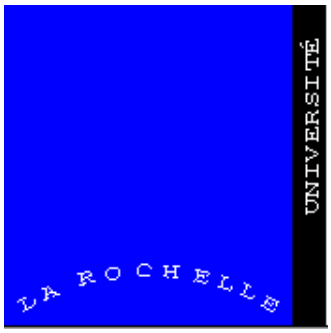

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Surface measures and related functional inequalities on configuration spaces

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Mots clés: Configuration spaces, Poisson measures, surface measures, coarea formulas, isoperimetry. Deviation inequalities.

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Surface measures and related functional inequalities on configuration spaces

Christian Houdré* Nicolas Privault

Abstract

Using finite difference operators, we define a notion of boundary and surface measure for configuration sets under Poisson measures. A Margulis-Russo type identity and a co-area formula are stated with applications to deviation inequalities and functional inequalities, and bounds are obtained on the associated isoperimetric constants.

Key words: Configuration spaces, Poisson measures, surface measures, co-area formulas, isoperimetry.
Mathematics Subject Classification. 60G57, 60H07, 28A75, 60D05.

1 Introduction

Isoperimetry consists in determining sets with minimal surface measure, among sets of given volume measure. In probability theory, isoperimetry is generally formulated by expressing the volume of sets via a probability measure, and surface measures using the expectation of an appropriate gradient norm. Gaussian isoperimetry is a well-known subject, see e.g. [15] for a review. A notion of surface measure on configuration spaces has been recently introduced in [6] using differential operators. Discrete isoperimetry is also possible on graphs and Markov chains, by defining the surface measure of a set A as an average of the number of elements in A that are connected to an element in A^c, cf. e.g. [10], [14], without requiring any smoothness on A. In this framework, an isoperimetric result has been obtained in [4], Prop. 3.6, for i.i.d. Poisson vectors in $\mathbb{N}^d$.

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In this paper we consider the problem of isoperimetry on configuration space in finite volume, i.e. on the space Ω of a.s. finite configurations $\omega = \{x_1, \ldots, x_n\}$, $n \geq 1$, of a metric space X. The configuration space Ω is equipped with a Poisson measure π with intensity σ, where σ is a finite diffuse Borel measure on X. Working with the configuration space instead of finite Poisson distributed i.i.d. vectors is similar to working with measurable functions on $\mathbb{R}$ instead of step functions. Each (π-a.s. finite) configuration $\omega \in \Omega$ has a set of "forward" neighbors of the form $\omega \cup \{x\}$, $x \in \omega^c = X \setminus \omega$, and a set of "backward" neighbors of the form $\omega \setminus \{x\}$, $x \in \omega$. A Markov chain and a graph of unbounded degree can both be constructed on Ω. In the Markov case one adds a point distributed according to the normalized intensity measure to a given configuration. In the graph case, a point chosen at random is removed from a given configuration. Such operations of additions and subtraction of points are also frequently used in statistical mechanics and in connection with logarithmic Sobolev inequalities, see e.g. [9]. Here they allow to construct two notions of neighbor (respectively denoted forward and backward) for a given configuration. It turns out that the graph and Markov kernels are mutually adjoint under the Poisson measure, and we will work with a symmetrized kernel in order to take both the graph and Markov structures into account. We emphasize that it is necessary here to use the graph and Markov approaches simultaneously (i.e. to consider both forward and backward neighbors), since considering only the Markov part or the graph part separately yields trivial values of the isoperimetric constants $h_p^{\pm} = 0$. In fact the classical discrete isoperimetric results that hold in our setting are those which are valid both in the Markov and graph cases. This notion of neighbors is used to define the inner and outer boundary and the surface measure π_s of arbitrary sets of configurations. Isoperimetry and the related isoperimetric constants are then studied by means of co-area formulas. We can define dimension free isoperimetric constants

$$h_1 = \inf_{0<\pi(A)<\frac{1}{2}} \frac{\pi_s(\partial A)}{\pi(A)},$$

and

$$h_\infty = \inf_{0<\pi(A)<\frac{1}{2}} \frac{\pi(\partial A)}{\pi(A)}.$$

Let $\lambda_2 = 1$ denote the optimal constant in the Poincaré inequality on configuration

space for the finite difference operator D. We have $\frac{1}{2} \leq h_1 \leq 2 + 2\sqrt{\sigma(X)}$, and

$$\max\left(\frac{1}{\sqrt{\pi\sigma(X)}}, \frac{1}{2\sigma(X)}\right) \leq h_\infty \leq 4\left(\frac{1}{\sigma(X)} + \frac{1}{\sqrt{\sigma(X)}}\right).$$

Margulis-Russo type identities are also obtained and yield asymptotic estimates for the probability of monotone sets.

Isoperimetry for graphs and Markov chains is often applied to determine bounds on the spectral gaps $\lambda_2, \lambda_\infty$, providing an estimate of the speed of convergence to equilibrium for stochastic algorithms ans in statistical mechanics. In such situations the values of the isoperimetric constants are easily computed as infima on finite sets. In the configuration space case the situation is different since λ_2 and λ_∞ are known and used to deduce bounds on the isoperimetric constants.

We proceed as follows. In Sect. 2 we construct a finite difference gradient on Poisson space and recall the associated integration by parts formulas, as well as the Clark formula. We also extend the isoperimetric result of [4] (see [3] on Gaussian space and [7] on Wiener space and path space), and state a Margulis-Russo type identity, in the general setting of configuration spaces under Poisson measures. In Sect. 3, a graph is constructed on configuration space by addition or deletion of configuration points. The inner and outer boundaries of subsets of configurations and their surface measures are defined in Sect. 4, e.g. a configuration $\omega \in A$ belongs to the inner boundary of A if it has "at least" a (forward or backward) neighbor in A^c. A deviation result in terms of the intensity parameter is obtained from the Margulis-Russo identity on Poisson space. Co-area formulas for the finite difference gradient, which differ from the Gauss type formulas of [11], are proved in Sect. 5. Boundary measures and surface measures are defined by averaging the norms of finite difference gradients, which represent the measure of the flow in and out a given set. An equivalence criterion for functional inequalities is also proved. In Sect. 6, the main isoperimetric constants are introduced, and bounds are stated on these constants. Sect. 7 is devoted to a generalization of Cheeger's inequality, following the arguments of [2], [12], [13].

2 Preliminaries

Let X be a metric space with Borel σ-algebra $\mathcal{B}(X)$ and let σ be a finite and diffuse measure on X. Let Ω denote the set of Radon measures

$$\Omega = \left\{ \omega = \sum_{i=1}^{i=N} \delta_{x_i} \ : \ (x_i)_{i=1}^{i=N} \subset X, \ x_i \neq x_j, \ \forall i \neq j, \ N \in \mathbb{N} \cup \{\infty\} \right\},$$

where δ_x denotes the Dirac measure at $x \in X$. For convenience of notation we identify $\omega = \sum_{i=1}^{i=n} \delta_{x_i}$ with the set $\omega = \{x_1, \ldots, x_n\}$. Let $\mathcal{F}$ denote the σ-algebra generated by all applications of the form $\omega \mapsto \omega(B)$, $B \in \mathcal{B}(X)$, and let π denote the Poisson measure with intensity σ on Ω, defined via

$$\pi(\{\omega \in \Omega \ : \ \omega(A_1) = k_1, \ldots, \omega(A_n) = k_n\}) = \prod_{i=1}^{i=n} \frac{\sigma(A_i)^{k_i}}{k_i!} e^{-\sigma(A_i)}, \qquad k_1, \ldots, k_n \in \mathbb{N},$$

on the σ-algebra $\mathcal{F}$ generated by sets of the form

$$\{\omega \in \Omega \ : \ \omega(A_1) = k_1, \ldots, \omega(A_n) = k_n\},$$

for $k_1, \ldots, k_n \in \mathbb{N}$, and disjoint $A_1, \ldots, A_n \in \mathcal{B}(X)$. Let $I_n(f_n)$ denote the multiple Poisson stochastic integral of the symmetric function $f_n \in L^2(X, \sigma)^{\circ n}$, defined as

$$I_n(f_n)(\omega) = \int_{\Delta_n} f_n(t_1, \ldots, t_n)(\omega(dt_1) - \sigma(dt_1)) \cdots (\omega(dt_n) - \sigma(dt_n)), \quad f_n \in L^2_\sigma(X^n)^{\circ n},$$

with $\Delta_n = \{(t_1, \ldots, t_n) \in X^n \ : \ t_i \neq t_j, \ \forall i \neq j\}$. We recall the isometry formula

$$E[I_n(f_n) I_m(g_m)] = n! \mathbf{1}_{\{n=m\}} \langle f_n, g_m \rangle_{L^2_\sigma(X)^{\circ n}},$$

see [19]. As is well-known, every square-integrable random variable $F \in L^2(\Omega^X, P)$ admits the Wiener-Poisson decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

in series of multiple stochastic integrals.

The gradient chosen here on Poisson space is a finite difference operator (see [6] for a different construction using derivation operators).

Definition 2.1 *For any $F : \Omega \longrightarrow \mathbb{R}$, let*

$$D_x F(\omega) = (F(\omega) - F(\omega + \delta_x)) 1_{\{x \in \omega^c\}} + (F(\omega) - F(\omega - \delta_x)) 1_{\{x \in \omega\}},$$

for all $\omega \in \Omega$ and $x \in X$.

Now, given $u\Omega \times X \to \mathbb{R}$ with sufficient integrability properties, we let

$$\delta_\sigma(u) = \int_X u(x,\omega)\sigma(dx) - \int_X u(x,\omega-\delta_x)\omega(dx),$$

and

$$\delta_\omega(u) = \int_X u(x,\omega)\omega(dx) - \int_X u(x,\omega+\delta_x)\sigma(dx),$$

The following relations are easily obtained:

$$\delta_\sigma(uF) = F\delta_\sigma(u) + \delta_\sigma(uDF) - \langle u, DF\rangle_{L^2(X,\sigma)}, \tag{1}$$

$$\delta_\omega(uF) = F\delta_\omega(u) + \delta_\omega(uDF) - \langle u, DF\rangle_{L^2(X,\omega)}, \tag{2}$$

and

$$\delta_\sigma(u) = \int_X u(x,\omega)(\sigma(dx)-\omega(dx)) + \int_X D_x u(x,\omega)\omega(dx), \tag{3}$$

$$\delta_\omega(u) = \int_X u(x,\omega)(\omega(dx)-\sigma(dx)) + \int_X D_x u(x,\omega)\sigma(dx). \tag{4}$$

As shown in Prop. 2.1 below, the operators δ_σ and δ_ω are adjoint of D, with respect to scalar products respectively given by σ and ω.

Proposition 2.1 *We have for $F : \Omega \to \mathbb{R}$ and $u : \Omega \times X \to \mathbb{R}$:*

$$E[F\delta_\sigma(u)] = E[\langle DF, u\rangle_{L^2(\sigma)}], \tag{5}$$

and

$$E[F\delta_\omega(u)] = E[\langle DF, u\rangle_{L^2(\omega)}], \tag{6}$$

provided the corresponding quantities are integrable.

Proof. The relation $E[\delta_\omega(u)] = 0$ can be seen as a consequence of Th. 1 or Cor. 1 in [20], and (6) follows from (2). This in turn implies $E[\delta_\sigma(u)] = 0$ from (3), and (5) using (1). □

Remark 2.1 *For simple processes the above result can be proved using the characteristic function of $\int_X h d\omega$ which satisfies*

$$E\left[\exp\left(iz\int_X h d\omega\right)\right] = \exp\int_X (e^{izh}-1)d\sigma, \quad z \in \mathbb{R}.$$

Differentiating each of those two expressions with respect to z yields

$$E\left[\int_X h d\omega \exp\left(iz\int_X h d\omega\right)\right] = E\left[\int_X h e^{izh} d\sigma \exp\left(iz\int_X h d\omega\right)\right],$$

hence

$$E\left[\int_X hd(\sigma-\omega)\exp\left(iz\int_X hd\omega\right)\right] = E\left[\langle h, 1-e^{izh}\rangle_{L^2(X,\sigma)}\exp\left(iz\int_X hd\omega\right)\right]$$
$$= E\left[\left\langle h, D\exp\left(iz\int_X hd\omega\right)\right\rangle_{L^2(X,\sigma)}\right],$$

where we used the relation $D_x\exp(iz\int_X hd\omega) = (1-e^{izh(x)})\exp(iz\int_X hd\omega)$, $\sigma(dx)$*-a.e. From* (4) *this implies* $E[\delta_\omega(u)] = 0$ *for all* u *of the form*

$$u = \sum_{i=1}^n 1_{A_i} e^{iz_1\omega(B_1)+\cdots+iz_n\omega(B_n)}.$$

We have

$$D_xF(\omega+\delta_x) = F(\omega+\delta_x) - F(\omega) = -D_xF(\omega), \quad x\notin\omega,$$

and

$$D_xF(\omega-\delta_x) = F(\omega-\delta_x) - F(\omega) = -D_xF(\omega), \quad x\in\omega.$$

Hence

$$\delta_\sigma DF(\omega) = \int_X (F(\omega)-F(\omega+\delta_x))\sigma(dx) - \int_X (F(\omega-\delta_x)-F(\omega))\omega(dx),$$

and

$$\delta_\omega DF(\omega) = \int_X (F(\omega)-F(\omega-\delta_x))\omega(dx) - \int_X (F(\omega+\delta_x)-F(\omega))\sigma(dx),$$

so that

$$\delta_\sigma DF(\omega) = \delta_\omega DF(\omega) = \int_X D_xF(\omega)\omega(dx) - \int_X D_xF(\omega)\sigma(dx) \tag{7}$$
$$= (\sigma(X)+\omega(X))F(\omega) - \int_X F(\omega+\delta_x)\sigma(dx) - \int_X F(\omega-\delta_x)\omega(dx).$$

From the definition of $I_n(f_n)$ it can also be easily shown that

$$\delta_\sigma DI_n(f_n) = \delta_\omega DI_n(f_n) = nI_n(f_n),$$

cf. e.g. [23]. It follows that the spectral gap of $\delta_\sigma D$ is $\lambda_2 = 1$, a fact which is recovered below by a different method. In the sequel we shall uniquely use the operator δ_σ, and denote it by δ. Let

$$D_x^+F(\omega) = \max(0, D_xF(\omega))$$
$$= (F(\omega)-F(\omega+\delta_x))^+1_{\{x\in\omega^c\}} + (F(\omega)-F(\omega-\delta_x))^+1_{\{x\in\omega\}}.$$

and

$$\begin{aligned} D_x^- F(\omega) &= -\min(0, D_x F(\omega)) \\ &= (F(\omega) - F(\omega + \delta_x))^- 1_{\{x \in \omega^c\}} + (F(\omega) - F(\omega - \delta_x))^- 1_{\{x \in \omega\}}. \end{aligned}$$

We have $D_x^+ F = D_x^-(-F)$, hence $|D_x F|^p = |D_x^+ F|^p + |D_x^- F|^p$, and

$$|DF(\omega)|_{L^p}^p = |D^+ F(\omega)|_{L^p}^p + |D^- F(\omega)|_{L^p}^p.$$

Lemma 2.1 *We have*

$$E[|D^+ F|_{L^p(\sigma)}^p] = E[|D^- F|_{L^p(\omega)}^p],$$

and

$$E[|D^- F|_{L^p(\sigma)}^p] = E[|D^+ F|_{L^p(\omega)}^p].$$

Proof. We have

$$\delta_\sigma D^+ F(\omega) = \delta_\omega D^+ F(\omega) = \int_X D_x^+ F(\omega)\omega(dx) - \int_X D_x^- F(\omega)\sigma(dx), \tag{8}$$

and

$$\delta_\sigma D^- F(\omega) = \delta_\omega D^- F(\omega) = \int_X D_x^- F(\omega)\omega(dx) - \int_X D_x^+ F(\omega)\sigma(dx), \tag{9}$$

which implies

$$E[|D^\pm F|_{L^p(\sigma)}^p] - E[|D^\mp F|_{L^p(\omega)}^p] = E[\delta_\sigma((D^\pm F)^p)] = 0.$$

□

Similarly, (7) will imply

$$E\left[\int_X D_x F \sigma(dx)\right] = -E\left[\int_X D_x F \omega(dx)\right]. \tag{10}$$

In the particular case $F = 1_{\{\omega(A)=k\}}$, Lemma 2.1 simply states the following easily verified equality:

$$E[|D^+ 1_{\{\omega(A)=k\}}|_{L^1(\sigma)}] = \sigma(A)E[1_{\{\omega(A)=k\}}] = (k+1)E[1_{\{\omega(A)=k+1\}}] = E[|D^- 1_{\{\omega(A)=k\}}|_{L^1(\omega)}].$$

We also have

$$E\left[|D^+ F|_{L^p(\frac{\sigma+\omega}{2})}^p\right] = E\left[|D^- F|_{L^p(\frac{\sigma+\omega}{2})}^p\right] = \frac{1}{2}E\left[|DF|_{L^p(\sigma)}^p\right] = \frac{1}{2}E\left[|DF|_{L^p(\omega)}^p\right],$$

in particular the Dirichlet forms $\mathcal{E}_\sigma(F,G)$ and $\mathcal{E}_\omega(F,G)$ defined as

$$\mathcal{E}_\sigma(F,F) = \frac{1}{2}E[|DF|^2_{L^2(\sigma)}], \quad \mathcal{E}_\omega(F,F) = \frac{1}{2}E[|DF|^2_{L^2(\omega)}]$$

coincide:

$$\mathcal{E}_\sigma(F,F) = \mathcal{E}_\omega(F,F).$$

This result can also be seen as a consequence of the relation $\delta_\sigma D = \delta_\omega D$.

The Clark formula given next yields the predictable representation of a random variable using the operator D. Take $X = [0,1]$ and σ the Lebesgue measure and let

$$N_t(\omega) = N_{[0,t]}(\omega) = \omega([0,t]), \quad t \in \mathbb{R}_+, \quad \omega \in \Omega,$$

i.e. $(N_t)_{t\in[0,1]}$ is a standard Poisson process under π.

Proposition 2.2 *([21], Th. 1) We have the following Clark formula, for $F \in L^2(\Omega,\pi)$:*

$$F = E[F] - \int_0^1 E[D_t F \mid \mathcal{F}_t] d\tilde{N}_t, \tag{11}$$

where the stochastic integral is taken in the Itô sense.

The formula is first proved for $F \in \mathrm{Dom}(D)$ and then extended to $L^2(\Omega)$ by continuity of $F \mapsto (E[D_t F \mid \mathcal{F}_t])_{t\in\mathbb{R}_+}$ from $L^2(\Omega,\pi)$ into $L^2(\Omega\times[0,1])$. The Clark formula (2.2) yields the Poincaré inequality:

$$\mathrm{var}(F) \leq E[|DF|^2_{L^2(\sigma)}], \quad F \in \mathrm{Dom}(D). \tag{12}$$

This inequality is in fact valid for an arbitrary Polish space X with diffuse measure σ. Note that if $F = 1_A$ then the Poincaré inequality implies

$$\pi(A)(1-\pi(A)) \leq \sigma(X),$$

in particular if $\sigma(X) \leq 1/4$ then we have either

$$\pi(A) \leq (1 - \sqrt{1-4\sigma(X)})/2$$

or

$$\pi(A) \geq (1 + \sqrt{1-4\sigma(X)})/2,$$

and if $\pi(A) \leq 1/2$ then

$$\pi(A) \leq 2\pi(A)(1-\pi(A)) \leq 2\sigma(X).$$

The following result gives a version of isoperimetry on Poisson space which is independent of dimension and generalizes the result of [4], p. 274. Let φ denote the standard Gaussian density, and let Φ denote its distribution function. Let $I(t) = \varphi(\Phi^{-1}(t))$, $0 \leq t \leq 1$ denote the Gaussian isoperimetric function, with the relations $I(x)I''(x) = -1$ and $I'(x) = -\Phi^{-1}(x)$, $x \in [0,1]$.

Proposition 2.3 *For every random variable $F : \Omega \to [0,1]$ we have*

$$I(E[F]) \leq E\left[\sqrt{I(F)^2 + 2|DF|^2_{L^2(\sigma)}}\right]. \tag{13}$$

Proof. Let X_n denote the $\mathbb{N}^n$-valued random variable defined as

$$X_n(\omega) = (\omega(A_1), \ldots, \omega(A_n)), \quad \omega \in \Omega.$$

If $F = f \circ X_n$ is a cylindrical functional we have

$$D_x F(\omega) = \sum_{k=1}^{k=n} 1_{A_k}(x)(f(X_n(\omega)) - f(X_n(\omega) + e_k)),$$

$f : \mathbb{N}^n \to \mathbb{R}$, where $(e_k)_{1 \leq k \leq n}$ denotes the canonical basis of $\mathbb{R}^n$. For the cylindrical functional F, (13) follows by application of Relation (3.13) in [4] and tensorization. The extension to general random variables can be done by martingale convergence, e.g. as in the proof of Th. 3.4 of [29]. □

This also implies that the optimal constant b_2 in the inequality

$$I(E[F]) \leq E\left[\sqrt{I(F)^2 + \frac{1}{b_2}|DF|^2_{L^2(\frac{\sigma+\omega}{2})}}\right]$$

satisfies $b_2 \geq 1$. Using the equivalence $I(\varepsilon) \simeq \varepsilon\sqrt{2\log 1/\varepsilon}$ and the Schwarz inequality, Relation (13) allows to recover the modified logarithmic Sobolev inequality of [1], [30]:

$$E[F \log F] - E[F] \log E[F] \leq \frac{1}{2} E\left[\frac{1}{F}|DF|^2_2\right].$$

Note that the analog Gaussian isoperimetry result can also be transferred to the Poisson space for the Carlen-Pardoux gradient [8], writing the exponential interjump times of the Poisson process as half sums of squared Gaussian random variables as in [22]. Let π_λ, $\lambda > 0$, denote the Poisson measure of intensity $\lambda\sigma(dx)$ on Ω, and let E_λ denote the expectation under π_λ. We refer to [18] for the following type of result, obtained by differentiation of the intensity parameter.

Proposition 2.4 *Assume that* $DF \in L^1(\pi_\lambda \otimes \sigma)$ *and* $F \in L^1(\pi_\lambda)$, $\lambda \in (a,b)$. *We have*

$$\frac{\partial}{\partial\lambda} E_\lambda[F] = -E_\lambda\left[\int_X D_x F \sigma(dx)\right] = E_\lambda\left[\int_X D_x F \omega(dx)\right], \quad \lambda \in (a,b).$$

Proof. Given the representation

$$F(\omega) = f_0 1_{\{|\omega|=0\}} + \sum_{n=1}^{\infty} 1_{\{|\omega|=n\}} f_n(x_1,\ldots,x_n),$$

where $\omega = \{x_1,\ldots,x_n\}$ when $|\omega| = n$, we have

$$E_\lambda[F] = e^{-\lambda\sigma(X)} f_0 + e^{-\lambda\sigma(X)} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_X \cdots \int_X f_n(x_1,\ldots,x_n)\sigma(dx_1)\cdots\sigma(dx_n),$$

and

$$\begin{aligned}
\frac{\partial}{\partial\lambda} E_\lambda[F] &= -\sigma(X) E_\lambda[F] \\
&\quad + e^{-\lambda\sigma(X)} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \int_X \cdots \int_X f_n(x_1,\ldots,x_n)\sigma(dx_1)\cdots\sigma(dx_n) \\
&= -\sigma(X) E_\lambda[F] + E_\lambda\left[\int_X F(\omega+\delta_x)\sigma(dx)\right] \\
&= -E_\lambda\left[\int_X D_x F(\omega)\sigma(dx)\right].
\end{aligned}$$

The second relation follows from (10). □

As a corollary we will obtain a Margulis-Russo type equality [16], [25] for monotone sets under Poisson measures.

Definition 2.2 *A measurable set* $A \subset \Omega$ *is called increasing if*

$$\omega \in A \quad \Longrightarrow \quad \omega + \delta_x \in A, \quad \sigma(dx) - a.e. \tag{14}$$

It is called decreasing if

$$\omega \in A \quad \Longrightarrow \quad \omega - \delta_x \in A, \quad \omega(dx) - a.e. \tag{15}$$

Note that if A is decreasing then A^c is increasing but the converse is not true. In fact, saying that A is decreasing is equivalent to the following property on A^c:

$$\omega \in A^c \quad \Longrightarrow \quad \omega + \delta_x \in A^c, \quad \forall x \in \omega^c, \tag{16}$$

which is stronger than saying that A^c is increasing. The set A is said to be monotone if it is either increasing or decreasing. The sets $\{\omega(B) \geq n\}$, resp. $\{\omega(B) \leq n\}$, are naturally increasing, resp. decreasing. Another example of monotone set is given by

$$\left\{\omega \in \Omega \ : \ \int_X f d\omega > K\right\}, \quad K \in \mathbb{R},$$

which is increasing, resp. decreasing, if $f \geq 0$, resp. $f \leq 0$. Clearly, a set A is increasing, resp. decreasing, if and only if $D_x 1_A \leq 0$ (i.e. $D_x 1_A = -D_x^- 1_A$, or $D_x^+ 1_A = 0$) $\sigma(dx)$-a.e., resp. $\omega(dx)$-a.e. As a corollary of Prop. 2.4 we have:

Corollary 2.1 *Let $A \subset \Omega$ be an increasing set. We have*

$$\frac{\partial}{\partial \lambda} \pi_\lambda(A) = E_\lambda \left[\int_X D_x^- 1_A \sigma(dx) \right] = E_\lambda \left[\int_X D_x^+ 1_A \omega(dx) \right].$$

If $A \subset \Omega$ is decreasing we have

$$\frac{\partial}{\partial \lambda} \pi_\lambda(A) = -E_\lambda \left[\int_X D_x^- 1_A \omega(dx) \right] = -E_\lambda \left[\int_X D_x^+ 1_A \sigma(dx) \right].$$

We also have if A is monotone:

$$\frac{\partial}{\partial \lambda} \pi_\lambda(A) = E_\lambda \left[\| D 1_A \|_{L^1(\sigma)} \right] = E_\lambda \left[\| D 1_A \|_{L^1(\omega)} \right].$$

3 Forward-backward kernels and reversibility on configuration space

Given $\omega \in \Omega$, the set of forward neighbors of ω is defined to be

$$\mathcal{N}_\omega^+ = \{\omega + \delta_x \ : x \in \omega^c\},$$

and similarly the set of backward neighbors of ω is

$$\mathcal{N}_\omega^- = \{\omega - \delta_x \ : x \in \omega\}.$$

We let

$$\mathcal{N}_\omega = \mathcal{N}_\omega^+ \cup \mathcal{N}_\omega^-.$$

We define two measure kernels $K^+(\omega, d\tilde{\omega})$ and $K^-(d\tilde{\omega}, \omega)$ which are respectively supported by $\mathcal{N}_\omega^+$ and $\mathcal{N}_\omega^-$.

Definition 3.1 *Let for $A \in \mathcal{F}$:*

$$K^+(\omega, A) = \int_X 1_A(\omega + \delta_x) \sigma(dx), \qquad K^-(A, \omega) = \sum_{x \in \omega} 1_A(\omega - \delta_x).$$

It is a classical fact that since π is a Poisson measure, the image under $\omega + \delta_x \mapsto x$ of the measure

$$\pi(d\tilde{\omega} \mid \tilde{\omega} \in \mathcal{N}_\omega^+)$$

coincides with the (normalized) measure σ on X:

$$\frac{\sigma(B)}{\sigma(X)} = \pi(\{\tilde{\omega} \ : \ \tilde{\omega} = \omega + \delta_x \ : x \in B\} \mid \tilde{\omega} \in \mathcal{N}_\omega^+), \quad B \in \mathcal{B}(X).$$

Hence the forward kernel satisfies

$$K^+(\omega, d\tilde{\omega}) = \sigma(X)\pi(d\tilde{\omega} \mid \tilde{\omega} \in \mathcal{N}_\omega^+),$$

and $(\sigma(X))^{-1}K^+(\omega, d\tilde{\omega})$ is of Markov type. Similarly, the image under $\omega - \delta_x \mapsto x$ of the measure

$$\pi(d\tilde{\omega} \mid \tilde{\omega} \in \mathcal{N}_\omega^-)$$

coincides with the normalized counting measure on ω:

$$\frac{\omega(B)}{\omega(X)} = \pi(\{\tilde{\omega} \ : \ \tilde{\omega} = \omega - \delta_x, \ x \in B\} \mid \tilde{\omega} \in \mathcal{N}_\omega^-),$$

hence the backward kernel satisfies

$$K^-(d\tilde{\omega}, \omega) = \omega(X)\pi(d\tilde{\omega} \mid \tilde{\omega} \in \mathcal{N}_\omega^-) = \sum_{x \in \omega} \delta_{\omega - \delta_x}(d\tilde{\omega}),$$

and $(\omega(X))^{-1}K^-(d\tilde{\omega}, \omega)$ is Markovian provided $\omega \neq \emptyset$. The kernel $K^-(d\tilde{\omega}, \omega)$ itself is not Markovian, instead it is of graph type, i.e.

$$K^-(\{\tilde{\omega}\}, \omega) = \begin{cases} 1 & \text{if } \tilde{\omega} = \omega - \delta_x \text{ for some } x \in X \text{ (i.e. } \tilde{\omega} \in \mathcal{N}_\omega), \\ 0 & \text{otherwise (i.e. } \tilde{\omega} \notin \mathcal{N}_\omega). \end{cases}$$

We have for $p \in [1, \infty)$:

$$|DF(\omega)|^p_{L^p(\sigma)} = \int_X |F(\omega) - F(\omega + \delta_x)|^p \sigma(dx) = \int_\Omega |F(\omega) - F(\tilde{\omega})|^p K^+(\omega, d\tilde{\omega}),$$

and

$$|DF(\omega)|^p_{L^p(\omega)} = \int_X |F(\omega) - F(\omega - \delta_x)|^p \omega(dx) = \int_\Omega |F(\omega) - F(\tilde{\omega})|^p K^-(\omega, d\tilde{\omega}).$$

For $p = \infty$ we have

$$|DF(\omega)|_{L^\infty(\sigma)} = \text{ess sup}_{\ \sigma(dx)} |F(\omega) - F(\omega + \delta_x)| = \text{ess sup}_{\ K^+(\omega, d\tilde{\omega})} |F(\omega) - F(\tilde{\omega})|,$$

and

$$|DF(\omega)|_{L^\infty(\omega)} = \text{ess sup}_{\ \omega(dx)} |F(\omega) - F(\omega - \delta_x)| = \text{ess sup}_{\ K^-(\omega, d\tilde{\omega})} |F(\omega) - F(\tilde{\omega})|.$$

We also have

$$E\left[|D^+1_A|_{L^p(\frac{\sigma+\omega}{2})}\right] = \int_A \bar{K}(\omega, A^c)^{1/p}\pi(d\omega), \quad E\left[|D^-1_A|_{L^p(\frac{\sigma+\omega}{2})}\right] = \int_{A^c} \bar{K}(\omega, A)^{1/p}\pi(d\omega).$$

The following proposition shows a reversibility property, which is an analog of Lemma 2.1.

Proposition 3.1 *The kernels $K^+(\omega, d\tilde{\omega})$ and $K^-(d\tilde{\omega}, \omega)$ are mutually adjoint under $\pi(d\tilde{\omega})$, i.e.*

$$\pi(d\omega)K^+(\omega, d\tilde{\omega}) = K^-(d\omega, \tilde{\omega})\pi(d\tilde{\omega}).$$

Proof. We have

$$\begin{aligned}
\int_\Omega \int_\Omega F(\omega)G(\tilde{\omega})K^+(\omega, d\tilde{\omega})\pi(d\omega) &= \int_\Omega F(\omega)G(\omega + \delta_x)\pi(d\omega)\sigma(dx) \\
&= -E[F\langle DG, 1\rangle_{L^2(\sigma)}] + \sigma(X)E[FG] \\
&= -E[G\delta_\sigma(1_X F)] + \sigma(X)E[FG] \\
&= \int_\Omega G(\omega)\sum_{x\in\omega} F(\omega - \delta_x)\pi(d\omega) \\
&= \int_\Omega \int_\Omega G(\tilde{\omega})F(\omega)K^-(d\omega, \tilde{\omega})\pi(d\tilde{\omega}).
\end{aligned}$$

□

In particular we have $E[K^-F] = \sigma(X)E[F]$:

$$\int_\Omega \int_\Omega F(\tilde{\omega})K^-(d\tilde{\omega}, \omega)\pi(d\omega) = \sigma(X)\int_\Omega F(\omega)\pi(d\omega),$$

and $E[K^+F] = E[\omega(X)F]$:

$$\int_\Omega \int_\Omega F(\tilde{\omega})K^+(\omega, d\tilde{\omega})\pi(d\omega) = \int_\Omega \omega(X)F(\omega)\pi(d\omega),$$

which is Lemma 1.1 in [29] and is similar to the Mecke identity [17]. This also implies

$$\int_A K^+(\omega, A^c)\pi(d\omega) = E[|D^+1_A(\omega)|^p_{L^p(\sigma)}] = E[|D^-1_A(\omega)|^p_{L^p(\omega)}] = \int_{A^c} K^-(A, \omega)\pi(d\omega),$$

$$\int_{A^c} K^+(\omega, A)\pi(d\omega) = E[|D^-1_A(\omega)|^p_{L^p(\sigma)}] = E[|D^+1_A(\omega)|^p_{L^p(\omega)}] = \int_A K^-(A^c, \omega)\pi(d\omega).$$

The proof of Lemma 2.1 can be reformulated using reversibility of forward and backward kernels.

Proof. We have

$$\begin{aligned}
E[|D^\pm F|^p_{L^p(\sigma)}] &= \int_\omega ((F(\omega) - F(\tilde{\omega}))^\pm)^p K^+(\omega, d\tilde{\omega})\pi(d\omega) \\
&= \int_\omega ((F(\omega) - F(\tilde{\omega}))^\pm)^p K^-(d\omega, \tilde{\omega})\pi(d\tilde{\omega}) \\
&= \int_\omega ((F(\tilde{\omega}) - F(\omega))^\mp)^p K^-(d\omega, \tilde{\omega})\pi(d\tilde{\omega}) \\
&= E[|D^\mp F|^p_{L^p(\omega)}].
\end{aligned}$$

□

Let $\bar{K}(\omega, d\tilde{\omega})$ denote the symmetrized kernel

$$\bar{K}(\omega, d\tilde{\omega}) = \frac{K^+(\omega, d\tilde{\omega}) + K^-(d\tilde{\omega}, \omega)}{2}.$$

We have

$$|DF(\omega)|^p_{L^p(\frac{\omega+\sigma}{2})} = \frac{1}{2}|DF(\omega)|^p_{L^p(\sigma)} + \frac{1}{2}|DF(\omega)|^p_{L^p(\omega)} = \int_\Omega |F(\omega) - F(\tilde{\omega})|^p \bar{K}(\omega, d\tilde{\omega}),$$

and for $p = \infty$:

$$|DF(\omega)|_{L^\infty(\sigma+\omega)} = \text{ess sup}_{\ \bar{K}(\omega,d\tilde{\omega})} |F(\omega) - F(\tilde{\omega})|.$$

We also have

$$\begin{aligned} E\left[|D1_A|_{L^p(\frac{\sigma+\omega}{2})}\right] &= E\left[|D^+1_A|_{L^p(\frac{\sigma+\omega}{2})}\right] + E\left[|D^-1_A|_{L^p(\frac{\sigma+\omega}{2})}\right] \\ &= \int_A \bar{K}(\omega, A^c)^{1/p}\pi(d\omega) + \int_{A^c} \bar{K}(\omega, A)^{1/p}\pi(d\omega), \end{aligned}$$

since $D^+_x F D^-_x F = 0$, $x \in X$. Let

$$\Gamma^\pm(F, F) = \frac{1}{2}|D^\pm F|^2_{L^2(\frac{\sigma+\omega}{2})}.$$

We have

$$\Gamma^+(F, G)(\omega) = \frac{1}{2}\int_{\Omega\times\Omega} (F(\omega) - F(\tilde{\omega}))^+ (G(\omega) - G(\tilde{\omega}))^+ \bar{K}(\omega, d\tilde{\omega}),$$

$$\Gamma^-(F, G)(\omega) = \frac{1}{2}\int_{\Omega\times\Omega} (F(\omega) - F(\tilde{\omega}))^- (G(\omega) - G(\tilde{\omega}))^- \bar{K}(\omega, d\tilde{\omega}),$$

and

$$\mathcal{E}(F, F) = E[\Gamma^+(F, F)] = E[\Gamma^-(F, F)].$$

Proposition 3.2 *The Laplacian associated to the discrete Dirichlet form $\mathcal{E}(F, F)$ is $L = \frac{1}{2}\delta D$, with*

$$L = \frac{1}{2}\delta D = \frac{\sigma(X) + \omega(X)}{2} I_d - \bar{K}.$$

Proof. Again, reversibility can be employed. We have

$$\begin{aligned} \mathcal{E}(F, G) &= \int_{\Omega\times\Omega} (F(\omega) - F(\tilde{\omega}))(G(\omega) - G(\tilde{\omega}))K^+(\omega, d\tilde{\omega})\pi(d\omega) \\ &= \int_{\Omega\times\Omega} F(\omega)G(\omega)K^+(\omega, d\tilde{\omega})\pi(d\omega) + \int_{\Omega\times\Omega} F(\tilde{\omega})G(\tilde{\omega})K^-(d\omega, \tilde{\omega})\pi(d\tilde{\omega}) \\ &\quad - \int_{\Omega\times\Omega} F(\omega)G(\tilde{\omega})K^+(\omega, d\tilde{\omega})\pi(d\omega) - \int_{\Omega\times\Omega} G(\omega)F(\tilde{\omega})K^+(\omega, d\tilde{\omega})\pi(d\omega) \\ &= E[F((\sigma(X) + \omega(X))G - K^+G - K^-G)]. \end{aligned}$$

□

Note that in the case of cylindrical functionals, L is the generator of Glauber dynamics considered in statistical mechanics as in e.g. [9], and has the Poisson probability as invariant measure. Although $K^-(d\tilde{\omega},\omega)$ and $K^+(\omega,d\tilde{\omega})$ are not Markov, they leave the Poisson measure invariant under appropriate normalizations, for example for $A=\{\omega(X)=k\}$, we have $K^-(A,\omega)=(k+1)1_{\{\omega(X)=k+1\}}$, and

$$\frac{1}{\sigma(X)}\int_\Omega \pi(d\omega)K^-(A,\omega)=\frac{k+1}{\sigma(X)}\pi(\{\omega(X)=k+1\})=\pi(A).$$

In particular we have the following result.

Proposition 3.3 *The Poisson measure $\pi(d\omega)$ is a stationary distribution for the symmetrized normalized kernel*

$$\frac{2}{\sigma(X)+\tilde{\omega}(X)}\bar{K}(\omega,d\tilde{\omega}).$$

Proof. We have

$$\int_\Omega \pi(d\omega)\frac{2}{\sigma(X)+\omega(X)}\bar{K}(\omega,A)=\int_A \pi(d\omega)\frac{2}{\sigma(X)+\omega(X)}\bar{K}(\omega,\Omega)=\pi(A).$$

□

4 Inner and outer boundaries

We have

$$D_x^+1_A(\omega)=1_{\{\omega\in A \text{ and } \omega+\delta_x\in A^c\}}1_{\{x\in\omega^c\}}+1_{\{\omega\in A \text{ and } \omega-\delta_x\in A^c\}}1_{\{x\in\omega\}},$$

and

$$D_x^-1_A(\omega)=1_{\{\omega\in A^c \text{ and } \omega+\delta_x\in A\}}1_{\{x\in\omega^c\}}+1_{\{\omega\in A^c \text{ and } \omega-\delta_x\in A\}}1_{\{x\in\omega\}}.$$

Hence

$$|D^+1_A(\omega)|^p_{L^p(\sigma)}=1_A(\omega)\sigma(\{x\in X\ :\ \omega+\delta_x\in A^c\})=1_A(\omega)K^+(\omega,A^c),$$

and

$$|D^+1_A(\omega)|^p_{L^p(\omega)}=1_A(\omega)\omega(\{x\in X\ :\ \omega-\delta_x\in A^c\})=1_A(\omega)K^-(A^c,w),$$

i.e. for $\omega\in A$, $|D^+1_A(\omega)|^p_{L^p(\sigma)}$ is the measure $K^+(\omega,A^c)$ on $\mathcal{N}_\omega^+$ of the set of forward neighbors which belong to A^c, and $|D^+1_A(\omega)|^p_{L^p(\omega)}$ is the number (or measure $K^-(A^c,\omega)$ on $\mathcal{N}_\omega^-$) of backward neighbors which belong to A^c. We also have

$$|D^-1_A(\omega)|^p_{L^p(\sigma)}=1_{A^c}(\omega)\sigma(\{x\in[0,1]\ :\ \omega+\delta_x\in A\})=1_{A^c}(\omega)K^+(\omega,A),$$

and

$$|D^- 1_A(\omega)|^p_{L^p(\omega)} = 1_{A^c}(\omega)\omega(\{x \in [0,1] \ : \ \omega - \delta_x \in A\}) = 1_{A^c}(\omega)K^-(A,\omega).$$

i.e. for $\omega \in A^c$, $|D^- 1_A(\omega)|^p_{L^p(\sigma)}$ is the measure $K^+(\omega, A)$ on $\mathcal{N}^+_\omega$ of the set of forward neighbors of $\omega \in A^c$ which belong to A, and $|D^- 1_A(\omega)|^p_{L^p(\omega)}$ is the number (measure $K^-(A,\omega)$ on $\mathcal{N}^-_\omega$) of backward neighbors of $\omega \in A^c$ which belong to A.

Remark 4.1 *We have* $D^+_x 1_A = D^-_x 1_{A^c}$ *and* $|D_x 1_A| = |D_x 1_{A^c}|$, $x \in X$.

In particular,

$$D^+_x 1_{\{\omega(B)=k\}} = 1_B(x) 1_{\{\omega(B)=k\}},$$

and

$$D^-_x 1_{\{\omega(B)=k\}} = 1_B(x)1_\omega(x)1_{\{\omega(B)=k+1\}} + 1_B(x)1_{\omega^c}(x)1_{\{\omega(B)=k-1\}},$$

hence

$$|D^+ 1_{\{\omega(B)=k\}}|^p_{L^p(\sigma)} = \sigma(B)1_{\{\omega(B)=k\}}, \quad |D^+ 1_B(\omega)|^p_{L^p(\omega)} = k 1_{\{\omega(B)=k\}},$$

and

$$|D^- 1_{\{\omega(B)=k\}}|^p_{L^p(\sigma)} = \sigma(B)1_{\{\omega(B)=k-1\}}, \quad |D^- 1_{\{\omega(B)=k\}}|^p_{L^p(\omega)} = (k+1)1_{\{\omega(B)=k+1\}}.$$

Similarly,

$$\begin{aligned}
|D^+ 1_A(\omega)|_{L^\infty(\sigma)} &= 1_{\{\omega \in A \text{ and } \sigma(\{x \in X \, : \, \omega + \delta_x \in A^c\}) > 0\}} = 1_A(\omega) 1_{\{K^+(\omega, A^c) > 0\}}, \\
|D^+ 1_A(\omega)|_{L^\infty(\omega)} &= 1_{\{\omega \in A \text{ and } \exists x \in \omega \, : \, \omega - \delta_x \in A^c\}} = 1_A(\omega) 1_{\{K^-(A^c, \omega) > 0\}}, \\
|D^- 1_A(\omega)|_{L^\infty(\sigma)} &= 1_{\{\omega \in A^c \text{ and } \sigma(\{x \in X \, : \, \omega + \delta_x \in A\}) > 0\}} = 1_{A^c}(\omega) 1_{\{K^+(\omega, A) > 0\}}, \\
|D^- 1_A(\omega)|_{L^\infty(\omega)} &= 1_{\{\omega \in A^c \text{ and } \exists x \in \omega \, : \, \omega - \delta_x \in A\}} = 1_{A^c}(\omega) 1_{\{K^-(A, \omega) > 0\}},
\end{aligned}$$

i.e. $|D^+ 1_A(\omega)|_{L^\infty(\sigma)} = 1$, resp. $|D^- 1_A(\omega)|_{L^\infty(\sigma)} = 1$, if and only if $\omega \in A$, resp. $\omega \in A^c$, has "at least" a forward neighbor in A^c, resp. A, and $|D^+ 1_A(\omega)|_{L^\infty(\omega)} = 1$, resp. $|D^- 1_A(\omega)|_{L^\infty(\omega)} = 1$, if and only if $\omega \in A$, resp. $\omega \in A^c$, has at least a backward neighbor in A^c, resp. A. The following definitions are stated independently of $p \in [1, \infty]$.

Definition 4.1 *Let* $p \in [1, \infty]$.

The inner and outer boundaries of A are defined as:

$$\partial_{\rm in} A = \{\omega \in A \ : \ \bar{K}(\omega, A^c) > 0\} = \{|D^+ 1_A(\omega)|_{L^p(\sigma+\omega)} > 0\},$$

and

$$\partial_{\rm out} A = \{\omega \in A^c \ : \ \bar{K}(\omega, A) > 0\} = \{|D^- 1_A(\omega)|_{L^p(\sigma+\omega)} > 0\}.$$

The boundary of A is defined as:

$$\begin{aligned} \partial A &= \partial_{\rm in} A \cup \partial_{\rm out} A \\ &= \{\omega \in \Omega \ : \ |D 1_A(\omega)|_{L^p(\sigma+\omega)} > 0\} \\ &= \{\omega \in \Omega \ : \ \bar{K}(\omega, A) + \bar{K}(\omega, A^c) > 0\}. \end{aligned}$$

For instance,

$$\partial_{\rm in}\{\omega(B) = k\} = \{\omega(B) = k\},$$

$$\partial_{\rm out}\{\omega(B) = k\} = \{\omega(B) = k-1\} \cup \{\omega(B) = k+1\},$$

$$\partial\{\omega(B) = k\} = \{k-1 \leq \omega(B) \leq k+1\}.$$

In particular, Prop. 2.3 shows that the isoperimetric function $p \mapsto \inf_{\pi(A)=p} \pi_s(\partial A)$ on Poisson space is greater than $1/\sqrt{2}$ times the Gaussian isoperimetric function I. We have $D_x^+ 1_A = D_x^- 1_{A^c}$, hence $\partial_{\rm in} A = \partial_{\rm out} A^c$ and $\partial A = \partial A^c$. We may also define the interior A° of A as

$$A^\circ = \{\omega \ : \ |D^+ 1_A(\omega)|_{L^p(\frac{\sigma+\omega}{2})} = 0\} = \{\omega \in A \ : \ \bar{K}(\omega, A) = 0\} = A \setminus \partial_{\rm in} A,$$

and the closure $\bar{A}$ of A as

$$\begin{aligned} \bar{A} &= \{\omega \in A^c \ : \ |D^- 1_A(\omega)|_{L^p(\frac{\sigma+\omega}{2})} = 0\}^c \\ &= A \cup \{\omega \in \Omega \ : \ \bar{K}(A, \omega) > 0\} = ((A^c)^\circ)^c = A \cup \partial_{\rm out} A. \end{aligned}$$

More refined definitions of inner and outer boundaries are possible, by distinguishing between "forward" and "backward" neighbors. Note however that defining the norms and boundaries with respect to K^+ only, resp. K^- only, leads to $\partial_{\rm out}\{\omega(B) \leq k\} = \emptyset$ since $|D^- 1_{\{\omega(B)\leq k\}}|_{L^p(\sigma)} = 0$, resp. $\partial_{\rm in}\{\omega(B) \geq k\} = \emptyset$ since $|D^+ 1_{\{\omega(B)\geq k\}}|_{L^p(\sigma)} = 0$, i.e. the isoperimetric constants $h_p^\pm$ defined below have trivial zero value. We have

$$\pi(\partial_{\rm in} A) = E[|D^+ 1_A|_{L^\infty(\sigma+\omega)}] = \pi(\{\omega \in A \ : \ \bar{K}(\omega, A^c) > 0\}),$$

$$\pi(\partial_{\rm out}A) = E[|D^-1_A|_{L^\infty(\sigma+\omega)}] = \pi(\{\omega\in A^c \ : \ \bar{K}(\omega,A)>0\}),$$

and

$$\begin{aligned}\pi(\partial A) &= E[|D1_A|_{L^\infty(\sigma+\omega)}] = E[|D^+1_A|_{L^\infty(\sigma+\omega)}] + E[|D^-1_A|_{L^\infty(\sigma+\omega)}]\\ &= \pi(\{\omega\in A \ : \ \bar{K}(\omega,A^c)>0\}) + \pi(\{\omega\in A^c \ : \ \bar{K}(\omega,A)>0\}).\end{aligned}$$

In discrete settings the surface measure $\pi_s(\partial A)$ of ∂A is not defined via a Minkowski content of the form

$$\pi_s(\partial A) = \liminf_{r\to 0}\frac{1}{r}(\pi(\{\omega\in\omega \ : \ d(\omega,A)<r\}) - \pi(A)).$$

Nevertheless, the surface measure of $\partial_{\rm in}A$, resp. $\partial_{\rm out}A$, can defined by averaging $1_A(\omega)\bar{K}(\omega,A^c)^{1/2} = |D^+1_A(\omega)|_{L^2(\frac{\sigma+\omega}{2})}$, resp. $1_{A^c}(\omega)\bar{K}(\omega,A)^{1/2} = |D^-1_A(\omega)|_{L^2(\frac{\sigma+\omega}{2})}$ with respect to the Poisson measure $\pi(d\omega)$.

Definition 4.2 *Let*

$$\pi_s(\partial_{\rm in}A) = E[|D^+1_A(\omega)|_{L^2(\frac{\sigma+\omega}{2})}] = \int_A \bar{K}(\omega,A^c)^{1/2}\pi(d\omega),$$

and

$$\pi_s(\partial_{\rm out}A) = E[|D^-1_A(\omega)|_{L^2(\frac{\sigma+\omega}{2})}] = \int_{A^c} \bar{K}(\omega,A)^{1/2}\pi(d\omega).$$

The above quantities represent average numbers of points in A, resp. A^c, which have a neighbor in A^c, resp. A, the Poisson measure playing here the role of a uniform measure. The surface measure of ∂A is

$$\begin{aligned}\pi_s(\partial A) &= \pi_s(\partial_{\rm in}A) + \pi_s(\partial_{\rm out}A)\\ &= E\left[|D^+1_A|_{L^2(\frac{\sigma+\omega}{2})}\right] + E\left[|D^-1_A|_{L^2(\frac{\sigma+\omega}{2})}\right] = E\left[|D1_A|_{L^2(\frac{\sigma+\omega}{2})}\right]\\ &= \int_A \bar{K}(\omega,A^c)^{1/2}\pi(d\omega) + \int_{A^c}\bar{K}(\omega,A)^{1/2}\pi(d\omega).\end{aligned}$$

As a consequence of the Margulis-Russo identity Cor. 2.1 we obtain asymptotic deviation bounds on $\pi_\lambda(A)$ when A is a monotone set.

Proposition 4.1 *Assume that there exists $\theta>0$ such that $\pi_\theta(A)=1/2$. Let A be an increasing subset of Ω, and let*

$$\Delta^- = \inf_{\partial_{\rm out}A}\|D^-1_A\|_{L^1(\sigma)}.$$

We have for $\lambda>\theta$:

$$\pi_\lambda(A) \le \Phi\left(\sqrt{2\lambda\Delta^-} - \sqrt{2\theta\Delta^-}\right),$$

and for $\lambda < \theta$:

$$\pi_\lambda(A) \geq \Phi\left(\sqrt{2\lambda\Delta^-} - \sqrt{2\theta\Delta^-}\right).$$

If A is decreasing we let

$$\Delta^+ = \inf_{\partial_{\text{in}} A} \|D^+ 1_A\|_{L^1(\sigma)},$$

then

$$\pi_\lambda(A) \leq \Phi\left(\sqrt{2\theta\Delta^+} - \sqrt{2\lambda\Delta^+}\right), \quad \lambda > \theta,$$

and

$$\pi_\lambda(A) \geq \Phi\left(\sqrt{2\theta\Delta^+} - \sqrt{2\lambda\Delta^+}\right), \quad \lambda < \theta.$$

Proof. We adapt an argument of [27], [28] to the Poisson case. We have

$$\begin{aligned}
E_\lambda[\|D^- 1_A\|_{L^2(\sigma)}] &= E_\lambda[1_{\{\|D^- 1_A\|_{L^\infty(\sigma)} > 0\}} \|D^- 1_A\|_{L^2(\sigma)}] \\
&\leq \pi_\lambda(\{\|D^- 1_A\|_{L^\infty(\sigma)} > 0\})^{1/2} E_\lambda[\|D^- 1_A\|^2_{L^2(\sigma)}]^{1/2} \\
&\leq \pi_\lambda(\partial_{\text{out}} A)^{1/2} E_\lambda[\|D^- 1_A\|_{L^1(\sigma)}]^{1/2} \\
&\leq \frac{1}{\sqrt{\Delta^-}} E_\lambda[\|D^- 1_A\|_{L^1(\sigma)}].
\end{aligned}$$

Let $f(\lambda) = \pi_\lambda(A)$. Using (13) we get

$$\begin{aligned}
f'(\lambda) &= E_\lambda[\|D^- 1_A\|_{L^1(\sigma)}]. \\
&\geq \sqrt{\Delta^-} E_\lambda[\|D^- 1_A\|_{L^2(\sigma)}] \\
&\geq \sqrt{\frac{\Delta^-}{2\lambda}} I(f(\lambda)) \\
&= \frac{-\sqrt{\Delta^-}}{\sqrt{2\lambda} I''(f(\lambda))}.
\end{aligned}$$

Hence for $\lambda > \theta$,

$$\begin{aligned}
\Phi^{-1}(f(\lambda)) &= \Phi^{-1}(f(\lambda)) - \Phi^{-1}(f(\theta)) \\
&= I'(f(\theta)) - I'(f(\lambda)) \\
&= \int_\lambda^\theta I''(f(t)) f'(t) dt \\
&\leq -\int_\lambda^\theta \frac{\sqrt{\Delta^-}}{\sqrt{2t}} dt \\
&= \sqrt{2\Delta^-}(\sqrt{\lambda} - \sqrt{\theta}),
\end{aligned}$$

and finally

$$f(\lambda) \leq \Phi\left(\sqrt{2\lambda\Delta^-} - \sqrt{2\theta\Delta^-}\right).$$

If A is decreasing and $\lambda > \theta$ we similarly show that

$$E_\lambda[\|D^+ 1_A\|_{L^2(\sigma)}] \leq \pi_\lambda(\partial_{\rm in} A)^{1/2} E_\lambda[\|D^+ 1_A\|_{L^1(\sigma)}]^{1/2} \leq \frac{1}{\sqrt{\Delta^+}} E_\lambda[\|D^+ 1_A\|_{L^1(\sigma)}],$$

$$\begin{aligned} f'(\lambda) &= -E_\lambda[\|D^+ 1_A\|_{L^1(\sigma)}]. \\ &\leq \sqrt{\Delta^+} E_\lambda[\|D^+ 1_A\|_{L^2(\sigma)}] \\ &\leq \frac{\sqrt{\Delta^+}}{\sqrt{2\lambda} I''(f(\lambda))}, \end{aligned}$$

and

$$\Phi^{-1}(f(\lambda)) \leq \int_\lambda^\theta \frac{\sqrt{\Delta^+}}{\sqrt{2t}} dt = \sqrt{2\Delta^+}(\sqrt{\theta} - \sqrt{\lambda}).$$

The case $\lambda < \theta$ is treated in a similar way. □

When $\lambda < \theta$ and Δ^- is large, the lower bound is equivalent to

$$\frac{1}{\sqrt{2\pi}(\sqrt{2\theta\Delta^-} - \sqrt{2\lambda\Delta^-})} e^{-(\sqrt{2\lambda\Delta^-} - \sqrt{2\theta\Delta^-})^2/2}.$$

As an example, for the increasing set $\{\omega(B) \geq n\}$ we have

$$\partial_{\rm out}\{\omega(B) \geq n\} = \{\omega(B) = n-1\},$$

and

$$D_x 1_{\{\omega(B)\geq n\}} = -D_x^- 1_{\{\omega(B)\geq n\}} = -1_B(x) 1_{\{\omega(B)=n-1\}},$$

hence

$$\|D1_{\{\omega(B)\geq n\}}\|_{L^1(\sigma)} = \sigma(B) 1_{\{\omega(B)=n-1\}} = \sigma(B) 1_{\partial_{\rm out}\{\omega(B)\geq n\}},$$

and $\Delta^- = \sigma(B)$. For the decreasing set $\{\omega(B) \leq n\}$ we have

$$\partial_{\rm in}\{\omega(B) \leq n\} = \{\omega(B) = n\},$$

and

$$D_x 1_{\{\omega(B)\leq n\}} = -D_x^- 1_{\{\omega(B)\leq n\}} = -1_B(x) 1_{\{\omega(B)=n\}},$$

hence

$$\|D1_{\{\omega(B)\leq n\}}\|_{L^1(\sigma)} = \sigma(B) 1_{\{\omega(B)=n\}} = \sigma(B) 1_{\partial_{\rm in}\{\omega(B)=n\}},$$

and $\Delta^+ = \sigma(B)$.

5 Co-area formulas

For $p = \infty$ the next Lemma shows that

$$E[|D^+F|_{L^\infty(\sigma+\omega)}] = \int_{-\infty}^{+\infty} \pi(\partial_{\rm in}\{F > t\})dt,$$

$$E[|D^-F|_{L^\infty(\sigma+\omega)}] = \int_{-\infty}^{+\infty} \pi(\partial_{\rm out}\{F > t\})dt.$$

Lemma 5.1 *We have*

$$E[|D^\pm F|_{L^\infty(\sigma+\omega)}] = \int_{-\infty}^{+\infty} E[|D^\pm 1_{\{F>t\}}|_{L^\infty(\sigma+\omega)}]dt,$$

and

$$E\left[|D^+F|_{L^\infty(\sigma+\omega)}\right] + E\left[|D^-F|_{L^\infty(\sigma+\omega)}\right] = \int_{-\infty}^{\infty} E\left[|D1_{\{F>t\}}|_{L^\infty(\sigma+\omega)}\right] dt.$$

Proof. The notations ess sup $_{\tilde\omega\in\mathcal{N}_\omega}$ and ess inf $_{\tilde\omega\in\mathcal{N}_\omega}$ denote respectively ess sup $_{\bar K(\omega,d\tilde\omega)}$ and ess inf $_{\bar K(\omega,d\tilde\omega)}$. We have

$$|D^+F(\omega)|_{L^\infty(\sigma+\omega)} = \text{ess sup}_{\,\tilde\omega\in\mathcal{N}_\omega} - F(\tilde\omega))^+ = F(\omega) - \text{ess inf}_{\,\tilde\omega\in\mathcal{N}_\omega} F(\tilde\omega),$$

hence

$$\begin{aligned}
E[|D^+F|_{L^\infty(\sigma+\omega)}] &= E[F] - E[\text{ess inf}_{\,\tilde\omega\in\mathcal{N}_\omega} F(\tilde\omega)] \\
&= \int_{-\infty}^{+\infty} \pi(\{F > t\})dt - \int_{-\infty}^{+\infty} \pi(\text{ess inf}_{\,\tilde\omega\in\mathcal{N}_\omega} F(\tilde\omega) > t)dt \\
&= \int_{-\infty}^{+\infty} \pi(\{F > t\})dt - \int_{-\infty}^{+\infty} \pi(\{\text{ess inf}_{\,\tilde\omega\in\mathcal{N}_\omega} F(\tilde\omega) > t \text{ and } F(\omega) > t\})dt \\
&= \int_{-\infty}^{+\infty} \pi(\{F(\omega) > t \text{ and } (\sigma+\omega)(\{x \in X \ : \ F(\omega \pm \delta_x) \le t\}) > 0\})dt \\
&= \int_{-\infty}^{+\infty} \pi(\{\omega \in \Omega \ : \ (\sigma+\omega)(\{x \in X \ : F(\omega) > t \text{ and } F(\omega \pm \delta_x) \le t\}) > 0\})dt \\
&= \int_{-\infty}^{+\infty} \pi(\{\omega \in \Omega \ : \ (\sigma+\omega)(\{x \in X \ : 1_{\{F(\omega)>t\}} - 1_{\{F(\omega\pm\delta_x)>\}} = 1\}) > 0\})dt \\
&= \int_{-\infty}^{+\infty} \pi(\{\omega \in \Omega \ : \ |D^+1_{\{F>t\}}|_{L^\infty(\sigma+\omega)} = 1\})dt \\
&= \int_{-\infty}^{+\infty} E[|D^+1_{\{F>t\}}|_{L^\infty(\sigma+\omega)}]dt.
\end{aligned}$$

The proof for D^- is similar. Finally we have, since $D_x^+ F D_x^- F = 0$:

$$\begin{aligned} & E[|D^+F|_{L^\infty(\sigma+\omega)}] + E[|D^-F|_{L^\infty(\sigma+\omega)}] \\ &= \int_{-\infty}^{\infty} E[|D^+ 1_{\{F>t\}}|_{L^\infty(\sigma+\omega)}]dt + \int_{-\infty}^{\infty} E[|D^- 1_{\{F>t\}}|_{L^\infty(\sigma+\omega)}]dt \\ &= \int_{-\infty}^{\infty} E[|D^+ 1_{\{F>t\}}|_{L^\infty(\sigma+\omega)} + |D^- 1_{\{F>t\}}|_{L^\infty(\sigma+\omega)}]dt \\ &= \int_{-\infty}^{\infty} E[|D 1_{\{F>t\}}|_{L^\infty(\sigma+\omega)}]dt. \end{aligned}$$

□

The next Lemma states a co-area formula in L^1.

Lemma 5.2 *We have*

$$E[|D^\pm F|_{L^1(\sigma)}] = \int_{-\infty}^{+\infty} E[|D^\pm 1_{\{F>t\}}|_{L^1(\sigma)}]dt,$$

$$E[|D^\pm F|_{L^1(\omega)}] = \int_{-\infty}^{+\infty} E[|D^\pm 1_{\{F>t\}}|_{L^1(\omega)}]dt.$$

Proof. We have for all $a, b \in \mathbb{R}$:

$$(b-a)^\pm = \int_{-\infty}^{\infty} (1_{\{a>t\}} - 1_{\{b>t\}})^\pm dt,$$

hence

$$D_x^\pm F = \int_{-\infty}^{\infty} D_x^\pm 1_{\{F>t\}} dt.$$

□

As a consequence we have

$$E\left[|D^\pm F|_{L^1(\frac{\sigma+\omega}{2})}\right] = \int_{-\infty}^{+\infty} E\left[|D^\pm 1_{\{F>t\}}|_{L^1(\frac{\sigma+\omega}{2})}\right] dt,$$

and

$$E\left[|DF|_{L^1(\frac{\sigma+\omega}{2})}\right] = \int_{-\infty}^{+\infty} E\left[|D 1_{\{F>t\}}|_{L^1(\frac{\sigma+\omega}{2})}\right] dt.$$

Proposition 5.1 *We have*

$$E[\Gamma^\pm(F,F)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E[\Gamma^\pm(1_{\{F>t\}}, 1_{\{F>s\}})]dsdt.$$

Proof. Again we use the identity

$$D_x^\pm F = \int_{-\infty}^{+\infty} D_x^\pm 1_{\{F>t\}} dt.$$

□

We close this section with an application of co-area formulas to an equivalence result on functional inequalities. Let $\mathcal{G}$ be a non-empty set of functions on Ω, and let

$$\mathcal{L}(F) = \sup_{G_1, G_2 \in \mathcal{G}} E[F^+ G_1 + F^- G_2]. \tag{17}$$

Several functionals have the representation (17), for example the entropy

$$\mathcal{L}(F) = \mathrm{Ent}|F| = E[|F| \log |F|] - E[|F|] \log E[|F|] = \sup_{E[e^G] \leq 1} E[|F| G],$$

the variance

$$\mathcal{L}(F) = E[(F - E[F])^2] = \mathrm{var}(F) = \inf_{a \in \mathbb{R}} E[(F-a)^2],$$

and

$$\mathcal{L}(F) = E[|F - m(F)|] = \inf_{a \in \mathbb{R}} E[|F-a|],$$

where $m(F)$ is by definition a median of F. The co-area formula implies the following equivalence, as in [12], [24]. The norm $|\cdot|_p$ denotes either $|\cdot|_{L^1(\sigma)}$ or $|\cdot|_{L^1(\omega)}$ when $p = 1$, and $|\cdot|_{L^\infty(\sigma+\omega)}$ when $p = \infty$.

Theorem 5.1 *Let $c \geq 0$. The following are equivalent:*

(i) $c\mathcal{L}(F) \leq E[|D^\pm F|_p]$, for all $F : \Omega \to \mathbb{R}$,

(ii) $c\mathcal{L}(1_A) \leq E[|D^\pm 1_A|_p]$ and $c\mathcal{L}(-1_A) \leq E[|D^\pm(-1_A)|_p]$, for all $A \in \mathcal{F}$,

with $p = 1, \infty$.

Proof. We follow the proof of [12]. In order to show $(ii) \Rightarrow (i)$ we note that for all $G_1, G_2 \in \mathcal{G}$,

$$\begin{aligned}
E[|D^\pm F|_p] &= \int_0^\infty E[|D^\pm 1_{\{F>t\}}|_p] dt + \int_{-\infty}^0 E[|D^\pm 1_{\{F>t\}}|_p] dt \\
&\geq c \int_0^\infty E[G_1 1_{\{F>t\}}] dt + \int_{-\infty}^0 E[|D^\pm(-1_{\{F \leq t\}})|_p] dt \\
&\geq cE[G_1 F^+] - c \int_{-\infty}^0 E[1_{\{F \leq t\}} G_2] dt \\
&= cE[G_1 F^+] + cE[F^- G_2],
\end{aligned}$$

hence

$$E[|D^\pm F|_p] \geq c \sup_{G_1, G_2 \in \mathcal{G}} (E[G_1 F^+] + E[F^- G_2]) \geq c\mathcal{L}(F).$$

□

6 Some explicit computations

In this section we define the main isoperimetric constants and establish some bounds on these constants.

Definition 6.1 *Let for $p \in [1, \infty]$:*

$$h_p^{\pm} = \inf_{0<\pi(A)<\frac{1}{2}} \frac{E\left[|D^{\pm}1_A|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A)}, \qquad h_p = \inf_{0<\pi(A)<\frac{1}{2}} \frac{E\left[|D1_A|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A)}.$$

We have

$$h_1^+ = h_1^- = \inf_{0<\pi(A)<\frac{1}{2}} \frac{1}{\pi(A)} \int_A \bar{K}(\omega, A^c)\pi(d\omega) = \inf_{0<\pi(A)<\frac{1}{2}} \frac{1}{\pi(A)} \int_{A^c} \bar{K}(\omega, A)\pi(d\omega),$$

$$h_2^+ = \inf_{0<\pi(A)<\frac{1}{2}} \frac{\pi_s(\partial_{\text{in}}A)}{\pi(A)}, \quad h_2^- = \inf_{0<\pi(A)<\frac{1}{2}} \frac{\pi_s(\partial_{\text{out}}A)}{\pi(A)}, \quad h_2 = \inf_{0<\pi(A)<\frac{1}{2}} \frac{\pi_s(\partial A)}{\pi(A)}$$

and

$$h_\infty^+ = \inf_{0<\pi(A)<\frac{1}{2}} \frac{\pi(\partial_{\text{in}}A)}{\pi(A)}, \quad h_\infty^- = \inf_{0<\pi(A)<\frac{1}{2}} \frac{\pi(\partial_{\text{out}}A)}{\pi(A)}, \quad h_\infty = \inf_{0<\pi(A)<\frac{1}{2}} \frac{\pi(\partial A)}{\pi(A)}.$$

The following is a functional version of $h_p^{\pm}$.

Definition 6.2 *Let for $p \in [1, \infty]$:*

$$\tilde{h}_p^{\pm} = \inf_{0<\pi(A)<1} \frac{E\left[|D^{\pm}1_A|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A)\pi(A^c)}, \qquad \tilde{h}_p = \inf_{0<\pi(A)<1} \frac{E\left[|D1_A|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A)\pi(A^c)}.$$

Note that in the definition of the isoperimetric constants we need to integrate with respect to $\omega + \sigma$, otherwise integrating with respect to ω or σ only would lead to vanishing isoperimetric constants, since

$$\{|D^+1_{\{\omega(B)\leq k\}}|_{L^p(\omega)} > 0\} = \emptyset,$$

and

$$\{|D^-1_{\{\omega(B)\geq k\}}|_{L^p(\sigma)} > 0\} = \emptyset.$$

The next proposition follows the presentation of [26].

Proposition 6.1 *We have*

a) $h_1 = 2h_1^+ = 2h_1^-$,

b) $\tilde{h}_p^+ = \tilde{h}_p^-$, $p = 1, \infty$,

c) $\min(h_p^+, h_p^-) < \tilde{h}_p^{\pm} < 2\min(h_p^+, h_p^-) \leq h_p^+ + h_p^- \leq h_p < \tilde{h}_p < 2h_p, \quad p \in [1, +\infty]$.

Proof. For the first statement we use Lemma 2.1, which implies

$$E\left[|D^+F|^p_{L^p(\frac{\sigma+\omega}{2})}\right] = E\left[|D^-F|^p_{L^p(\frac{\sigma+\omega}{2})}\right] = \frac{1}{2}E\left[|DF|^p_{L^p(\frac{\sigma+\omega}{2})}\right].$$

The second statement follows from Remark 4.1. The last statement follows from the inequalities, if $0 < \pi(A) < 1/2$:

$$\begin{aligned} h_p^+ &\leq \frac{E\left[|D^+1_A|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A)} \leq \frac{E\left[|D^+1_A|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A)\pi(A^c)} \\ &= \frac{E\left[|D^{\pm}1_A|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A)\pi(A^c)} \leq 2\frac{E\left[|D^{\pm}1_A|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A)}, \end{aligned}$$

and similarly if $1/2 \leq \pi(A) < 1$:

$$\begin{aligned} h_p^- &\leq \frac{E\left[|D^-1_{A^c}|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A^c)} \leq \frac{E\left[|D^-1_{A^c}|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A)\pi(A^c)} \\ &= \frac{E\left[|D^{\pm}1_{A^c}|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A)\pi(A^c)} \leq 2\frac{E\left[|D^{\pm}1_{A^c}|_{L^p(\frac{\sigma+\omega}{2})}\right]}{\pi(A^c)}. \end{aligned}$$

□

Definition 6.3 *Let for* $p \in [1, \infty]$*:*

$$k_p^{\pm} = \inf_{F \neq C} \frac{E\left[|D^{\pm}F|_{L^p(\frac{\sigma+\omega}{2})}\right]}{E[(F - m(F))^{\pm}]}, \qquad k_p = \inf_{F \neq C} \frac{E\left[|DF|_{L^p(\frac{\sigma+\omega}{2})}\right]}{E[|F - m(F)|]}.$$

Proposition 6.2 *We have* $h_1^{\pm} = k_1^{\pm}$, $h_\infty^{\pm} = k_\infty^{\pm}$, $h_1 = k_1$, *and* $k_\infty \leq h_\infty \leq 2k_\infty$.

Proof. First of all we note that since $m(1_A) = 0$ if $\pi(A) \leq 1/2$, we have

$$k_p^{\pm}\pi(A) = k_p^{\pm}E[(1_A - m(1_A))^{\pm}] \leq E\left[|D^{\pm}1_A|_{L^p(\frac{\sigma+\omega}{2})}\right],$$

hence $h_p^{\pm} \geq k_p^{\pm}$, $p = 1, \infty$, and similarly

$$k_p\pi(A) = k_pE[|1_A - m(1_A)|] \leq E\left[|D1_A|_{L^p(\frac{\sigma+\omega}{2})}\right],$$

hence $h_p \geq k_p$, $p = 1, \infty$, From the co-area formulas Lemmas 5.1 and 5.2 we have for $p = 1, \infty$, since $\pi(F > m(F)) \leq 1/2$:

$$\begin{aligned} E\left[|D^+F|_{L^p(\frac{\sigma+\omega}{2})}\right] &= \int_{-\infty}^{\infty} E\left[|D^+1_{\{F>t\}}|_{L^p(\frac{\sigma+\omega}{2})}\right] dt \\ &\geq h_p^+ \int_{-\infty}^{\infty} \pi(\{F > t\})dt \\ &\geq h_p^+ \int_{m(F)}^{\infty} \pi(\{F > t\})dt \\ &= h_p^+ \int_{0}^{\infty} \pi(\{F - m(F) > t\})dt \\ &\geq h_p^+ E[(F - m(F))^+]. \end{aligned}$$

Hence $k_p^+ \geq h_p^+$. Similarly we obtain

$$E\left[|D^-F|_{L^p(\frac{\sigma+\omega}{2})}\right] = E\left[|D^+(-F)|_{L^p(\frac{\sigma+\omega}{2})}\right] \geq h_p^- E[(-F-m(-F))^+] = h_p^- E[(F-m(F))^-],$$

hence $k_p^- \geq h_p^-$, and

$$\begin{aligned} E\left[|DF|_{L^1(\frac{\sigma+\omega}{2})}\right] &= \int_{-\infty}^{m(F)} E\left[|D1_{\{F>t\}}|_{L^1(\frac{\sigma+\omega}{2})}\right] dt + \int_{m(F)}^{\infty} E\left[|D1_{\{F>t\}}|_{L^1(\frac{\sigma+\omega}{2})}\right] dt \\ &= \int_{0}^{\infty} E\left[|D1_{\{-F+m(F)>t\}}|_{L^1(\frac{\sigma+\omega}{2})}\right] dt + \int_{0}^{\infty} E\left[|D1_{\{F-m(F)>t\}}|_{L^1(\frac{\sigma+\omega}{2})}\right] dt \\ &\geq h_1 \int_{-\infty}^{0} \pi(\{-F + m(F) > t\})dt + h_1 \int_{0}^{\infty} \pi(\{F - m(F) > t\})dt \\ &\geq h_1 E[(F - m(F))^-] + h_1 E[(F - m(F))^+] \\ &= h_1 E[|F - m(F)|], \end{aligned}$$

hence $k_1 \geq h_1$. From Lemma 5.1 we also have

$$\begin{aligned} 2E[|DF|_{L^\infty(\sigma+\omega)}] &\geq E[|D^+F|_{L^\infty(\sigma+\omega)}] + E[|D^-F|_{L^\infty(\sigma+\omega)}] \\ &= \int_{-\infty}^{\infty} E[|D1_{\{F>t\}}|_{L^\infty(\sigma+\omega)}]dt \\ &= \int_{-\infty}^{m(F)} E[|D1_{\{F>t\}}|_{L^\infty(\sigma+\omega)}]dt + \int_{m(F)}^{\infty} E[|D1_{\{F>t\}}|_{L^\infty(\sigma+\omega)}]dt \\ &= \int_{0}^{\infty} E[|D1_{\{-F+m(F)>t\}}|_{L^\infty(\sigma+\omega)}]dt + \int_{0}^{\infty} E[|D1_{\{F-m(F)>t\}}|_{L^\infty(\sigma+\omega)}]dt \\ &\geq h_\infty \int_{-\infty}^{0} \pi(\{-F + m(F) > t\})dt + h_\infty \int_{0}^{\infty} \pi(\{F - m(F) > t\})dt \\ &\geq h_\infty E[(F - m(F))^-] + h_\infty E[(F - m(F))^+] \\ &\geq h_\infty E[|F - m(F)|], \end{aligned}$$

hence $2k_\infty \geq h_\infty$. □

Remark 6.1 *The above proof also implies, if* $F \geq 0$ *and* $\pi(F > 0) \leq 1/2$*:*

$$h_p^+ E[F] \leq E\left[|D^+ F|_{L^p(\frac{\sigma+\omega}{2})}\right],$$

and

$$h_p^- E[F] \leq E\left[|D^- F|_{L^p(\frac{\sigma+\omega}{2})}\right].$$

The following is the definition of the Poincaré constants.

Definition 6.4 *Let for* $p \in [1, \infty]$*:*

$$\lambda_p^{\pm} = \inf_{F \neq C} \frac{E\left[|D^{\pm} F|^2_{L^p(\frac{\sigma+\omega}{2})}\right]}{\mathrm{var}(F)}, \qquad \lambda_p = \inf_{F \neq C} \frac{E\left[|DF|^2_{L^p(\frac{\sigma+\omega}{2})}\right]}{\mathrm{var}(F)}.$$

Remark that $\lambda_p^+ = \lambda_p^-$, $p \in [1, \infty]$, since $D_x^+ F = D_x^-(-F)$, and $\tilde{h}_1^+ \geq \lambda_2^+$. We have

$$E\left[|DF|^2_{L^2(\frac{\sigma+\omega}{2})}\right] = \frac{1}{2} E\left[|DF|^2_{L^2(\sigma)}\right] = \frac{1}{2} E\left[|DF|^2_{L^2(\omega)}\right],$$

hence

$$\lambda_2 = 2 \inf_{F \neq C} \frac{\mathcal{E}(F, F)}{\mathrm{var}(F)}.$$

Th. 5.1 shows that

$$\lambda_\infty^{\pm} = \inf_{\pi(A) > 0} \frac{E[|D^{\pm} 1_A|_{L^\infty(\sigma+\omega)}]}{\mathrm{var} 1_A}.$$

Definition 6.5 *Let for* $p \in [1, \infty]$*:*

$$\tilde{k}_p^{\pm} = \inf_{F \neq C} \frac{E[|D^{\pm} F|_{L^p(\frac{\sigma+\omega}{2})}]}{E[(F - E[F])^{\pm}]}, \qquad \tilde{k}_p = \inf_{F \neq C} \frac{E[|DF|_{L^p(\frac{\sigma+\omega}{2})}]}{E[|F - E[F]|]}.$$

When $\sigma(X) < \pi/4$, Relation (22) also improves the lower bound on h_∞ given in [4] in the cylindrical (i.e. finite dimensional) case.

Proposition 6.3 *We have*

$$\lambda_2 = 2\lambda_2^+ = 2\lambda_2^- = 1, \tag{18}$$

$$\lambda_\infty = \frac{1}{\sigma(X)}, \tag{19}$$

$$\frac{1}{2} \leq h_1 = 2h_1^+ = 2h_1^- \leq 2 + 2\sqrt{\sigma(X)}, \tag{20}$$

$$1/\sqrt{2\pi} \leq h_2, \tag{21}$$

$$\max\left(\frac{1}{\sqrt{\pi\sigma(X)}}, \frac{1}{2\sigma(X)}\right) \leq h_\infty \leq \frac{4}{\sigma(X)} + \frac{4}{\sqrt{\sigma(X)}}, \tag{22}$$

$$\lambda_\infty^+ \leq \frac{h_\infty^+}{2} = \frac{k_\infty^+}{2} \leq \sqrt{\lambda_\infty} = \frac{1}{\sqrt{\sigma(X)}}, \tag{23}$$

$$\lambda_\infty^- \leq \frac{h_\infty^-}{2} \leq \sqrt{\lambda_\infty} = \frac{1}{\sigma(X)} + \sqrt{\frac{2}{\sigma(X)}}, \tag{24}$$

$$k_2^+ \leq 2, \tag{25}$$

$$h_2^+ \leq \sqrt{1 + \sqrt{\sigma(X)}}. \tag{26}$$

Proof. The Clark formula and Lemma 2.1 show that when $\pi(A) \leq 1/2$,

$$\begin{aligned}\frac{1}{2}\pi(A) &\leq \mathrm{var}(1_A) \leq E[|D1_A|^2_{L^2(\sigma)}] = E[|D1_A|_{L^1(\sigma)}] \\ &= 2E\left[|D^+1_A|_{L^1(\frac{\sigma+\omega}{2})}\right] = 2E\left[|D^-1_A|_{L^1(\frac{\sigma+\omega}{2})}\right] = E\left[|D1_A|_{L^1(\frac{\sigma+\omega}{2})}\right]\end{aligned}$$

hence

$$h_1 = 2h_1^- = 2h_1^+ \geq 1/2,$$

which proves the first part of (20). We have

$$\begin{aligned}h_1^+\pi(A) &\leq E[|D^+1_A|_{L^1(\frac{\sigma+\omega}{2})}] = \frac{1}{2}E[|D1_A|_{L^1(\sigma)}] \\ &\leq \frac{1}{2}\sigma(X)E[|D1_A|_{L^\infty(\sigma)}] \leq \frac{1}{2}\sigma(X)E[|D1_A|_{L^\infty(\sigma+\omega)}],\end{aligned}$$

hence $h_1^+ \leq \sigma(X)h_\infty/2$, which yields the second part of (20) from (22) (proved below). On the other hand,

$$\begin{aligned}\mathrm{var}F &\leq E[|DF|^2_{L^2(\sigma)}] = E[|DF|^2_{L^2(\omega)}] = E\left[|DF|^2_{L^2(\frac{\sigma+\omega}{2})}\right] = 2E\left[|D^\pm F|^2_{L^2(\frac{\sigma+\omega}{2})}\right] \\ &\leq \sigma(X)E[|D^\pm F|^2_{L^\infty(\sigma+\omega)}],\end{aligned}$$

hence $\lambda_2 = 2\lambda_2^- = 2\lambda_2^+ \geq 1$ and $\lambda_\infty \geq 1/\sigma(X)$. Letting $F(\omega) = \omega(X)$, we have

$$D_x F = 1_{\{x\in\omega\}} - 1_{\{x\in\omega^c\}},$$

and

$$\mathrm{var}(F) = \sigma(X) = E[|DF|^2_{L^2(\sigma)}] = E[|DF|^2_{L^2(\frac{\sigma+\omega}{2})}] = \sigma(X) E[|DF|^2_{L^\infty(\sigma+\omega)}],$$

which shows $\lambda_2 \leq 1$ and $\lambda_\infty \leq 1/\sigma(X)$, and concludes (18) and (19). Relation (21) follows from Th. 2.3: applying (13) to $F = 1_A$ we get, since $I(1_A) = 0$:

$$E[|D1_A|_{L^2(\frac{\sigma+\omega}{2})}] \geq \frac{1}{\sqrt{2}} E[|D1_A|_{L^2(\sigma)}] \geq \frac{1}{2} I(\pi(A)) \geq \frac{2}{\sqrt{2\pi}} \pi(A)(1-\pi(A)) \geq \frac{1}{\sqrt{2\pi}} \pi(A),$$

hence $h_2 \geq 1/\sqrt{2\pi}$, and

$$E[|D1_A|_{L^\infty(\sigma+\omega)}] \geq E[|D1_A|_{L^\infty(\sigma)}] \geq \frac{1}{\sqrt{\sigma(X)}} E[|D1_A|_{L^2(\sigma)}] \geq \frac{1}{\sqrt{\pi\sigma(X)}} \pi(A),$$

where we used the inequality

$$I(t) \geq \sqrt{\frac{2}{\pi}} I_{\mathrm{var}}(t),$$

with $I_{\mathrm{var}}(t) = t(1-t)$, $0 \leq t \leq 1$, hence $h_\infty \geq 1/\sqrt{\pi\sigma(X)}$. We have if $\pi(A) < 1/2$:

$$\lambda_\infty \pi(A) \leq 2\lambda_\infty \pi(A)\pi(A^c) \leq 2E[|D1_A|^2_{L^\infty(\sigma+\omega)}] = 2E[|D1_A|_{L^\infty(\sigma+\omega)}],$$

hence $\lambda_\infty \leq 2h_\infty$ which, with Th. 6.2 and $h_\infty \geq 1/\sqrt{\pi\sigma(X)}$, proves Relation (22). Similarly for $\pi(A) \leq 1/2$ we have

$$\lambda^\pm_\infty \pi(A) \leq 2\lambda^\pm_\infty \pi(A)\pi(A^c) \leq 2E[|D^\pm 1_A|^2_{L^\infty(\sigma+\omega)}] = 2E[|D^\pm 1_A|_{L^\infty(\sigma+\omega)}],$$

hence $\lambda^\pm_\infty \leq 2h^\pm_\infty$, and (23), (24) hold from Th. 6.1. Relation (25) follows from Th. 7.1 below. We also have

$$\begin{aligned}
(h_2^+)^2 \pi(A)^2 &\leq E\left[|D^+ 1_A|_{L^2(\frac{\sigma+\omega}{2})}\right]^2 \\
&= E\left[1_A |D^+ 1_A|_{L^2(\frac{\sigma+\omega}{2})}\right]^2 \\
&\leq \pi(A) E\left[|D^+ 1_A|^2_{L^2(\frac{\sigma+\omega}{2})}\right] \\
&= \pi(A) E\left[|D^+ 1_A|_{L^1(\frac{\sigma+\omega}{2})}\right],
\end{aligned}$$

hence $(h_2^+)^2 \leq h_1^+$, which proves (26). □

Clearly the logarithmic Sobolev constants

$$l^\pm_p = \inf_{0<\pi(A)<\frac{1}{2}} \frac{E\left[|D^\pm 1_A|_{L^p(\frac{\sigma+\omega}{2})}\right]}{-\pi(A)\log\pi(A)}, \quad \text{and} \quad l_p = \inf_{0<\pi(A)<\frac{1}{2}} \frac{E\left[|D1_A|_{L^p(\frac{\sigma+\omega}{2})}\right]}{-\pi(A)\log\pi(A)}$$

vanish, $p \in [1, +\infty]$, since

$$l_\infty = \inf_{0<\pi(A)<\frac{1}{2}} \frac{\pi(\partial A)}{-\pi(A)\log\pi(A)} = 0,$$

(take $A_k = \{\omega(B) \geq k\}$), i.e. from Th. 5.1 the classical logarithmic Sobolev inequality does not hold on Poisson space. In other terms the optimal constant ρ_p in the inequality

$$\rho_p \mathrm{Ent}[F^2] \leq E[|DF|^2_{L^p(\sigma)}].$$

is equal to 0 for all $p \geq 1$, cf. [15].

Theorem 6.1 *We have*

$$k_\infty^+ = h_\infty^+ \leq 2\sqrt{\lambda_\infty} = \frac{2}{\sqrt{\sigma(X)}} \quad \textit{and} \quad k_\infty^- = h_\infty^- \leq \frac{2}{\sigma(X)}\left(1 + \sqrt{2\sigma(X)}\right).$$

Proof. Note that if $F \geq 0$,

$$\begin{aligned}
|D^+F^2(\omega)|_{L^\infty(\sigma+\omega)} &= \text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}(F^2(\omega) - F^2(\tilde\omega)) \\
&= \text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}(F^2(\omega) - F^2(\tilde\omega))1_{\{F(\omega)\geq F(\tilde\omega)\}} \\
&= \text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}(F(\omega) - F(\tilde\omega))(F(\omega) + F(\tilde\omega))1_{\{F(\omega)\geq F(\tilde\omega)\}} \\
&\leq 2\text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}(F(\omega) - F(\tilde\omega))F(\omega) \\
&= 2|D^+F|_{L^\infty(\sigma+\omega)}F(\omega).
\end{aligned}$$

If $\pi(\{F > 0\}) \leq 1/2$, then by Remark 6.1 applied to F^2,

$$\begin{aligned}
(h_\infty^+)^2 E[F^2]^2 &\leq E[|D^+F^2|_{L^\infty(\sigma)}]^2 \\
&\leq 4E[F|D^+F|_{L^\infty(\sigma)}]^2 \\
&\leq 4E[|D^+F|^2_{L^\infty(\sigma+\omega)}]E[F^2],
\end{aligned}$$

hence

$$\frac{(h_\infty^+)^2}{4}E[F^2] \leq E[|D^+F|^2_{L^\infty(\sigma)}].$$

In the general case we may assume that $m(F) = 0$, i.e.

$$\pi(\{F > 0\}) \leq 1/2, \quad \text{and} \quad \pi(\{F < 0\}) \leq 1/2.$$

We have

$$\pi(\{F^+ > 0\}) \leq 1/2, \quad \text{and} \quad \pi(\{F^- < 0\}) \leq 1/2,$$

hence

$$\frac{(h_\infty^+)^2}{4}E[(F^+)^2] \leq E[|D^+F^+|^2_{L^\infty(\sigma)}],$$

and

$$\frac{(h_\infty^+)^2}{4}E[(F^-)^2] \leq E[|D^+F^-|^2_{L^\infty(\sigma)}].$$

We have

$$\begin{aligned}|D^+F^+(\omega)|_{L^\infty(\sigma)} &= \text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F^+(\omega)-F^+(\tilde{\omega}))\\ &= \text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}|F(\omega)-F(\tilde{\omega})|1_{\{F(\omega)>0\}},\end{aligned}$$

and

$$\begin{aligned}|D^+F^-(\omega)|_{L^\infty(\sigma)} &= \text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F^-(\omega)-F^-(\tilde{\omega}))\\ &= \text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}|F(\omega)-F(\tilde{\omega})|1_{\{F(\omega)<0\}}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{(h_\infty^+)^2}{4}\text{var}F &\leq \frac{(h_\infty^+)^2}{4}E[F^2] = \frac{(h_\infty^+)^2}{4}E[F^2 1_{\{F>0\}}] + \frac{(h_\infty^+)^2}{4}E[F^2 1_{\{F<0\}}]\\ &\leq E[1_{\{F>0\}}|DF|^2_{L^\infty(\sigma)}] + E[1_{\{F<0\}}|DF|^2_{L^\infty(\sigma)}],\end{aligned}$$

from which $\lambda_\infty \geq (h_\infty^+)^2/4$. The second statement has a similar proof:

$$\begin{aligned}|D^-F^2|_{L^\infty(\sigma)} &= \text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F^2(\tilde{\omega})-F^2(\omega))\\ &= \text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F(\tilde{\omega})-F(\omega))(F(\tilde{\omega})+F(\omega))1_{\{F(\tilde{\omega})\geq F(\omega)\}}\\ &= \text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F(\tilde{\omega})-F(\omega))^2 + 2(F(\tilde{\omega})-F(\omega))F(\omega)1_{\{F(\tilde{\omega})\geq F(\omega)\}}\\ &\leq \text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F(\tilde{\omega})-F(\omega))^2 + 2(F(\tilde{\omega})-F(\omega))F(\omega)1_{\{F(\tilde{\omega})\geq F(\omega)\}}.\end{aligned}$$

By Remark 6.1,

$$\begin{aligned}h_\infty^- E[F^2] &\leq E\left[\text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F(\tilde{\omega})-F(\omega))^2 + 2(F(\tilde{\omega})-F(\omega))F(\omega)1_{\{F(\tilde{\omega})\geq F(\omega)\}}\right]\\ &\leq E\left[\text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F(\tilde{\omega})-F(\omega))^2\right]\\ &\quad +2E[F^2]^{1/2}E\left[\text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F(\tilde{\omega})-F(\omega))^2 1_{\{F(\tilde{\omega})\geq F(\omega)\}}\right]^{1/2},\end{aligned}$$

hence

$$(\sqrt{1+h_\infty^-}-1)^2 E[F^2] \leq E\left[\text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F(\tilde{\omega})-F(\omega))^2 1_{\{F(\tilde{\omega})\geq F(\omega)\}}\right]. \qquad (27)$$

In the general case, if 0 is a median of F we have, applying (27) to F^+ and F^-:

$$\begin{aligned}E[F^2] &= E[(F^+)^2] + E[(F^-)^2]\\ &\leq E\left[\text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F^+(\tilde{\omega})-F^+(\omega))^2 1_{\{F^+(\tilde{\omega})\geq F^+(\omega)\}}\right]\\ &\quad +E\left[\text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}(F^-(\tilde{\omega})-F^-(\omega))^2 1_{\{F^-(\tilde{\omega})\geq F^-(\omega)\}}\right]\\ &\leq 2E\left[\text{ess sup}_{\tilde{\omega}\in\mathcal{N}_\omega}|F(\tilde{\omega})-F(\omega)|^2\right],\end{aligned}$$

hence

$$\frac{1}{\sigma(X)} = \lambda_\infty \geq \frac{(\sqrt{1+h_\infty^-}-1)^2}{2}.$$

□

Theorem 6.2 *We have*

$$\lambda_\infty = \frac{1}{\sigma(X)} \geq \frac{(\sqrt{h_\infty+1}-1)^2}{4}.$$

Proof. We have if $m(F)=0$:

$$\begin{aligned} 2E[|DF|_\infty] &\geq \int_{-\infty}^{+\infty} \pi(\partial\{F>t\})dt \\ &\geq h_\infty \int_{-\infty}^{+\infty} \min(\pi(\{F>t\}),\pi(\{F\leq t\}))dt = h_\infty E[F]. \end{aligned}$$

Applying the above inequality to $(F^+)^2$ we have

$$\begin{aligned} h_\infty E[F^{+2}] &\leq 2E[|D(F^+)^2|_\infty] \\ &\leq 2E[\text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}|F^+(\omega)-F^+(\tilde\omega)|(F^+(\omega)+F^+(\tilde\omega))] \\ &\leq 2E[\text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}|F^+(\omega)-F^+(\tilde\omega)|(F^+(\tilde\omega)-F^+(\omega)) \\ &\quad +2|F^+(\omega)-F^+(\tilde\omega)|F^+(\omega)] \\ &\leq 2E[\text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}(F^+(\omega)-F^+(\tilde\omega))^2] \\ &\quad +4E[\text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}|F^+(\omega)-F^+(\tilde\omega)|F^+(\omega)] \\ &\leq 2E[\text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}(F(\omega)-F(\tilde\omega))^2] \\ &\quad +4E[\text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}|F(\omega)-F(\tilde\omega)|F^+(\omega)]. \end{aligned}$$

Similarly we have

$$h_\infty E[(F^-)^2] \leq 2E[\text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}(F(\omega)-F(\tilde\omega))^2]+4E[\text{ess sup}_{\tilde\omega\in\mathcal{N}_\omega}|F(\omega)-F(\tilde\omega)|F^-(\omega)].$$

Hence

$$\begin{aligned} h_\infty E[F^2] &\leq h_\infty E[(F^+)^2]+h_\infty E[(F^-)^2] \\ &\leq 4E[|DF|_\infty^2]+4E[|DF|_\infty|F|] \\ &\leq 4E[|DF|_\infty^2]+4E[|DF|_\infty^2]^{1/2}E[F^2]^{1/2}, \end{aligned}$$

which implies

$$E[|DF|_\infty^2] \geq E[F^2]\frac{(\sqrt{h_\infty+1}-1)^2}{4}.$$

In the general case we use the fact that $\text{var}F \leq E[(F-m(F))^2]$. Relation (21) is proved in Prop. 2.3. □

7 A remark on Cheeger's inequality

This section follows the presentation of [13] and [26], adapting it to the configuration space case. Let $N : \mathbb{R} \to \mathbb{R}$ be a Young function, i.e. N is convex, even, non-negative, with $N(0) = 0$ and $N(x) > 0$ for all $x \neq 0$. Let

$$C_N = \sup_{x>0} \frac{xN'(x)}{N(x)} < \infty.$$

The Orlicz norm of F is defined as

$$\|F\|_N = \inf \left\{ \lambda > 0 \ : \ E\left[N\left(\frac{F}{\lambda} \right) \right] \leq 1 \right\}.$$

Theorem 7.1 *For all F such that $m(F) = 0$ we have*

$$\|F\|_N \leq \frac{C_N}{k_p^+} \| |DF|_{L^p(\frac{\sigma+\omega}{2})} \|_N,$$

and

$$E[N(F)] \leq E\left[N\left(\frac{C_N}{k_p^+} |DF|_{L^p(\frac{\sigma+\omega}{2})} \right) \right].$$

For $p = 1$ we have $h_1^+ = k_1^+$ hence

$$E[N(F - m(F))] \leq E\left[N\left(\frac{C_N}{h_1^+} |DF|_{L^1(\frac{\sigma+\omega}{2})} \right) \right].$$

If $N(x) = x^p$ we have $C_N = p$ and $\|F\|_N = \|F\|_p$, hence for some constant $C(p)$,

$$C(p)\|F - E[F]\|_p \leq \|F - m(F)\|_p \leq \frac{p}{k_2^+} \| |DF|_{L^2(\frac{\sigma+\omega}{2})} \|_p.$$

For $p = 2$ we have $C(2) = 1$, hence

$$\mathrm{var} F \leq \frac{4}{(k_2^+)^2} E\left[|DF|^2_{L^2(\frac{\sigma+\omega}{2})} \right],$$

and

$$k_2^+ \leq 2.$$

In the particular case $N(x) = x^p$ we have the following better result.

Theorem 7.2 *For all F such that $m(F) = 0$ we have*

$$E[|F|^p] \leq E\left[\left(\frac{p}{h_1} |DF|_{L^1(\frac{\sigma+\omega}{2})} \right)^p \right],$$

and

$$\|F\|_p \leq \frac{p}{h_1} \| |DF|_{L^1(\frac{\sigma+\omega}{2})} \|_p.$$

We also have the following.

Proposition 7.1 *Let* $I_{\mathrm{var}}(t) = t(1-t)$, $0 \leq t \leq 1$ *and let* $\tilde{b}_p$ *denote the optimal constant in the inequality*

$$I_{\mathrm{var}}(E[F]) \leq E\left[\sqrt{I_{\mathrm{var}}(F)^2 + \frac{1}{\tilde{b}_p}|DF|^2_{L^2(\frac{\sigma+\omega}{2})}}\right].$$

We have $\tilde{b}_p \geq \left(1 - \frac{1}{\sqrt{2}}\right) k_p^+$.

8 Appendix

In this appendix we state the proofs of Th. 7.1, Th. 7.2 and Prop. 7.1, which are based on classical arguments, cf. [5], [26].

Proof. of Th. 7.1. By the mean value theorem we have

$$E[|D^+N(F)|_{L^p(\frac{\sigma+\omega}{2})}] \leq E[N'(F)|D^+F|_{L^p(\frac{\sigma+\omega}{2})}].$$

On the other hand, if $\|F\|_N = 1$,

$$\begin{aligned}
k_p^+ E[N(F)] &= k_p^+ E[N(F^+)] + k_p^+ E[N(F^-)] \\
&\leq E\left[|D^+N(F^+)|_{L^p(\frac{\sigma+\omega}{2})}\right] + E\left[|D^+N(F^-)|_{L^p(\frac{\sigma+\omega}{2})}\right] \\
&\leq E\left[N'(F^+)|D^+F^+|_{L^p(\frac{\sigma+\omega}{2})}\right] + E\left[N'(F^-)|D^+F^-|_{L^p(\frac{\sigma+\omega}{2})}\right] \\
&\leq E\left[N'(|F|)|D^+F|_{L^p(\frac{\sigma+\omega}{2})}\right] \\
&\leq C_N \||D^+F|_{L^p(\frac{\sigma+\omega}{2})}\|_N E[N(F)],
\end{aligned}$$

where we used the generalization of the Hölder inequality

$$E[N'(|F|)G] \leq E[N'(|F|)|F|]$$

which holds since $1 = E[N(G)] \leq E[N(|F|)] = 1$, cf. Lemma 2.1 of [5], applied to $|F|$ and

$$G = |D^+F|_{L^p(\frac{\sigma+\omega}{2})} \left(\||D^+F|_{L^p(\frac{\sigma+\omega}{2})}\|_N\right)^{-1}.$$

Hence

$$k_p^+ \leq C_N \||D^+F|_{L^p(\frac{\sigma+\omega}{2})}\|_N.$$

Since $\|F\|_N = 1$, we have

$$\|F\|_N \leq \frac{C_N}{k_p^+} \||D^+F|_{L^p(\frac{\sigma+\omega}{2})}\|_N,$$

for all F with $m(F) = 0$. The second statement is proved by application of the preceding to $N_\alpha(x) = N(x)/\alpha$, $\alpha > 0$, as in Th. 3.1 of [5].

Proof. of Th. 7.2. We note that

$$E\left[|D|F|^p|_{L^1(\frac{\sigma+\omega}{2})}\right] \leq pE\left[|F|^{p-1}|DF|_{L^1(\frac{\sigma+\omega}{2})}\right],$$

and apply an argument similar to the proof of Th. 7.1, with $C_N = p$.

Proof. of Prop. 7.1. The proof is identical to Theorem 4.11 in [26]. The generalization of Cheeger's inequality applied to $N(x) = \sqrt{1+x^2} - 1$ gives $C_N = 2$ and

$$E[N(F)] \leq E\left[N\left(\frac{2}{k_p^+}|DF|_{L^p(\sigma+\omega)}\right)\right].$$

We have with $c = \sqrt{2} - 1$ and $c_1 = k_p^+/2$:

$$\begin{aligned} cI_{\text{var}}(E[F]) &= c\text{var}(F) + cE[F(1-F)] \\ &\leq cE[F(1-F)] + E[\sqrt{1+F^2} - 1] \\ &\leq cE\left[\sqrt{c^2(F(1-F))^2 + |DF|^2_{L^p(\frac{\sigma+\omega}{2})}/c_1^2}\right], \end{aligned}$$

hence

$$I_{\text{var}}(E[F]) \leq E[\sqrt{I_{\text{var}}(F)^2 + |DF|^2_{L^p(\frac{\sigma+\omega}{2})}/(cc_1)^2}].$$

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