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Nonexistence results of solutions to systems of semilinear differential inequalities on the Heisenberg group

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Abstract

Denoting $\Delta_{\mathbb{H}}$ the laplacian on the $(2N+1)-$ dimensional Heisenberg group $\mathbb{H}^N$ and $|\eta|_{\mathbb{H}}^4 = \tau^2 + \sum_{i=1}^N (x_i^2 + y_i^2)^2$ for $\eta = (x_1, ..., x_N, y_1, ..., y_N, \tau) \in \mathbb{H}^N$, we give some nonexistence results of solutions to the system of inequalities of the type

$$(\mathrm{ES}_m) \quad \begin{cases} -\Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}, & \eta \in \mathbb{H}^N, \quad 1 \leq i \leq m, \\ u_{m+1} = u_1, & \end{cases}$$

in $\mathbb{H}^N$, with $m \in \mathbb{N}^*$, $p_{m+1} = p_1$, $\gamma_{m+1} = \gamma_1$ and $a_i \in L^\infty$, $1 \leq i \leq m$. These nonexistence results hold for N less than critical exponents which depend on p_i and γ_i, $1 \leq i \leq m$. Our result concerning (ES_m), for the case $m = 2$, improve the one obtained by Pohozaev and Véron [12].

1 Introduction

For the reader's convenience, we recall some background facts used here. The Heisenberg group $\mathbb{H}^N$, whose points will be denoted by $\eta = (x, y, \tau)$ is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the group operation $\circ$ defined by

$$\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(< x \, , \, \tilde{y} > - < \tilde{x} \, , \, y >)),$$

where $< , >$ is the usual inner product in $\mathbb{R}^N$. The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}^N$ is obtained, from the vector fields $X_i = \partial_{x_i} + 2y_i \, \partial_\tau$ and $Y_i = \partial_{y_i} - 2x_i \, \partial_\tau$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} (X_i^2 + Y_i^2). \tag{1}$$

Observe that the vector field $T = \partial_\tau$ does not appear in (1). This fact lets presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$[X_i, Y_j] = -4T, \quad j, \, k \in \{1, 2, ..., N\}. \tag{2}$$

The relation (2) proves that $\mathbb{H}^N$ is a nilpotent Lie group of order 2. Incidently, (2) constitues an abstract version of the canonical relations of commutation of Heisenberg between momentums and positions. Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right).$$

A natural group of dilatations on $\mathbb{H}^N$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \quad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where

$$Q = 2N + 2$$

is the homogeneous dimension of $\mathbb{H}^N$.

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}^N$ and homogeneous w.r.t the dilatations δ_λ. More precisely, we have

$$\forall (\eta, \tilde{\eta}) \in \mathbb{H}^N \times \mathbb{H}^N, \quad \Delta_{\mathbb{H}}(u(\eta \circ \tilde{\eta})) = (\Delta_{\mathbb{H}} u)(\eta \circ \tilde{\eta})$$

and

$$\Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_\lambda.$$

It is natural to define a distance from η to the origin by

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \sum_{i=1}^{N} (x_i^2 + y_i^2)^2 \right)^{1/4}.$$

In [12], Pohozaev and Véron gave another proof of a result of Birindelli, Capuzzo-Dolcetta and Cutri [1] concerning the nonexistence of weak solutions of the differential inequality

$$\Delta_{\mathbb{H}}(au) + |\eta|_{\mathbb{H}}^{\gamma} |v|^p \le 0 \quad \text{in} \quad \mathbb{H}^N$$

for $\gamma > -2$, $1 < p \le (Q + \gamma)/(Q - 2)$ and $a \in L^\infty(\mathbb{H}^N)$.
They then addressed the question of nonexistence of weak solutions of the system (ES_2). They showed that this system admits no solution defined in $\mathbb{H}^N$ whenever $\gamma_i > -2$ and $1 < p_i \le (Q + \gamma_i)/(Q - 2)$, $i = 1, 2$. The estimates on p_i, $i = 1, 2$, are obtained using Young's inequality and they are not optimal. Using the Hölder inequality, we obtain better estimates on p_i, $1 \le i \le m$. The same strategy is suitable to study the systems (PS_m) and (HS_m).
In the first version of this paper which has been submitted but not published, we presented also new and sharp results for the the two following systems

$$(\mathrm{PS}_m) \quad \begin{cases} \frac{\partial u_i}{\partial t} - \Delta_{\mathbb{H}}(a_i u_i) \ge |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}, & \eta \in \mathbb{H}^N, \quad 1 \le i \le m, \\ u_{m+1} = u_1, \end{cases}$$

$$(\mathrm{HS}_m) \quad \begin{cases} \frac{\partial^2 u_i}{\partial t^2} - \Delta_{\mathbb{H}}(a_i u_i) \ge |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}, & \eta \in \mathbb{H}^N, \quad 1 \le i \le m, \\ u_{m+1} = u_1. \end{cases}$$

In [2], the authors based on the previous version of this article, presented results for systems of evolution type with higher order time derivatives. Their results are generalized versions of our previous results on (PS_m) and (HS_m).
For interesting results on elliptic equations and systems, we refer to the recent articles Kartsatos and Kurta [7, 8, 9], and Mitidieri and Pohozaev [11] .

To render the presentation very clear, we start with the case of systems of two inequalities

2 Systems of two Inequalities

In this section, we will treat the case $m = 2$ and consider the system

$$(\mathrm{ES}_2)\quad \begin{cases} -\Delta_{\mathbb{H}}(a_1 u) & \geq & |\eta|_{\mathbb{H}}^{\gamma_1}\,|v|^{p_1}, \\ -\Delta_{\mathbb{H}}(a_2 v) & \geq & |\eta|_{\mathbb{H}}^{\gamma_2}\,|u|^{p_2}, \end{cases}$$

where a_i, $i \in \{1,2\}$, be measurable and bounded functions defined on $\mathbb{H}^N$, $p_i > 1$ and γ_i, $i = 1, 2$ real numbers. We identify points in $\mathbb{H}^N$ with points in $\mathbb{R}^{2N+1}$. We also recall that the Haar measure on $\mathbb{H}^N$ is identical to the Lebesgue measure $d\eta = dx\,dy\,d\tau$ in $\mathbb{R}^{2N+1} = \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$. In the sequel, the integral $\int_{\mathbb{R}^{2N+1}}$ will be simply denoted by $\int$, the measure of integration however will be specified.

Definition 1. *Let a_1 and a_2 be two bounded measurable functions on $\mathbb{R}^{2N+1}$. A weak solution (u, v) of the system (ES_2) on $\mathbb{R}^{2N+1}$ is a pair of locally integrable functions (u, v) such that*

$$\begin{cases} u & \in & L^{p_2}_{loc}\left(\mathbb{R}^{2N+1}, |\eta|_{\mathbb{H}}^{\gamma_2}\, d\eta\right), \\ v & \in & L^{p_1}_{loc}\left(\mathbb{R}^{2N+1}, |\eta|_{\mathbb{H}}^{\gamma_1}\, d\eta\right) \end{cases}$$

satisfying

$$\int_{\mathbb{R}^{2N+1}} \left(a_1 u \Delta_{\mathbb{H}}\varphi + |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1}\varphi\right)\, d\eta \leq 0, \tag{3}$$

and

$$\int_{\mathbb{R}^{2N+1}} \left(a_2 v \Delta_{\mathbb{H}}\varphi + |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2}\varphi\right)\, d\eta \leq 0 \tag{4}$$

for any nonnegative test function $\varphi \in C^2_c(\mathbb{R}^{2N+1})$.

Theorem 1. *Assume that*

$$Q \leq Q_e^* = 2 + \frac{1}{p_1 p_2 - 1}\max\{(\gamma_1 + 2) + p_1(\gamma_2 + 2)\,;\, p_2(\gamma_1 + 2) + (\gamma_2 + 2)\}.$$

Then there is no nontrivial weak solution (u, v) of the system (ES_2).

Proof. Let $\varphi_R \in \mathcal{D}(\mathbb{H}^N)$ be a nonnegative function such that

$$\varphi_R(\eta) = \Phi^\lambda\left(\frac{\tau^2 + |x|^4 + |y|^4}{R^4}\right), \tag{5}$$

where $\lambda >> 1$, $R > 0$ and $\Phi \in \mathcal{D}([0,+\infty[)$ is the "standard cut-off function"

$$0 \leq \Phi(r) \leq 1, \quad \Phi(r) = \begin{cases} 1 & \text{if} \quad 0 \leq r \leq 1, \\ 0 & \text{if} \quad r \geq 2. \end{cases}$$

Note that $\text{supp}(\varphi_R)$ is a subset of

$$\Omega_R = \{\eta \equiv (x, y, \tau) \in \mathbb{H}^N; \;\; 0 \leq \tau^2 + |x|^4 + |y|^4 \leq 2R^4\}$$

and $\text{supp}(\Delta_{\mathbb{H}}\varphi_R)$ is included in

$$\mathcal{C}_R = \{\eta \equiv (x, y, \tau) \in \mathbb{H}^N; \;\; R^4 \leq \tau^2 + |x|^4 + |y|^4 \leq 2R^4\}.$$

Let

$$\rho = \frac{\tau^2 + |x|^4 + |y|^4}{R^4},$$

then

$$\begin{aligned} \Delta_{\mathbb{H}}\varphi_R(\eta) = & \frac{4(N+4)\Phi'(\rho)}{R^4}\lambda\Phi^{\lambda-1}(\rho)(|x|^2+|y|^2) + \\ & \frac{16\Phi''(\rho)}{R^8}\lambda\Phi^{\lambda-1}(\rho)\left((|x|^6+|y|^6)+\tau^2(|x|^2+|y|^2)+2\tau < x\,,\, y > (|x|^2-|y|^2)\right) + \\ & \frac{16\Phi'^2(\rho)}{R^8}\lambda(\lambda-1)\Phi^{\lambda-2}(\rho)\left((|x|^6+|y|^6)+\frac{\tau^2}{4}(|x|^2+|y|^2)+2\tau < x\,,\, y > (|x|^2-|y|^2)\right). \end{aligned}$$

It follows that there is a positive constant $C > 0$, independent of R, such that

$$\forall \eta \in \Omega_R, \quad |\Delta_{\mathbb{H}}\varphi_R(\eta)| \leq \frac{C}{R^2}. \tag{6}$$

Let (u, v) be a nontrivial weak solution of (ES_2). Using (3) and (4) with $\varphi = \varphi_R$ one has

$$\begin{aligned} \int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_R\, d\eta & \leq -\int a_1 u\Delta_{\mathbb{H}}\varphi_R\, d\eta \leq ||a_1||_{L^\infty}\int |u|\,|\Delta_{\mathbb{H}}\varphi_R|\, d\eta \\ & \leq ||a_1||_{L^\infty}\left(\int |\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\right)^{1/p_2}\left(\int |\Delta_{\mathbb{H}}\varphi_R|^{p_2'}\left(\varphi_R|\eta|_{\mathbb{H}}^{\gamma_2}\right)^{1-p_2'}\right)^{1/p_2'} \end{aligned} \tag{7}$$

and

$$\begin{aligned} \int |\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\, d\eta & \leq -\int a_2 v\Delta_{\mathbb{H}}\varphi_R\, d\eta \\ & \leq ||a_2||_{L^\infty}\left(\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_R\right)^{1/p_1}\left(\int |\Delta_{\mathbb{H}}\varphi_R|^{p_1'}\left(\varphi_R|\eta|_{\mathbb{H}}^{\gamma_1}\right)^{1-p_1'}\right)^{1/p_1'} \end{aligned} \tag{8}$$

thanks to the Hölder inequality. Setting

$$I(R) = \int |\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\, d\eta \quad \text{and} \quad J(R) = \int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_R\, d\eta,$$

we have

$$J(R) \le C_1\, I(R)^{1/p_2}\, \mathcal{A}_{p_2,\gamma_2}(R)^{1/p_2'}, \tag{9}$$

where

$$\mathcal{A}_{p_2,\gamma_2}(R) = \int |\Delta_{\mathbb{H}}\varphi_{_R}|^{p_2'}\, (\varphi_{_R}|\eta|_{\mathbb{H}}^{\gamma_2})^{1-p_2'}\; d\eta,$$

and C_1 is a positive constant independent on R. Similarly, we have

$$I(R) \le C_2\, J(R)^{1/p_1}\, \mathcal{A}_{p_1,\gamma_1}(R)^{1/p_1'}, \tag{10}$$

where

$$\mathcal{A}_{p_1,\gamma_1}(R) = \int |\Delta_{\mathbb{H}}\varphi_{_R}|^{p_1'}\, (\varphi_{_R}|\eta|_{\mathbb{H}}^{\gamma_1})^{1-p_1'}\; d\eta,$$

and C_2 is a positive constant independent on R.

Note that for λ sufficiently large, the integrals $\mathcal{A}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, are convergent. Indeed, in the expression $\mathcal{A}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, we have $|\eta|_{\mathbb{H}} \ge R^4$, and the exponent of $\varphi_{_R}$ is positive for λ large enough.

In order to estimate the integrals $\mathcal{A}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, we introduce the scaled variables

$$\left\{\begin{array}{lll} \tilde{\tau} & = & R^{-2}\,\tau, \\ \tilde{x} & = & R^{-1}\,x, \\ \tilde{y} & = & R^{-1}\,y. \end{array}\right. \tag{11}$$

Using the fact that $\mathrm{supp}\varphi_{_R} \subset \Omega_R$, we conclude that

$$\mathcal{A}_{p_i,\gamma_i}(R) \le C\, R^{2N+2-2p_i'+\gamma_i(1-p_i')}, \quad i \in \{1,2\}. \tag{12}$$

Using (10) and (12) in (9), we obtain

$$J(R)^{1-\frac{1}{p_1p_2}} \le C\, \mathcal{A}_{p_1,\gamma_1}(R)^{\frac{1}{p_1'p_2}}\, \mathcal{A}_{p_2,\gamma_2}(R)^{\frac{1}{p_2'}} \le C R^{\sigma_J},$$

where

$$\begin{aligned} \sigma_J &= \frac{1}{p_2'}(2N+2-2p_2+\gamma_2(1-p_2')) + \frac{1}{p_1'p_2}(2N+2-2p_1+\gamma_1(1-p_1')) \\ &= Q\left(1-\frac{1}{p_1p_2}\right) - \frac{(2p_2+2+\gamma_2)p_1+\gamma_1}{p_1p_2}. \end{aligned}$$

Similarly, we have

$$I(R)^{1-\frac{1}{p_1p_2}} \le C\, \mathcal{A}_{p_1,\gamma_1}(R)^{\frac{1}{p_1'}}\, \mathcal{A}_{p_2,\gamma_2}(R)^{\frac{1}{p_1p_2'}} \le C R^{\sigma_J},$$

where

$$\sigma_I = Q\left(1-\frac{1}{p_1p_2}\right) - \frac{(2p_1+2+\gamma_1)p_2+\gamma_2}{p_1p_2}.$$

Now, we require $\sigma_I \le 0$ or $\sigma_J \le 0$ which is equivalent to

$$\begin{aligned} Q \le Q_e^* &= \frac{1}{p_1p_2-1}\max\{p_1(2(p_2+1)+\gamma_2)+\gamma_1\,;\, p_2(2(p_1+1)+\gamma_1)+\gamma_2\} \\ &= 2+\frac{1}{p_1p_2-1}\max\{(\gamma_1+2)+p_1(\gamma_2+2)\,;\, p_2(\gamma_1+2)+(\gamma_2+2)\}. \end{aligned}$$

In this case, the integrals $I(R)$ and $J(R)$, increasing in R, are bounded uniformly w.r.t. R. Using the monotone convergence theorem, we deduce that $|\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}$ and $|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}$ are in $L^1(\mathbb{R}^{2N+1})$. Note that instead of (7) we have more precisely

$$\begin{aligned}\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_R\, d\eta \ &\leq\ ||a_1||_{L^\infty}\left(\int_{\mathcal{C}_R}|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\, d\eta\right)^{1/p_2}\mathcal{A}_{p_2,\gamma_2}(R)^{1/p_2'}\\ &\leq\ C\int_{\mathcal{C}_R}|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\, d\eta.\end{aligned}$$

Finally, using the dominated convergence theorem, we obtain that

$$\lim_{R\to+\infty}\int_{\mathcal{C}_R}|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\, d\eta = 0.$$

Hence

$$\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\, d\eta\ = 0,$$

which implies that $v\equiv 0$ and $u\equiv 0$ via (8). This contradicts the fact that (u,v) is a nontrivial weak solution of (ES$_2$), which achieves the proof. □

Remark 1. *The critical exponent Q_e^* can be written as*

$$Q_e^* = 2+\max\{X_1, X_2\},$$

where the vector $(X_1, X_2)^T$ is the solution of the linear system

$$\begin{pmatrix} -1 & p_1\\ p_2 & -1\end{pmatrix}\begin{pmatrix} X_1\\ X_2\end{pmatrix} = \begin{pmatrix}\gamma_1+2\\ \gamma_2+2\end{pmatrix}. \tag{13}$$

Comment

In their paper, Pohozaev and Véron [12] showed that if

$$1 < p_j \leq \frac{Q+\gamma_j}{Q-2},\qquad j\in\{1,2\}, \tag{14}$$

then the system (ES$_2$) has no nontrivial weak solution. The condition (14) is equivalent to

$$Q \leq 2+\min\left\{\frac{\gamma_1+2}{p_1-1};\, \frac{\gamma_2+2}{p_2-1}\right\}. \tag{15}$$

Theorem 1 gives a better estimate of the exponent. Indeed,

$$\frac{(\gamma_1+2)+p_1(\gamma_2+2)}{p_1p_2-1} - \frac{\gamma_2+2}{p_2-1} = -\frac{p_2(\gamma_1+2)+(\gamma_2+2)}{p_1p_2-1} + \frac{\gamma_1+2}{p_1-1},$$

which implies that

$$\max\left\{\frac{(\gamma_1+2)+p_1(\gamma_2+2)}{p_1p_2-1};\, \frac{p_2(\gamma_1+2)+(\gamma_2+2)}{p_1p_2-1}\right\} \geq \min\left\{\frac{\gamma_1+2}{p_1-1};\, \frac{\gamma_2+2}{p_2-1}\right\}.$$

3 Systems of m semilinear inequalities

In this section, we give generalizations of the last results to systems with m inequalities, $m \in \mathbb{N}^*$.

Let $(X_1, X_2, ..., X_m)$ be the solution of the linear system

$$\begin{pmatrix} 1 & -p_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -p_2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -p_{m-1} \\ -p_m & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{m-1} \\ X_m \end{pmatrix} = \begin{pmatrix} -\gamma_1 - 2 \\ -\gamma_2 - 2 \\ \vdots \\ -\gamma_{m-1} - 2 \\ -\gamma_m - 2 \end{pmatrix}, \tag{16}$$

where $p_i > 1$ and γ_i are given real numbers, $i \in \{1, 2, ..., m\}$.
Consider the system

$$(ES_m) \left\{ \begin{array}{l} -\Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad x \in \mathbb{H}^N, \quad 1 \leq i \leq m, \\ \\ u_{m+1} = u_1, \end{array} \right.$$

where $p_{m+1} = p_1$, $\gamma_{m+1} = \gamma_1$.

Definition 2. *Let a_i, $i \in \{1, 2, ..., m\}$, be m bounded measurable functions on $\mathbb{R}^{2N+1}$. A weak solution $(u_1, ..., u_m)$ of the system (ES_m) on $\mathbb{R}^{2N+1}$ is a vector of locally integrable functions $(u_1, ..., u_m)$ such that*

$$u_i \in L^{p_i}_{loc}\left(\mathbb{R}^{2N+1}, |\eta|_{\mathbb{H}}^{\gamma_i}\, d\eta\right), \quad i \in \{1, 2, ..., m\},$$

satisfying

$$\int_{\mathbb{R}^{2N+1}} (a_i u \Delta_{\mathbb{H}}\varphi + |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}} \varphi)\, d\eta \leq 0, \quad i \in \{1, 2, ..., m-1\} \tag{17}$$

and

$$\int_{\mathbb{R}^{2N+1}} (a_m u_m \Delta_{\mathbb{H}}\varphi + |\eta|_{\mathbb{H}}^{\gamma_1} |u|^{p_1} \varphi)\, d\eta \leq 0 \tag{18}$$

for any nonnegative test function $\varphi \in C_c^2(\mathbb{R}^{2N+1})$.

Theorem 2. *If $Q \leq 2 + \max\{X_1, X_2, ..., X_m\}$ then the system (ES_m) has no nontrivial solution.*

Proof. In order to simplify the proof, we treat only the case $m = 3$; the general case can be established in the same manner.

Let (u_1, u_2, u_3) be a nontrivial weak solution of (ES_m). The inequalities (17) and (18), with $\varphi = \varphi_R$ defined by (5), imply that

$$\int |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} \varphi_R\, d\eta \leq ||a_3||_{L^\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_3} |u_3|^{p_3} \varphi_R \right)^{1/p_3} \left(\int |\Delta_{\mathbb{H}} \varphi_R|^{p_3'} (\varphi_R |\eta|_{\mathbb{H}}^{\gamma_3})^{1-p_3'} \right)^{1/p_3'},$$

$$\int |\eta|_{\mathbb{H}}^{\gamma_2} |u_2|^{p_2} \varphi_R \, d\eta \leq ||a_1||_{L^\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_1} |u_1|^{p_1} \varphi_R \right)^{1/p_1} \left(\int |\Delta_{\mathbb{H}} \varphi_R|^{p_1'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_1}\right)^{1-p_1'} \right)^{1/p_1'}$$

and

$$\int |\eta|_{\mathbb{H}}^{\gamma_3} |u_3|^{p_3} \varphi_R \, d\eta \leq ||a_2||_{L^\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_2} |u_2|^{p_2} \varphi_R \right)^{1/p_2} \left(\int |\Delta_{\mathbb{H}} \varphi_R|^{p_2'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_2}\right)^{1-p_2'} \right)^{1/p_2'}.$$

Let

$$\begin{aligned} I_i(R) &= \int |\eta|_{\mathbb{H}}^{\gamma_i} |u_i|^{p_i} \varphi_R \, d\eta, \quad 1 \leq i \leq 3, \\ \mathcal{A}_i(R) &= \int |\Delta_{\mathbb{H}} \varphi_R|^{p_i'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_i}\right)^{1-p_i'}, \quad 1 \leq i \leq 3, \end{aligned}$$

then there is a positive constant C such that

$$\begin{cases} I_1 \leq C\, I_3^{1/p_3} \mathcal{A}_3^{1/p_3'}, \\ I_2 \leq C\, I_1^{1/p_1} \mathcal{A}_1^{1/p_1'}, \\ I_3 \leq C\, I_2^{1/p_2} \mathcal{A}_2^{1/p_2'}. \end{cases}$$

Whence, the estimates

$$\begin{cases} I_1^{1-\frac{1}{p_1p_2p_3}} \leq C \mathcal{A}_1^{\frac{1}{p_1'p_2p_3}} \mathcal{A}_2^{\frac{1}{p_2'p_3}} \mathcal{A}_3^{\frac{1}{p_3'}}, \\ I_2^{1-\frac{1}{p_1p_2p_3}} \leq C \mathcal{A}_1^{\frac{1}{p_1'}} \mathcal{A}_2^{\frac{1}{p_1p_2'p_3}} \mathcal{A}_3^{\frac{1}{p_1p_3'}}, \\ I_3^{1-\frac{1}{p_1p_2p_3}} \leq C \mathcal{A}_1^{\frac{1}{p_1'p_2}} \mathcal{A}_2^{\frac{1}{p_2'}} \mathcal{A}_3^{\frac{1}{p_1p_2p_3'}}, \end{cases}$$

hold true.

In order to estimate the expressions I_i, $1 \leq i \leq 3$, we use the scaled variables (11) and obtain

$$I_i^{1-\frac{1}{p_1p_2p_3}} \leq C R^{\sigma_i}, \quad 1 \leq i \leq 3,$$

where

$$\begin{cases} \sigma_1 = \left(1 - \frac{1}{p_1p_2p_3}\right)\left(Q - 2 - \frac{(\gamma_1+2)+p_1(\gamma_2+2)+p_1p_2(\gamma_3+2)}{p_1p_2p_3-1}\right), \\ \sigma_2 = \left(1 - \frac{1}{p_1p_2p_3}\right)\left(Q - 2 - \frac{p_2p_3(\gamma_1+2)+(\gamma_2+2)+p_2(\gamma_3+2)}{p_1p_2p_3-1}\right), \\ \sigma_3 = \left(1 - \frac{1}{p_1p_2p_3}\right)\left(Q - 2 - \frac{p_3(\gamma_1+2)+p_1p_3(\gamma_2+2)+(\gamma_3+2)}{p_1p_2p_3-1}\right). \end{cases}$$

Now, we require that, at least, one of σ_i, $1 \leq i \leq 3$, is less than zero, which is equivalent to $Q \leq 2 + \max\{X_1, X_2, X_3\}$, where the vector $(X_1, X_2, X_3)^T$ is the solution of

$$\begin{pmatrix} 1 & -p_1 & 0 \\ 0 & 1 & -p_2 \\ -p_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -\gamma_1 - 2 \\ -\gamma_2 - 2 \\ -\gamma_3 - 2 \end{pmatrix}.$$

Following the arguments used in the proof of Theorem 1, we conclude that $(u_1, u_2, u_3) \equiv (0, 0, 0)$. This ends the proof by contradiction. $\square$

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