \neq 0$. Then, the matrix $\mathcal{B}_0(x)$ below satisfies the hypothesis of transversality

$$\begin{array}{cl}
\begin{array}{ccc}
\overbrace{\hphantom{0 \ \cdots \ \frac{x^{m_{l_0+1}-1}}{(m_{l_0+1}-1)!}}}^{m_{l_0+1}} & \overbrace{\hphantom{1 \ \cdots \ \frac{x^{m_{l_0+2}-1}}{(m_{l_0+2}-1)!}}}^{m_{l_0+2}} \qquad\qquad & \overbrace{\hphantom{1 \ \cdots \ \frac{x^{m_{n_1n_2}-1}}{(m_{n_1n_2}-1)!}}}^{m_{n_1n_2}}
\end{array} & \\
\underbrace{\begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
1 & \cdots & \frac{x^{m_{l_0+1}-1}}{(m_{l_0+1}-1)!} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & \frac{x^{m_{l_0+2}-1}}{(m_{l_0+2}-1)!} & \vdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & \frac{x^{m_{n_1n_2}-1}}{(m_{n_1n_2}-1)!}
\end{pmatrix}}_{m} & \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} l_0 \\ \\ \\ \\ \\ \\ \end{array} .
\end{array}$$

In this case, the matrix of $\mathcal{M}_{m,m}$ determined in the hypothesis of transversality is the matrix identity I_m.

To the formal system (41), we associate the holomorphical system

$$\varepsilon^h \frac{d\mathcal{T}}{dx} = \mathcal{A}(x,\varepsilon)\mathcal{T} + \mathcal{B}_0(x)a + \sum_{|\vec{v}|=2} f_{\vec{v}}(x,\varepsilon)\mathcal{T}^{\vec{v}} + \mathcal{A}^{12}(x,\varepsilon) \tag{43}$$

where $\mathcal{A}$, $f_{\vec{v}}$ and $\mathcal{A}^{12}$ are defined by the coefficient matrix A of (17). Therefore, the system (43), where we denote by $\mathcal{F}(x,\mathcal{T},a,\varepsilon)$ the right hand side, satisfies the following hypotheses

- the function $\mathcal{F}$ is holomorphic in the variables x, $\mathcal{T}$ and a in the neighbourhood $\mathcal{D} = \mathcal{D}_x(0) \times \mathcal{D}_{\mathcal{T}}(0) \times \mathcal{D}_a(0)$ of $(0,0,0)$ and $\mathcal{F}(x,0,0,0) = 0$ for all x in $\mathcal{D}_x(0)$,
- the function $\mathcal{F}$ is holomorphic for ε in $\mathcal{S}$ and $\mathcal{F}$ is asymptotic of Gevrey order $1/h$ to the formal series
$$\sum_{k=0}^{+\infty} \hat{\mathcal{F}}_k\,(x,\mathcal{T},a)\,\varepsilon^k$$
as $\varepsilon \longrightarrow 0$ in $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$. The functions $\hat{\mathcal{F}}_k$ are holomorphic in $\mathcal{D}$,
- the leading matrix is given by
$$\hat{\mathcal{A}}_0(x) = \frac{\partial \mathcal{F}}{\partial \mathcal{T}}\,(x,0,0,0)$$
and satisfies $\det\left(\hat{\mathcal{A}}_0(x)\right) = x^m k(x)$ with $k(0) \neq 0$,
- the matrix $\mathcal{B}_0(x)$ satisfies the hypothesis of transversality.

Theorem (M. Canalis-Durand, J.P. Ramis, R. Schäfke, Y. Sibuya) *The systems (43) and (41) have a unique formal solution*
$$\hat{\mathcal{T}}(x,\varepsilon) = \sum_{r=1}^{+\infty} \hat{\mathcal{T}}_r(x)\varepsilon^r, \qquad \hat{a}(\varepsilon) = \sum_{r=1}^{+\infty} \hat{c}_r\varepsilon^r, \tag{44}$$
where the $\hat{\mathcal{T}}_r$ are holomorphic in $\mathcal{D}_x(0)$ and the $\hat{c}_r$ are in $\mathbb{C}^m$. More, $\hat{\mathcal{T}}$ is of Gevrey order $1/h$ uniformly for x in $\mathcal{D}_x(0)$ and $\hat{a}$ is of Gevrey order $1/h$ too.

This theorem is a key of our proof, I encourage the interested reader to look for more details in [CDRSS00].
Thanks to the previous result, we know that the formal system (41) admits a unique couple $\left(\hat{\mathcal{T}},\hat{a}\right)$ of solutions. But in the first part of our proof we have checked the uniqueness of the formal solution $\hat{\mathcal{T}}$ of the formal system (39). If we choose $\hat{a} = 0$, system (41) is formally equivalent to (39), therefore $\left(\hat{\mathcal{T}},0\right)$ is the unique formal solution of (41). Using again the theorem above we conclude that $\hat{\mathcal{T}}$, consequently $\hat{P}$ is of Gevrey order $1/h$ uniformly for x in $\mathcal{D}_x(0)$.

Consider the block-diagonal matrix $\hat{B}$. The two independent systems for $\hat{B}$ are
$$\left\{\begin{array}{l} \hat{B}^{11} = \hat{A}^{11} + \hat{A}^{12}\hat{P}^{21} \\ \hat{B}^{22} = \hat{A}^{22} + \hat{A}^{21}\hat{P}^{12}. \end{array}\right. \tag{45}$$
As the coefficients $\hat{P}_r(x)$ of the formal series $\hat{P}$ have no singularity at $x=0$, it is obvious, according to (45), that the coefficients $\hat{B}_r(x)$ of the formal series $\hat{B}$ have also no singularity at $x=0$. By means of the Gevrey properties of the series $\hat{A}$ and $\hat{P}$, we obtain that $\hat{B}$ is also of Gevrey order $1/h$ uniformly for x in $\mathcal{D}_x(0)$ (cf. [Bal94]). This yields the theorem. ♣

By this formal splitting theorem, the initial formal system of ordinary differential equations of one level

$$\varepsilon^h \frac{dy}{dx} = \hat{A}(x,\varepsilon)y$$

is reduced to the formal system

$$\varepsilon^h \frac{dw}{dx} = \hat{B}(x,\varepsilon)w$$

where $\hat{B}(x,\varepsilon)$ is a block-diagonal matrix. The formal fundamental solution $\hat{W}$ of the new system has the form

$$\hat{W}(x,\varepsilon) = \left(\sum_{r=0}^{+\infty} \hat{W}_r(x)\varepsilon^r\right) \exp \begin{pmatrix} q_1(x,\varepsilon)I_{n_1} & 0 \\ 0 & Q_2(x,\varepsilon) \end{pmatrix}.$$

This reduction of the initial problem is however only formal, we need more to prove the conjecture. So in the following paragraph we are going to prove a similar splitting theorem in the holomorphic setting.

3.7.2 Holomorphic splitting theorem.

Theorem 6. *Under the assumptions of theorem 1., there exists a block-matrix P, holomorphic in x and ε for x in a neighbourhood $\mathcal{D}_x(0)$ of 0, ε in a sector $\mathcal{S}$ (of sufficiently small opening), of the form*

$$P(x,\varepsilon) = \begin{pmatrix} I_{n_1} & P^{12}(x,\varepsilon) \\ P^{21}(x,\varepsilon) & I_{n_2} \end{pmatrix},$$

and which is asymptotic of Gevrey order $1/h$ to the formal series $\hat{P}$ of theorem 5 as $\varepsilon \longrightarrow 0$ in $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$. With the following property: the transformation $y = P(x,\varepsilon)w$ changes the system

$$\varepsilon^h \frac{dy}{dx} = A(x,\varepsilon)y \tag{46}$$

into the new system

$$\varepsilon^h \frac{dw}{dx} = B(x,\varepsilon)w \tag{47}$$

with B a block-diagonal matrix, holomorphic in x and ε for x in $\mathcal{D}_x(0)$, ε in $\mathcal{S}$ and asymptotic of Gevrey order $1/h$ to the formal series $\hat{B}$ of theorem 5 as $\varepsilon \longrightarrow 0$ in $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$.

<u>Proof</u> : If P is solution of the system of ordinary differential equations

$$\varepsilon^h \frac{dP}{dx} = A(x,\varepsilon)P - PB(x,\varepsilon), \tag{48}$$

then the transformation $y = P(x,\varepsilon)w$ changes (46) into (47). Using the same block-matrix notations as those of the proof of theorem 5, we show that the system (48) is equivalent to the following system

$$\begin{cases} B^{11} = A^{11} + A^{12}P^{21} \\ B^{22} = A^{22} + A^{21}P^{12} \\ \varepsilon^h \dfrac{dP^{12}}{dx} = A^{11}P^{12} - P^{12}A^{22} - P^{12}A^{21}P^{12} + A^{12} \\ \varepsilon^h \dfrac{dP^{21}}{dx} = A^{22}P^{21} - P^{21}A^{11} - P^{21}A^{12}P^{21} + A^{21}. \end{cases}$$

As in the proof of theorem 5, the third system of differential equations (similarly for the last), written in the vectorial form, gives the following new system

$$\varepsilon^h \frac{d\mathcal{T}}{dx} = \mathcal{A}(x,\varepsilon)\mathcal{T} + \sum_{|\vec{v}|=2} f_{\vec{v}}(x,\varepsilon)\mathcal{T}^{\vec{v}} + \mathcal{A}^{12}(x,\varepsilon). \tag{49}$$

Exactly as for the system (39), we associate to (49) the following system with a vector $a = (a_1, \ldots, a_m)$ of m additional parameters

$$\varepsilon^h \frac{d\mathcal{T}}{dx} = \mathcal{A}(x,\varepsilon)\mathcal{T} + \mathcal{B}_0(x)a + \sum_{|\vec{v}|=2} f_{\vec{v}}(x,\varepsilon)\mathcal{T}^{\vec{v}} + \mathcal{A}^{12}(x,\varepsilon). \tag{50}$$

Theorem (M. Canalis-Durand, J.P. Ramis, R. Schäfke, Y. Sibuya) *If (50) satisfies the hypothesis of transversality, then for $\mathcal{S}$ a sector in the ε-plane of sufficiently small opening, there exists a solution $(\mathcal{T}, a)$, holomorphic in x and ε for x in a neighbourhood $\mathcal{D}_x(0)$ of 0, ε in the sector $\mathcal{S}$, and asymptotic of Gevrey order $1/h$ to the formal solution $\left(\hat{\mathcal{T}}, \hat{a}\right)$ of (41) as $\varepsilon \longrightarrow 0$ in $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$.*

This theorem is the main result of the paper [CDRSS00]. The proof of this theorem uses the same tools which appear in section 3.6 for the proof of theorem 4.

We recall that in our case the formal solution $\hat{a}(\varepsilon)$ is the null series $\sum_{r=0}^{+\infty} 0.\varepsilon^r$. According to [CDRSS00] by using the formal Borel transform of Gervey order $1/h$ on $\hat{a}(\varepsilon)$ and then the truncated Laplace transform of Gervey order $1/h$, we construct a quasi-solution $\tilde{a}(\varepsilon)$ which is here the identically null function. Therefore thanks to the above theorem we obtain the existence of a solution $\mathcal{T}$ of (49) and so consequently a matrix solution P satisfying the properties of theorem 6. Now, we define the matrix B by

$$\begin{cases} B^{11}(x,\varepsilon) := A^{11}(x,\varepsilon) + A^{12}(x,\varepsilon)P^{21}(x,\varepsilon) \\ B^{22}(x,\varepsilon) := A^{22}(x,\varepsilon) + A^{21}(x,\varepsilon)P^{12}(x,\varepsilon) \\ B^{12}(x,\varepsilon) :\equiv 0 \\ B^{21}(x,\varepsilon) :\equiv 0 \end{cases}$$

for x in $\mathcal{D}_x(0)$ and for ε in $\mathcal{S}$. Then this block-diagonal matrix B is holomorphic in x and ε, for x in $\mathcal{D}_x(0)$, ε in $\mathcal{S}$ and B is asymptotic of Gevrey order $1/h$ to the formal series $\hat{B}$ of theorem 5 as $\varepsilon \longrightarrow 0$ in $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$. The holomorphic splitting theorem is then proved. ♣

Applying this holomorphic splitting theorem, we deduce that the system of one level

$$\varepsilon^h \frac{dy}{dx} = A(x,\varepsilon)y$$

is equivalent to the following two independent systems :

- The system of n_1 singularly perturbed ordinary differential equations

$$\varepsilon^h \frac{dw_1}{dx} = B^{11}(x,\varepsilon)w_1 \tag{51}$$

  with B^{11} a matrix-valued function, holomorphic in x and ε, for x in $\mathcal{D}_x(0)$ and for ε in $\mathcal{S}$. The coefficient matrix B^{11} admits the formal series $\hat{B}^{11}$, defined by theorem 5, as asymptotic representation of Gevrey order $1/h$ as $\varepsilon \longrightarrow 0$ in the sector $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$. Furthermore the system (51) admits a formal fundamental solution $\hat{W}_1$ of the form

$$\hat{W}_1(x,\varepsilon) = \left(I_{n_1} + \sum_{r=h}^{+\infty} \hat{W}_{1,r}(x)\varepsilon^r \right) \exp\left(q_1(x,\varepsilon)I_{n_1}\right).$$

Therefore, the system (51) is an essentially scalar system.

- The system of n_2 singularly perturbed ordinary differential equations

$$\varepsilon^h \frac{dw_2}{dx} = B^{22}(x,\varepsilon)w_2 \tag{52}$$

  with B^{22} a matrix-valued function, holomorphic in x and ε, for x in $\mathcal{D}_x(0)$ and for ε in $\mathcal{S}$. The coefficient matrix B^{22} admits the formal series $\hat{B}^{22}$, defined by theorem 5, as asymptotic representation of Gevrey order $1/h$ as $\varepsilon \longrightarrow 0$ in the sector $\mathcal{S}$ uniformly for x in $\mathcal{D}_x(0)$. And the formal fundamental solution $\hat{W}_2$ of (52) has the form

$$\hat{W}_2(x,\varepsilon) = \left(I_{n_2} + \sum_{r=h}^{+\infty} \hat{W}_{2,r}(x)\varepsilon^r \right) \exp\left(Q_2(x,\varepsilon)\right).$$

If the system (52) is also essentially scalar, this means that the diagonal matrix Q_2 can be written as $Q_2(x,\varepsilon) = q_2(x,\varepsilon)I_{n_2}$, then we stop the splitting process. Otherwise, we apply again the splitting theorem to the system (52). Thus we define similarly to (28) two sets N_1 and N_2

$$N_1 := \{j \in \{n_1+1,\ldots,n\} \mid q_j(x,\varepsilon) \equiv q_{n_1+1}(x,\varepsilon)\}$$

and

$$N_2 := \{n_1+1,\ldots,n\} \setminus N_1.$$

We can suppose, without loose of generality, that the diagonal matrix Q_2 has the form

$$Q_2(x,\varepsilon) = \begin{pmatrix} Q_{2,1}(x,\varepsilon) & 0 \\ 0 & Q_{2,2}(x,\varepsilon) \end{pmatrix}$$

where $Q_{2,1}(x,\varepsilon) = q_{n_1+1}(x,\varepsilon)I$ and $Q_{2,2}(x,\varepsilon) = \mathrm{diag}_{j\in N_2}(q_j(x,\varepsilon))$. We have the same hypothesis as those of section 3.5, the difference is the nature of the coefficient matrix $B^{22}(x,\varepsilon)$ in the system (52). Indeed, the function B^{22} is asymptotic of Gevrey oder $1/h$ to the formal series $\hat{B}^{22}$. The theorem 5 of formal splitting remains the same for this system, because we work with the formal system, but we need to adapt the last theorem of holomorphic splitting. There is no difficulty to extend the theorem to a coefficient matrix which is holomorphic and of Gevrey type, because the proof of theorem 6 uses the main result of the paper [CDRSS00] which is given for such class of functions.
At the end of this splitting process, we yield a set of independent systems of different sizes but they are all essentially scalar systems, so we come back to the section 3.6. Therefore we prove the conjecture for systems with levels less or equal to one.

3.8 Systems of several levels.

3.8.1 Systems of two levels.

We start with a system of singularly perturbed ordinary differential equations satisfying the same hypothesis as those of the system (4). We apply to this system the theorem of simplification, therefore we yield a system which satisfies the hypothesis enumerated in section 3.5. As our system is of two levels

$$\mathcal{V} = \{s_1, s_2\} \cup \{+\infty\}$$

where $0 \le s_1 < s_2 \le h-1$. Using the suitable transformation, we can amount us to the case where $s_1 = 0$. The idea of the proof is the same, we start by the definition of two sets N_1 and N_2

$$N_1 := \left\{ i \in \{1,\dots,n\} \mid \mathrm{val}_\varepsilon\left(\varepsilon^h \frac{dq_{1i}}{dx}(x,\varepsilon)\right) \in \{s_2\} \cup \{+\infty\} \right\}$$

and

$$N_2 := \{1,\dots,n\} \setminus N_1.$$

We denote by n_1, respectively n_2, the number of their elements. Without loose of generality we can suppose that the diagonal matrix Q has the form (29). As in the section below we apply to the system the formal and holomorphic splitting theorems to cut it into two independent systems :

- The first system, as (52), is a problem of less than two levels, so we go back to the section 3.7.
- The second is a system whose set $\mathcal{V}$ is included in $\{s_1, s_2\} \cup \{+\infty\}$. If we have again a problem of two levels then we restart the splitting process else we have a problem of less than two levels, so we go back to the section 3.7.

As each splitting decrease strictly the size of the system, after a finite number of splittings we obtain a set of independent systems which have less than two levels.

We apply theorems 6 and 7 to systems with two levels, the proofs uses only at one stage the hypothesis concerning the number of levels: in the formal splitting theorem for formula (30) and (35), but in our case for a problem of two levels they do not change.

3.8.2 Systems of three or more levels.

For such systems the process is the same as above, the new fact which appears is that we started with functions that are of Gevrey order $1/h$, we split the systems using the division:

$$N_1 := \left\{ i \in \{1, \ldots, n\} \mid \mathrm{val}_\varepsilon \left(\varepsilon^h \frac{dq_{1i}}{dx}(x, \varepsilon) \right) \neq 0 \right\}$$

and

$$N_2 = \{1, \ldots, n\} \setminus N_1,$$

as often as necessary. Therefore we reduce the set $\mathcal{V}$ which means that we reduce the number of levels. Remark, that we split the system into a chronological way, because we work with functions of Gevrey order $1/h$ after the first step of splitting and the new system has the form

$$\varepsilon^{h-s_1} \frac{dy}{dx} = A(x, \varepsilon) y,$$

to apply again the splitting theorems we remark that a function of Gevrey order $1/h$ is also of Gevrey order $1/(h - s_1)$. So a new splitting is possible
This completes the proof of W.Wasow's conjecture. ♣

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