Multiple solutions to a nonlinear elliptic equation involving Paneitz type operators

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Abstract

This paper deals with an elliptic equation involving Paneitz type operators on compact Riemannian manifolds with concave-convex nonlinearities and a real parameter. Nonlocal and multiple existence results are established. Characteristic values of the real parameter are introduced and their role in the change of the energy sign and the existence of positive solutions are highlighted.

1 Introduction

In this paper, we study a nonlinear elliptic equation involving Paneitz type operators on compact Riemannian manifolds. The nonlinearity considered here is concave-convex. The simultaneous effect of the concave and the convexe terms has been initially investigated by Ambrosetti-Brézis-Cerami [1] in the Euclidian case for the Laplace operator. Since, elliptic problems with this kind of nonlinearities was extensively studied by several authors with different classes of domains and with more general differential operators like the p-Lapalcian. We can refer the reader to the valuable survey article by Ambrosetti-Garcia Azorero-Peral [2] and the references therein.

The aim of this work is to establish nonlocal and multiple existence results (w.r.t a real parameter) to an elliptic equation involving the Paneitz-Branson operator with concave-convex nonlinear terms. Also, characteristic values of the real parameter are introduced (under variational form) and some of their specific properties are carried out.

Now, let (M, g) be a smooth 4-dimensional Riemannian manifold and let S_g, Rc_g be the scalar curvature and the Ricci curvature of g respectively. The Paneitz operator, introduced by Paneitz [24], defined on (M, g) is the fourth order operator:

$$P_g^4 u := \Delta_g^2 u - \operatorname{div}_g \left(\frac{2}{3} S_g g - 2Rc_g \right) du,$$

where $\Delta_g u = -\mathrm{div}_g \nabla u$ is the Laplacian of u and du is the differential of u, both with respect to the metric g.

The Paneitz operator was generalized to higher dimensions by Branson [5]. Given (M, g) a n-dimensional Riemannian manifold ($n \geq 5$), the Paneitz-Branson operator P_g^n is defined as follows

$$P_g^n u := \Delta_g^2 u - \mathrm{div}_g \left(\frac{(n-2)^2 + 4}{2(n-1)(n-2)} S_g g - \frac{4}{n-2} Rc_g \right) du + \frac{n-4}{2} Q_g^n u,$$

where the $Q-$curvature Q_g^n is given by

$$Q_g^n = \frac{1}{2(n-1)} \Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |Rc_g|^2.$$

Geometrically, the $Q-$curvature can be seen, for the Paneitz-Branson operator, as the analogue of the scalar curvature for the conformal Laplacian. Notice that when (M, g) is Einstein, the Paneitz-Branson operator is reduced to

$$P_g^n u = \Delta_g^2 u + \alpha_n \Delta_g u + \beta_n u,$$

where

$$\alpha_n := \frac{n^2 - 2n - 4}{2n(n-1)} S_g \quad \text{and} \quad \beta_n = \frac{(n-4)(n^4 - 4)}{16n(n-1)^2} S_g^2.$$

This is a special case of what one usually refers to as a Paneitz-Branson operator with constant coefficients, namely an operator which expresses as

$$P_g u = \Delta_g^2 u + \alpha \Delta_g u + \beta u,$$

where α and β are real numbers. In this direction, we can refer the reader to Djadli-Hebey-Ledoux [11], Esposito-Robert [15], Felli-Hebey-Robert [16], Hebey [19, 20], Hebey-Robert [18], and finally to Robert [27].

In this paper, we study the existence of multiple solutions to

$$P_g u = \lambda a(x) |u|^{q-2} u + b(x) |u|^{r-2} u,$$

with respect to the positive real parameter λ, where (M, g) is a smooth compact Riemannian manifold of dimension $n \geq 5$, a and b are positive continuous functions on M. This problem is stated in the framework of the Sobolev space $H_2^2(M)$ consisting of functions u in $L^2(M)$ wich are such that $|\nabla u|$ and $|\nabla^2 u|$ are also in $L^2(M)$. We limit ourseves to the case of subcritical concave-convex nonlinearity, that is

$$1 < q < 2 < r < 2^{\#},$$

where $2^{\#} := 2n/(n-4)$ is the critical exponent for the embedding of the Sobolev space H_2^2 in L^p- spaces. Also, we will assume along this work that [18]

$$\frac{\alpha^2}{4} \geq \beta > 0. \tag{1}$$

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In this situation, it is clear that the Paneitz-Branson operator P_g is coercive, *i.e.* there is $C > 0$ such that for any $u \in H_2^2(M)$,

$$\int_M (P_g u) u \, dv_g \geq C ||u||^2_{H_2^2(M)},$$

where

$$||u||_{H_2^2(M)} = \left\{ \int_M \left((\Delta_g u)^2 + |\nabla u|^2 + u^2 \right) dv_g \right\}^{1/2}$$

is the standard norm of $H_2^2(M)$. In the specific case where (M, g) is Einstein, the condition (1) is satisfied. Indeed,

$$\frac{\alpha_n^2}{4} - \beta_n = \frac{S_g^2}{n^2(n-1)^2}.$$

For further detailed discussions on this subject, we refer the reader to Beckner [3], Branson-Chang-Yang [6], Chang-Gursky-Yang [8, 9], Chang-Yang [10] and to Gursky [17], for the Paneitz operator. For the Paneitz-Branson operator, we mention the references described above [11, 15, 16, 19, 20, 18, 27].

Hereafter, the space $H_2^2(M)$ will be endowed with the norm $|| \, . \, ||$

$$||u|| = \left(\int_M (P_g u) u \, dv_g \right)^{1/2},$$

which is equivalent to norm $|| \, . \, ||_{H_2^2(M)}$. Following standard notations, $|| \, . \, ||_p$ will stand for the L^p−norm (with respect to the Riemannian measure dv_g).

Consider the following problem

$$\begin{cases} P_g u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{r-2}u & \text{in } M, \\ \\ 1 < q < 2 < r \leq 2^{\#}, \end{cases} \tag{2}$$

For solutions of (2) we understand critical points of the associated Euler-Lagrange (energy) functional $E_\lambda \in C^1(H_2^2(M))$, given by

$$E_\lambda(u) = \frac{1}{2}P(u) - \frac{\lambda}{q}Q(u) - \frac{1}{r}R(u),$$

where

$$P(u) = ||u||^2, \quad Q(u) = \int_M a|u|^q \, dv_g \text{ and } R(u) = \int_M b|u|^r \, dv_g.$$

In [4], Bernis, Garcia-Azorero and Peral studied the following equation

$$\Delta^2 u = \lambda |u|^{q-2}u + |u|^{2^{\#}-2}u$$

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in a smooth bounded domain in $\mathbb{R}^n$ with Dirichlet or Navier boundary conditions. Applying the Lusternik-Schnirelman theory, the authors showed the existence of infinitely many solutions for λ small enough (a local result w.r.t. the parameter λ). Using classical methods, the authors showed the existence of two positive solutions for $0 < \lambda < \Lambda$, even in the supercritical case. Notice that our result concerning the existence of infinitely many solutions is not local. Indeed, we show the existence of two disjoint and infinite sets of solutions to (2) for $0 < \lambda < \widehat{\lambda}$, where $\widehat{\lambda} > C(||a||_\infty, ||b||_\infty, q, r, M) > 0$. The first set consists of solutions with negative energy while the second set contains solutions with arbitrary energy. On the other hand, our proof concerning the existence of positive solutions gives a new argument for the construction of Palais-Smale sequences. Similar equation was studied also by El Hamidi [13] in the Euclidian case, with the $p-$Laplacian operator and nonlinear mixed boundary conditions.

We introduce the modified Euler-Lagrange functional [12, 13, 29], $\widetilde{E}_\lambda$ defined on $\mathbb{R} \times H_2^2(M)$ by

$$\widetilde{E}_\lambda(t, u) := E_\lambda(tu).$$

If u is an arbitrary element of $H_2^2(M)$, $\partial_t \widetilde{E}_\lambda(., u)$ stands for the first derivative of the real valued function: $t \mapsto \widetilde{E}_\lambda(t, u)$. Similarly, $\partial_{tt} \widetilde{E}_\lambda(., u)$ denotes the second derivative.

2 Existence of Positive solutions

In this section, we show the existence of two "branches" of positive solutions to (2). The idea of the approach is as follows: for every $u \in H_2^2(M) \setminus \{0\}$ and $\lambda > 0$, we determine the positive critical points (in terms of u and λ) of the real-valued function $t \mapsto \widetilde{E}_\lambda(t, u)$. Then, the variable t is substituted by these real critical points to obtain functionals, depending only on the variable u (and the parameter λ), defined on the Nehari manifold [23]. On shows easily that these new functionals are bounded below, which implies that we can minimize to obtain possible critical points of the Euler-Lagrange E_λ itself. However, the positivity of these critical points is not guaranteed. To show the existence of positive solutions, we carry out a new approach of an idea introduced, in our knowledge, by Djadli, Hebey and Ledoux [11] and by Van Der Vorst [28].

2.1 Technical lemmas

Lemma 1. *Let $u \in H_2^2(M) \setminus \{0\}$. There exists a unique $\lambda(u) > 0$ such that the real valued function $t \mapsto \partial_t \widetilde{E}_\lambda(t, u)$ has*

$$\begin{cases} \text{two positive zeros} & \text{if} \quad 0 < \lambda < \lambda(u), \\ \text{one positive zero} & \text{if} \quad \lambda = \lambda(u), \\ \text{no zero} & \text{if} \quad \lambda > \lambda(u). \end{cases}$$

Proof. Let $u \in H^2_2(M) \setminus \{0\}$ and let us write

$$\partial_t \widetilde{E}_\lambda(t, u) = t^{q-1} \widetilde{F}_\lambda(t, u),$$

where

$$\widetilde{F}_\lambda(t, u) = t^{2-q} P(u) - \lambda Q(u) - t^{r-q} R(u).$$

It follows that

$$\partial_{tt} \widetilde{E}_\lambda(t, u) = (q-1) t^{q-2} \widetilde{F}_\lambda(t, u) + t^{q-1} \partial_t \widetilde{F}_\lambda(t, u),$$

with

$$\partial_t \widetilde{F}_\lambda(t, u) = t^{2-q-1} \left\{ (2-q) P(u) - (r-q) t^{r-2} R(u) \right\}.$$

The real valued function $t \longmapsto \widetilde{F}_\lambda(t, u)$ is increasing for $t \in]0, t(u)[$, decreasing for $t \in]t(u), +\infty[$ and attains its unique maximum for $t = t(u)$, where

$$t(u) = \left(\frac{2-q}{r-q} \frac{P(u)}{R(u)} \right)^{\frac{1}{r-2}}. \tag{3}$$

Therefore, the function $t \longmapsto \widetilde{F}_\lambda(t, u)$ has

$$\begin{cases} \text{two positive zeros} & \text{if} \quad \widetilde{F}_\lambda(t(u), u) > 0, \\ \text{one positive zero} & \text{if} \quad \widetilde{F}_\lambda(t(u), u) = 0, \\ \text{no zero} & \text{if} \quad \widetilde{F}_\lambda(t(u), u) < 0. \end{cases}$$

Moreover, a direct computation gives

$$\widetilde{F}_\lambda(t(u), u) = \frac{r-2}{2-q} \left(\frac{2-q}{r-q} \frac{P(u)}{R(u)} \right)^{\frac{r-q}{r-2}} R(u) - \lambda Q(u).$$

It follows that

$$\begin{cases} \widetilde{F}_\lambda(t(u), u) > 0 & \text{if} \quad \lambda < \lambda(u), \\ \widetilde{F}_\lambda(t(u), u) = 0 & \text{if} \quad \lambda = \lambda(u), \\ \widetilde{F}_\lambda(t(u), u) < 0 & \text{if} \quad \lambda > \lambda(u), \end{cases}$$

where

$$\lambda(u) = \widehat{C} \frac{P^{\frac{r-q}{r-2}}(u)}{Q(u) R^{\frac{2-q}{r-2}}(u)}, \tag{4}$$

and

$$\widehat{C} = \frac{r-2}{2-q} \left(\frac{2-q}{r-q} \right)^{\frac{r-q}{r-2}}.$$

Hence, if $\lambda \in]0, \lambda(u)[$, the real valued function $t \longmapsto \partial_t \widetilde{E}_\lambda(t, u)$ has two positive zeros which we will denote by $\underline{t}(u, \lambda)$ and $\overline{t}(u, \lambda)$. Notice that

$$0 < \underline{t}(u, \lambda) < t(u) < \overline{t}(u, \lambda).$$

Since, $\widetilde{F}_\lambda(\underline{t}(u,\lambda),u) = \widetilde{F}_\lambda(\overline{t}(u,\lambda),u) = 0$, $\partial_t \widetilde{F}_\lambda(t,u) > 0$ for $t < t(u)$ and $\partial_t \widetilde{F}_\lambda(t,u) < 0$ for $t > t(u)$, we get

$$\partial_{tt}\widetilde{E}_\lambda(\underline{t}(u,\lambda),u) > 0 \quad \text{and} \quad \partial_{tt}\widetilde{E}_\lambda(\overline{t}(u,\lambda),u) < 0.$$

Consequently, the real valued function $t \longmapsto \widetilde{E}_\lambda(t,u)$, $t > 0$, attains its unique local minimum at $t = \underline{t}(u,\lambda)$ and its unique local maximum at $t = \overline{t}(u,\lambda)$. $\qquad\square$

Let us precise that for every $u \in H_2^2(M) \setminus \{0\}$ and $\lambda \in]0, \lambda(u)[$, $\underline{t}(u,\lambda)u$ and $\overline{t}(u,\lambda)u$ belong to the Nehari manifold [23] defined by

$$\mathcal{N} := \{v \in H_2^2(M) \setminus \{0\} \ : \ E_\lambda'(v)v = 0\}.$$

At this stage we introduce the characteristic value

$$\widehat{\lambda} := \inf_{u \in H_2^2(M) \setminus \{0\}} \lambda(u), \tag{5}$$

of the parameter λ. We will show below that the problem (2) possesses two "branches" of solutions for $\lambda \in]0, \widehat{\lambda}[$. It is interesting to remark that this result is not local with respect to the parameter λ. Indeed, let $S_q(M)$ and $S_r(M)$ be the best Sobolev constants of the embeddings $H_2^2(M) \subset L^q(M)$ and $H_2^2(M) \subset L^r(M)$ respectively. Then,

$$\begin{aligned}
\widehat{\lambda} &= \widehat{C} \inf_{u \in H_2^2(M) \setminus \{0\}} \left(\frac{[P(u)]^{q/2}}{Q(u)} \right) \left(\frac{[P(u)]^{r/2}}{R(u)} \right)^{\frac{2-q}{r-2}} \\
&\geq \frac{\widehat{C}}{\max\limits_{x \in M} a(x) \left[\max\limits_{x \in M} b(x) \right]^{\frac{2-q}{r-2}}} [S_q(M)]^{q/2} [S_r(M)]^{\frac{r(2-q)}{2(r-2)}} \\
&> 0.
\end{aligned}$$

Finally, remark that the function $\lambda(.)$ is homogeneous on $H_2^2(M) \setminus \{0\}$.
Now, we show an interesting lemma which will be very useful below to construct positive solutions from solutions with arbitrary sign.

Lemma 2. *Let* $\lambda \in]0, \widehat{\lambda}[$, *$v$ and w in $H_2^2(M) \setminus \{0\}$. If*

$$\begin{cases} \|v\| &= \|w\|, \\ |v| &\leq |w| \quad \text{in} \ \ M, \end{cases}$$

then

$$\widetilde{E}_\lambda(\underline{t}(v,\lambda),v) \geq \widetilde{E}_\lambda(\underline{t}(w,\lambda),w) \quad \text{and} \quad \widetilde{E}_\lambda(\overline{t}(v,\lambda),v) \geq \widetilde{E}_\lambda(\overline{t}(w,\lambda),w).$$

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Proof. Without loss of generality, we can assume $||v|| = ||w|| = 1$. Consider the application

$$\Phi_\lambda : \quad \begin{aligned}]0, +\infty[^3 &\longrightarrow \mathbb{R} \\ (\xi, \theta, \sigma) &\longmapsto \xi^2/2 - \lambda\theta\xi^q/q - \sigma\xi^r/r. \end{aligned}$$

According to the study done in Lemma 1, for every $\theta > 0$, $\sigma > 0$ and λ satisfying

$$0 < \lambda < \frac{\widehat{C}}{\theta\sigma^{\frac{2-q}{r-2}}},$$

the real-valued function $\Phi_\lambda(., \theta, \sigma)$ starts from $\Phi_\lambda(0, \theta, \sigma) = 0$, decreases to reach its unique local minimum for $\xi = \underline{\xi}(\theta, \sigma)$, increases to reach its unique local maximum for $\xi = \overline{\xi}(\theta, \sigma)$, and finally decreases towards $-\infty$ when ξ goes to $+\infty$. Using the fact that

$$\frac{\partial^2 \Phi_\lambda}{\partial \xi^2}(\underline{\xi}(\theta, \sigma), \theta, \sigma) > 0 \quad \text{and} \quad \frac{\partial^2 \Phi_\lambda}{\partial \xi^2}(\overline{\xi}(\theta, \sigma), \theta, \sigma) < 0,$$

the implicit function theorem implies that $\underline{\xi}(., .)$ and $\overline{\xi}(., .)$ are smooth in θ and σ. On the other hand, direct computations show that

$$\frac{\partial}{\partial\theta}\left[\Phi_\lambda(\underline{\xi}(\theta, \sigma), \theta, \sigma)\right] = -\frac{\lambda}{q}\left(\underline{\xi}(\theta, \sigma)\right)^q < 0,$$

$$\frac{\partial}{\partial\sigma}\left[\Phi_\lambda(\underline{\xi}(\theta, \sigma), \theta, \sigma)\right] = -\frac{1}{r}\left(\underline{\xi}(\theta, \sigma)\right)^r < 0.$$

Similarly, we have

$$\frac{\partial}{\partial\theta}\left[\Phi_\lambda(\overline{\xi}(\theta, \sigma), \theta, \sigma)\right] = -\frac{\lambda}{q}\left(\overline{\xi}(\theta, \sigma)\right)^q < 0,$$

$$\frac{\partial}{\partial\sigma}\left[\Phi_\lambda(\overline{\xi}(\theta, \sigma), \theta, \sigma)\right] = -\frac{1}{r}\left(\overline{\xi}(\theta, \sigma)\right)^r < 0.$$

This means that the critical values $\Phi_\lambda(\underline{\xi}(\theta, \sigma), \theta, \sigma)$ and $\Phi_\lambda(\overline{\xi}(\theta, \sigma), \theta, \sigma)$ decrease when θ or σ increases.

Now, let $0 < \theta_1 \leq \theta_2$, $0 < \sigma_1 \leq \sigma_2$ and $\lambda > 0$ such that

$$0 < \lambda < \frac{\widehat{C}}{\theta_i\sigma_i^{\frac{2-q}{r-2}}}, \quad i = 1, 2. \tag{6}$$

The relations (6) can be rewritten: $(\theta_i, \sigma_i) \in \Omega_\lambda$, $i = 1, 2$, where Ω_λ denotes the subset of $\mathbb{R}^2$ defined by

$$\Omega_\lambda = \left\{(\theta, \sigma) \in]0, +\infty[^2; \ \theta\sigma^{\frac{2-q}{r-2}} < \widehat{C}/\lambda\right\}.$$

Consider the curve (path) $\gamma \subset \Omega_\lambda$ defined by

$$\gamma := \{(\theta, \sigma_1); \ \theta_1 \leq \theta \leq \theta_2\} \cup \{(\theta_2, \sigma); \ \sigma_1 \leq \sigma \leq \sigma_2\}$$

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connecting (θ_1, σ_1) to (θ_2, σ_2). It follows from above that the functions $(\theta, \sigma) \longmapsto \Phi_\lambda(\underline{\xi}(\theta, \sigma), \theta, \sigma)$ and $(\theta, \sigma) \longmapsto \Phi_\lambda(\overline{\xi}(\theta, \sigma), \theta, \sigma)$ decrease when (θ, σ) describes the path γ from (θ_1, σ_1) to (θ_2, σ_2). Therefore, we get

$$
\begin{aligned}
\Phi_\lambda(\underline{\xi}(\theta_1, \sigma_1), \theta_1, \sigma_1) &\geq \Phi_\lambda(\underline{\xi}(\theta_2, \sigma_2), \theta_2, \sigma_2), \\
\Phi_\lambda(\overline{\xi}(\theta_1, \sigma_1), \theta_1, \sigma_1) &\geq \Phi_\lambda(\overline{\xi}(\theta_2, \sigma_2), \theta_2, \sigma_2).
\end{aligned}
$$

Here, $Q(v)$, $R(v)$, $Q(w)$ and $R(w)$ will play the role of θ_1, σ_1, θ_2 and σ_2 rspectively. Moreover, precise that for every $u \in H_2^2(M)$ such that $\|u\| = 1$, we have $\widetilde{E}_\lambda(t, u) = \Phi_\lambda(t, Q(u), R(u))$. Hence, we conclude that

$$
\widetilde{E}_\lambda(\underline{t}(v, \lambda), v) \geq \widetilde{E}_\lambda(\underline{t}(w, \lambda), w) \quad \text{and} \quad \widetilde{E}_\lambda(\overline{t}(v, \lambda), v) \geq \widetilde{E}_\lambda(\overline{t}(w, \lambda), w).
$$

This achieves the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Let us show now that the infimum of $\widetilde{E}_\lambda(\underline{t}(u, \lambda), u)$ and $\widetilde{E}_\lambda(\overline{t}(u, \lambda), u)$, when u describes $\in H_2^2(M) \setminus \{0\}$, exist in $\mathbb{R}$. Indeed, since $\partial_t \widetilde{E}_\lambda(\underline{t}(u, \lambda), u) = 0$ it follows that

$$
P(\underline{t}(u, \lambda)u) = \lambda Q(\underline{t}(u, \lambda)u) + R(\underline{t}(u, \lambda)u).
$$

Then

$$
E_\lambda(\underline{t}(u, \lambda)u) = \left(\frac{1}{2} - \frac{1}{r} \right) R(\underline{t}(u, \lambda)u) - \lambda \left(\frac{1}{q} - \frac{1}{2} \right) Q(\underline{t}(u, \lambda)u)
$$

Using the fact that $1 < q < 2 < r$, we get

$$
\lim_{\|\underline{t}(u, \lambda)u\| \to \infty} E_\lambda(\underline{t}(u, \lambda)u) = +\infty,
$$

which implies that the function $u \longmapsto E_\lambda(\underline{t}(u, \lambda)u)$, defined on $H_2^2(M) \setminus \{0\}$, is bounded below. In the same way, we get that the function $u \longmapsto E_\lambda(\overline{t}(u, \lambda)u)$, defined on $H_2^2(M) \setminus \{0\}$, is bounded below. Therefore, if we define

$$
\underline{\alpha}(\lambda) = \inf_{u \in H_2^2(M) \setminus \{0\}} \widetilde{E}_\lambda(\underline{t}(u, \lambda), u), \tag{7}
$$

$$
\overline{\alpha}(\lambda) = \inf_{u \in H_2^2(M) \setminus \{0\}} \widetilde{E}_\lambda(\overline{t}(u, \lambda), u), \tag{8}
$$

we have the following

Lemma 3. *Let $(u_n) \subset H_2^2(M) \setminus \{0\}$ be a minimizing sequence of (7) and $\underline{v}_n := \underline{t}(u_n, \lambda)u_n$. Then*

> **a.** $\limsup\limits_{n \to +\infty} \|\underline{v}_n\| < +\infty$
>
> **b.** $\liminf\limits_{n \to +\infty} \|\underline{v}_n\| > 0$.

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Similarly, let $(u_n) \subset H_2^2(M) \setminus \{0\}$ be a minimizing sequence of (8) and $\overline{v}_n := \overline{t}(u_n, \lambda)u_n$. Then

$$\textbf{c.} \quad \limsup_{n \to +\infty} ||\overline{v}_n|| < +\infty$$

$$\textbf{d.} \quad \liminf_{n \to +\infty} ||\overline{v}_n|| > 0.$$

Proof.

a. Let $(u_n) \subset H_2^2(M) \setminus \{0\}$ be a minimizing sequence of (7). Since $\partial_t \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n) = 0$, it follows that

$$P(\underline{v}_n) = \lambda Q(\underline{v}_n) + R(\underline{v}_n). \tag{9}$$

Similarly, since $\partial_{tt} \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n) > 0$, it follows that

$$P(\underline{v}_n) - \lambda(q - 1)Q(\underline{v}_n) - (r - 1)R(\underline{v}_n) > 0. \tag{10}$$

Combining (9) and (10), we obtain

$$E_\lambda(\underline{v}_n) := \frac{1}{2}P(\underline{v}_n) - \frac{\lambda}{q}Q(\underline{v}_n) - \frac{1}{r}R(\underline{v}_n) < 0,$$

for every n. Suppose that there is a subsequence of $(\underline{v}_n)$, still denoted by $(\underline{v}_n)$, such that $\lim_{n \to +\infty} ||\underline{v}_n|| = +\infty$. Since M is compact and a, b are positive and continuous on M, then there is $C > 0$ such that $Q(\underline{v}_n) \le CR(\underline{v}_n)$ for every n. Therefore, $\lim_{n \to +\infty} R(\underline{v}_n) = +\infty$. Using the fact that $0 < q < r$ we get $Q(\underline{v}_n) = o_n(R(\underline{v}_n))$, and consequently

$$P(\underline{v}_n) = R(\underline{v}_n)(1 + o_n(1)).$$

Thus,

$$E_\lambda(\underline{v}_n) = R(\underline{v}_n) \left(\frac{1}{2} - \frac{1}{r} + o_n(1) \right),$$

which implies that $E_\lambda(\underline{v}_n)$ tends to $+\infty$ as n goes to $+\infty$ and this is impossible. We conclude that $\limsup_{n \to +\infty} ||\underline{v}_n|| < +\infty$.

c. The same arguments with a minimizing sequence (u_n) of (8) show that $\limsup_{n \to +\infty} ||\overline{v}_n|| < +\infty$.

b. Let $(u_n) \subset H_2^2(M) \setminus \{0\}$ be a minimizing sequence of (7) and suppose that there is a subsequence of $(\underline{v}_n)$, still denoted by $(\underline{v}_n)$, such that $\lim_{n \to +\infty} ||\underline{v}_n|| = 0$. It follows that $\lim_{n \to +\infty} E_\lambda(\underline{v}_n) = 0$ *i.e.* $\underline{\alpha}(\lambda) = 0$, which is impossible since $\widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n) < 0$, for every n.

d. Let $(u_n) \subset H_2^2(M) \setminus \{0\}$ be a minimizing sequence of (8). Since $\partial_t \widetilde{E}_\lambda(\overline{t}(u_n, \lambda), u_n) = 0$ and $\partial_{tt} \widetilde{E}_\lambda(\overline{t}(u_n, \lambda), u_n) < 0$ it follows that

$$\begin{cases} P(\overline{v}_n) - \lambda Q(\overline{v}_n) - R(\overline{v}_n) = 0, \\ P(\overline{v}_n) - \lambda(q - 1)Q(\overline{v}_n) - (r - 1)R(\overline{v}_n) < 0. \end{cases}$$

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Combining the two last inequalities we obtain, for every n

$$(2-q)P(\overline{v}_n) < (r-q)R(\overline{v}_n) \le C'P(\overline{v}_n),$$

via the continuous embedding $H_2^2(M) \subset L^r(M)$. Then $(2-q) \le C''||\overline{v}_n||^{r-2}$. Now, suppose that there is a subsequence of $(\overline{v}_n)$, still denoted by $(\overline{v}_n)$, such that $\lim\limits_{n \to +\infty} ||\overline{v}_n|| = 0$. This implies that $2 - q \le 0$, which is impossible. $\qquad\square$

At this stage, we give an invariance result satisfied by $\underline{t}(u, \lambda)$ and $\overline{t}(u, \lambda)$ with respect to u. Indeed, for every real number $\gamma > 0$, we have

$$\widetilde{E}_\lambda\left(\gamma t, \frac{u}{\gamma}\right) = \widetilde{E}_\lambda(t, u),$$

$$\partial_t \widetilde{E}_\lambda\left(\gamma t, \frac{u}{\gamma}\right) = \frac{1}{\gamma}\partial_t \widetilde{E}_\lambda(t, u),$$

$$\partial_{tt} \widetilde{E}_\lambda\left(\gamma t, \frac{u}{\gamma}\right) = \frac{1}{\gamma^2}\partial_{tt} \widetilde{E}_\lambda(t, u).$$

The uniqueness of the local minimum and the local maximum of the real valued function $t \longmapsto \widetilde{E}_\lambda(t, u), t > 0, u \in H_2^2(M) \setminus \{0\}$, implies that

$$\underline{t}(u, \lambda) = \frac{1}{\gamma}\underline{t}\left(\frac{u}{\gamma}, \lambda\right), \tag{11}$$

$$\overline{t}(u, \lambda) = \frac{1}{\gamma}\overline{t}\left(\frac{u}{\gamma}, \lambda\right). \tag{12}$$

Hence, we get

$$\underline{\alpha}(\lambda) = \inf_{u \in \mathbb{S}} \widetilde{E}_\lambda(\underline{t}(u, \lambda), u), \tag{13}$$

$$\overline{\alpha}(\lambda) = \inf_{u \in \mathbb{S}} \widetilde{E}_\lambda(\overline{t}(u, \lambda), u), \tag{14}$$

where $\mathbb{S}$ is the unit sphere of $H_2^2(M)$.

2.2 Palais-Smale sequences and positive solutions

Theorem 1. *Let $(u_n) \subset \mathbb{S}$ be a minimizing sequence of* (13) *(resp. of* (14)*). Then, $(\underline{v}_n) :=$ $(\underline{t}(u_n, \lambda)u_n)$ (resp. $(\overline{v}_n) := (\overline{t}(u_n, \lambda)u_n)$) are Palais-Smale sequences for the functional E_λ.*

Proof. Let $(u_n) \subset \mathbb{S}$ be a minimizing sequence of (13). According to the previous lemma, $(\underline{v}_n)$ is bounded in $H_2^2(M)$. Moreover, for every $u \in H_2^2(M) \setminus \{0\}$ and $\lambda \in]0, \widehat{\lambda}[$, we know that $\partial_t \widetilde{E}_\lambda(\underline{t}(u, \lambda), u) = 0$ and $\partial_{tt} \widetilde{E}_\lambda(\underline{t}(u, \lambda), u) \ne 0$. The implicit function theorem

implies that $\underline{t}(u, \lambda)$ is C^1 with respect to u since $\widetilde{E}$ is. Let us introduce the C^1 functional $\underline{\mathcal{E}}_\lambda$ defined on $\mathbb{S}$ by

$$\underline{\mathcal{E}}_\lambda(u) = \widetilde{E}_\lambda(\underline{t}(u, \lambda), u) = E_\lambda(\underline{t}(u, \lambda)u).$$

Then

$$\underline{\alpha}(\lambda) = \inf_{u \in \mathbb{S}} \underline{\mathcal{E}}_\lambda(u) \quad \text{and} \quad \lim_{n \to +\infty} \underline{\mathcal{E}}_\lambda(u_n) = \underline{\alpha}(\lambda).$$

Using the Ekeland variational principle on the complete manifold $(\mathbb{S}, || \; ||)$ to the functional $\underline{\mathcal{E}}_\lambda$, we conclude that

$$|\underline{\mathcal{E}}'_\lambda(u_n)(\varphi_n)| \leq \frac{1}{n} ||\varphi_n||, \quad \text{for every} \quad \varphi_n \in \mathrm{T}_{u_n} \mathbb{S},$$

where $\mathrm{T}_{u_n} \mathbb{S}$ is the tangent space to $\mathbb{S}$ at the point u_n. Moreoever, for every $\varphi_n \in \mathrm{T}_{u_n} \mathbb{S}$, one has

$$\begin{aligned}
\underline{\mathcal{E}}'_\lambda(u_n)(\varphi_n) &= \partial_t \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n) \underline{t}'(u_n, \lambda)(\varphi_n) + \partial_u \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n)(\varphi_n), \\
&= \partial_u \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n)(\varphi_n),
\end{aligned}$$

since $\partial_t \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n) = 0$, where $\underline{t}'(u_n, \lambda)$ denotes the derivative of $\underline{t}(., \lambda)$ with respect to its first variable at the point (u_n, λ).
On the other hand, let

$$\begin{aligned}
\pi : \quad H_2^2(M) \setminus \{0\} &\longrightarrow \mathbb{R} \times \mathbb{S} \\
u &\longmapsto \left(||u||, \frac{u}{||u||} \right) := (\pi_1(u), \pi_2(u)).
\end{aligned}$$

Applying Hölder's inequality, we get for every $(u, \varphi) \in (H_2^2(M) \setminus \{0\}) \times H_2^2(M)$:

$$\begin{cases} |\pi_1'(u)(\varphi)| &\leq \quad ||\varphi||, \\ ||\pi_2'(u)(\varphi)|| &\leq \quad 2\frac{||\varphi||}{||u||}. \end{cases}$$

>From Lemma 3, there is a positive constant C such that

$$\underline{t}(u_n, \lambda) \geq C, \quad \forall \, n \in \mathbb{N}.$$

Then for every $\varphi \in H_2^2(M)$, there are $\varphi_n^1 = \pi_1'(u_n)(\varphi) \in \mathbb{R}$ and $\varphi_n^2 = \pi_2'(u_n)(\varphi) \in \mathrm{T}_{u_n} \mathbb{S}$ such that $|\varphi_n^1| \leq ||\varphi||, ||\varphi_n^2|| \leq \frac{2}{C}||\varphi||$. This allows to obtain

$$\begin{aligned}
E_\lambda'(\underline{t}(u_n, \lambda)u_n)(\varphi) &= \partial_t \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n)(\varphi_n^1) + \partial_u \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n)(\varphi_n^2), \\
&= \partial_u \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n)(\varphi_n^2), \\
&= \underline{\mathcal{E}}'_\lambda(u_n)(\varphi_n^2).
\end{aligned}$$

More precisely, we get

$$\begin{aligned}
E_\lambda'(\underline{t}(u_n, \lambda)u_n)(\varphi) &\leq \frac{1}{n}||\varphi_n^2|| \\
&\leq \frac{2}{nC}||\varphi||.
\end{aligned}$$

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In other words
$$\lim_{n\to\infty} ||E'_\lambda(\underline{v}_n)||_* = 0,$$
which achieves the first claim. The same arguments can be used to show that $(\overline{v}_n)$ is a Palais-Smale sequence for the functional E_λ. This ends the proof. $\square$

Theorem 2. *Let $1 < q < 2 < r < 2^\#$ and $\lambda \in]0, \widehat{\lambda}[$. Then the problem (2) has at least two positive solutions.*

   Proof. We will adopt the notations used in the previous lemmas. As mentioned in Theorem 1: $E_\lambda(\underline{v}_n)$ converges to $\underline{\alpha}(\lambda)$, $||E'_\lambda(\underline{v}_n)||_*$ converges to 0 as n tends to $+\infty$ and $(\underline{v}_n)$ is bounded in $H_2^2(M)$. Passing if necessary to a subsequence, we have

$$\begin{aligned}
\underline{v}_n &\rightharpoonup \underline{v} \quad \text{in} \quad H_2^2(M), \\
\underline{v}_n &\to \underline{v} \quad \text{in} \quad L^r(M), \quad (\text{also in } L^q(M)), \\
\underline{v}_n &\to \underline{v} \quad \text{a.e} \quad M.
\end{aligned}$$

Let $w_n = \underline{v}_n - \underline{v}$, then using a lemma due to Brézis-Lieb [7], we get

$$\begin{aligned}
P(w_n) &= P(\underline{v}_n) - P(\underline{v}) + o_n(1), \\
Q(w_n) &= Q(\underline{v}_n) - Q(\underline{v}) + o_n(1), \\
R(w_n) &= R(\underline{v}_n) - R(\underline{v}) + o_n(1).
\end{aligned}$$

It follows that

$$\begin{aligned}
E_\lambda(w_n) &= E_\lambda(\underline{v}_n) - E_\lambda(\underline{v}) + o_n(1), \\
E'_\lambda(w_n) &= E'_\lambda(\underline{v}_n) - E'_\lambda(\underline{v}) + o_n(1),
\end{aligned}$$

and consequently $E'_\lambda(w_n)w_n \longrightarrow 0$ as $n \longrightarrow +\infty$, which implies that

$$||w_n||^2 = \lambda Q(w_n) + R(w_n) + o_n(1).$$

Therefore, $||w_n|| \to 0$ as $n \to +\infty$. Hence, $\underline{v}_n$ converges strongly to some $\underline{v}$ in $H_2^2(M) \setminus \{0\}$. In other words, there is $\underline{v} \in H_2^2(M) \setminus \{0\}$ such that

$$\overline{t}(u_n, \lambda)u_n \longrightarrow \underline{v} \quad \text{in} \quad H_2^2(M).$$

Since (u_n) belongs to the unit sphere $\mathbb{S}$, then

$$\begin{cases}
\overline{t}(u_n, \lambda) &\longrightarrow \quad \underline{t} := ||\underline{v}|| > 0, \\
u_n &\longrightarrow \quad \underline{u} := \underline{v}/||\underline{v}|| \quad \text{in} \quad H_2^2(M).
\end{cases}$$

Notice that

$$\begin{cases}
\partial_t \widetilde{E}_\lambda(\underline{t}, \underline{u}) = \lim_{n\to\infty} \partial_t \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n) = 0, \\
\\
\partial_{tt} \widetilde{E}_\lambda(\underline{t}, \underline{u}) = \lim_{n\to\infty} \partial_{tt} \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n) \geq 0.
\end{cases}$$

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But $\partial_{tt}\widetilde{E}_\lambda(\underline{t}, \underline{u}) = 0$ can not occur, since $0 < \lambda < \widehat{\lambda}$ and because of Lemma 1. Hence,

$$\begin{cases} \partial_t \widetilde{E}_\lambda(\underline{t}, \underline{u}) = \lim_{n \to \infty} \partial_t \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n) = 0, \\[2mm] \partial_{tt} \widetilde{E}_\lambda(\underline{t}, \underline{u}) = \lim_{n \to \infty} \partial_{tt} \widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n) > 0, \end{cases}$$

which implies that

$$\underline{t} = \underline{t}(\underline{u}, \lambda).$$

Notice that $\underline{u}$ is not necessary positive. At this stage, let $\underline{w} \in H_2^2(M)$ be the solution of

$$\left(\Delta_g + \frac{\alpha}{2}\right) \underline{w} = \left|\left(\Delta_g + \frac{\alpha}{2}\right) \underline{u}\right| \quad \text{in } M.$$

It is clear that $\left(\Delta_g + \frac{\alpha}{2}\right)(\underline{w} - \underline{u}) \geq 0$ and $\left(\Delta_g + \frac{\alpha}{2}\right)(\underline{w} + \underline{u}) \geq 0$. Since the manifold M is compact, the maximum principle for the operator $\Delta_g + \frac{\alpha}{2}$ implies that $\underline{w} \geq |\underline{u}|$. Using the fact that

$$||u||^2 := \int_M \left\{\left(\Delta_g u + \frac{\alpha}{2}u\right)^2 + \left(\beta - \frac{\alpha^2}{4}\right)u^2\right\} dv_g, \quad \forall u \in H_2^2(M),$$

and that $\beta - \alpha^2/4 \leq 0$, we get

$$||\underline{w}|| \leq ||\underline{u}||.$$

Using Lemma 2, we conclude that

$$E_\lambda(\underline{t}(\underline{w}, \lambda)\underline{w}) \leq E_\lambda(\underline{t}(\underline{u}, \lambda)\underline{u}).$$

We obtain that $\underline{t}(\underline{w}, \lambda)\underline{w}$ is a nontrivial nonnegative solution to (2). Finally, since $\underline{w} \geq 0$, $\underline{w} \neq 0$ then applying again the maximum principle for the operator $\Delta_g + \frac{\alpha}{2}$, we deduce that $\underline{w}$ is positive. The same arguments can be used for $\overline{v}_n$ to obtain a second positive solution $\overline{t}(\overline{w}, \lambda)\overline{w}$, which achieves the proof. $\qquad\Box$

2.3 Behaviour of the energy

In this subsection, we give results concerning the sign of the energy for the positive solutions to (2). To be more precise in the sequel, $\underline{t}(\underline{w}_\mu, \mu)\underline{w}_\mu$ and $\overline{t}(\overline{w}_\mu, \mu)\overline{w}_\mu$ will stand for the positive solutions found above, when the value of the parameter λ is equal to μ.

Theorem 3. *Let* $1 < q < 2 < r < 2^\#$. *Then*

    a. $E_\lambda\left(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda\right) < 0$ *for* $\lambda \in]0, \widehat{\lambda}[$,

    b. $\begin{cases} E_\lambda\left(\overline{t}(\overline{w}_\lambda, \lambda)\overline{w}_\lambda\right) > 0 & \text{for } \lambda \in]0, \lambda_0[, \\ E_\lambda\left(\overline{t}(\overline{w}_\lambda, \lambda)\overline{w}_\lambda\right) < 0 & \text{for } \lambda \in]\lambda_0, \widehat{\lambda}[, \end{cases}$

where

$$\lambda_0 := \frac{q}{r}\left(\frac{r}{2}\right)^{\frac{r-q}{r-2}} \widehat{\lambda}.$$

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Proof.

a. Let us recall that $\partial_t \widetilde{E}_\lambda \left(\underline{t}(\underline{w}_\lambda, \lambda), \underline{w}_\lambda \right) = 0$ and $\partial_{tt} \widetilde{E}_\lambda \left(\underline{t}(\underline{w}_\lambda, \lambda), \underline{w}_\lambda \right) > 0$. Then

$$
\begin{cases}
P(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda) - \lambda Q(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda) - R(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda) = 0, \\[2mm]
P(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda) - \lambda(q-1)Q(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda) - (r-1)R(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda) > 0.
\end{cases}
$$

Using the fact that $1 < q < 2 < r$, we get

$$
\frac{1}{2}P(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda) - \frac{\lambda}{q}Q(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda) - \frac{1}{r}R(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda) < 0,
$$

and consequently $E_\lambda(\underline{t}(\underline{w}_\lambda, \lambda)\underline{w}_\lambda) < 0$.

b. Let u be an arbitrary element of $H_2^2(M) \setminus \{0\}$ and let us write

$$
\widetilde{E}_\lambda(t, u) = t^q \widetilde{G}_\lambda(t, u), \text{ where } \widetilde{G}_\lambda(t, u) = t^{2-q}\frac{P(u)}{2} - \lambda\frac{Q(u)}{q} - t^{r-q}\frac{R(u)}{r}.
$$

It follows that

$$
\partial_t \widetilde{E}_\lambda(t, u) = q t^{q-1} \widetilde{G}_\lambda(t, u) + t^q \partial_t \widetilde{G}_\lambda(t, u),
$$

with

$$
\partial_t \widetilde{G}_\lambda(t, u) = t^{2-q-1} \left\{ \frac{2-q}{2}P(u) - \frac{r-q}{r}t^{r-2}R(u) \right\}.
$$

The real valued function $t \longmapsto \widetilde{G}_\lambda(t, u)$ is increasing on $]0, t_0(u)[$, decreasing on $]t_0(u), +\infty[$ and reaches its unique maximum for $t = t_0(u)$, where

$$
t_0(u) = \left(\frac{r}{2} \right)^{\frac{1}{r-2}} t(u), \tag{15}
$$

and $t(u)$ is defined in (3). On the other hand, a direct computation gives

$$
\widetilde{G}_\lambda(t_0(u), u) = \left(\frac{2-q}{r-q}\frac{P(u)}{R(u)} \right)^{\frac{r-q}{r-2}} R(u) - \lambda Q(u).
$$

Similarly, $\widetilde{G}_\lambda(t_0(u), u) > 0$ (resp. $\widetilde{G}_\lambda(t_0(u), u) < 0$) if $\lambda < \lambda_0(u)$ (resp. $\lambda > \lambda_0(u)$) and $\widetilde{G}_{\lambda_0(u)}(t_0(u), u) = 0$, where

$$
\lambda_0(u) = \frac{q}{r} \left(\frac{r}{2} \right)^{\frac{r-q}{r-2}} \lambda(u), \tag{16}
$$

and $\lambda(u)$ is given by (4). Thus, we get

$$
\begin{cases}
\widetilde{E}_\lambda(t_0(u), u) > 0 & \text{if} \quad \lambda < \lambda_0(u), \\
\widetilde{E}_\lambda(t_0(u), u) = 0 & \text{if} \quad \lambda = \lambda_0(u), \\
\widetilde{E}_\lambda(t_0(u), u) < 0 & \text{if} \quad \lambda > \lambda_0(u).
\end{cases} \tag{17}
$$

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Let us consider the increasing real valued function

$$
\begin{aligned}
]0,1[&\longrightarrow \mathbb{R}\\
t &\longmapsto (\ln t)/(1-t).
\end{aligned}
$$

Then, for every two real numbers x, y such that $0 < x < y < 1$, one has

$$
\ln\left(\frac{1}{x}\right) > \frac{1-x}{1-y}\ln\left(\frac{1}{y}\right) = \ln\left(\left(\frac{1}{y}\right)^{\frac{1-x}{1-y}}\right).
$$

Therefore,

$$
0 < x\left(\frac{1}{y}\right)^{\frac{1-x}{1-y}} < 1.
$$

In the specific case: $x = q/r$ and $y = 2/r$ we get

$$
0 < \frac{q}{r}\left(\frac{r}{2}\right)^{\frac{r-q}{r-2}} < 1,
$$

which gives $0 < \lambda_0(u) < \lambda(u)$.

Moreover, for every $u \in H_2^2(M)\setminus\{0\}$, one has $\widetilde{G}_{\lambda_0(u)}(t,u) < 0$ for $t \in]0,+\infty[\setminus\{t_0(u)\}$ and $\widetilde{G}_{\lambda_0(u)}(t_0(u),u) = 0$. Hence, the real valued function $t \longmapsto \widetilde{E}_{\lambda_0(u)}(t,u)$, $(t > 0)$, reaches its unique maximum at $t = t_0(u)$ and we obtain the following relation

$$
\bar{t}(u,\lambda_0(u)) = t_0(u). \tag{18}
$$

Classical arguments of variational calculus show that $\lambda_0(u)$ is weakly lower semi-continuous on $H_2^2(M)\setminus\{0\}$. Then, the value

$$
\lambda_0 := \inf_{u \in H_2^2(M)\setminus\{0\}} \lambda_0(u) \tag{19}
$$

is achieved on $H_2^2(M)\setminus\{0\}$. Since $\lambda_0(u)$ is homogeneous in u, we can assume that there is some $u^* \in \mathbb{S}$ such that $\lambda_0 = \lambda_0(u^*)$.

Now, let $\lambda \in]0,\lambda_0[$. Then, for every $u \in H_2^2(M)\setminus\{0\}$ one has $\lambda < \lambda_0(u)$ and consequently $\widetilde{E}_\lambda(t_0(u),u) > 0$ holds true from (17). But, $t \longmapsto \widetilde{E}_\lambda(t,u)$, $(t > 0)$ reaches its unique maximum for $t = \bar{t}(u,\lambda)$, hence $\widetilde{E}_\lambda(\bar{t}(u,\lambda),u) > 0$, for every $u \in H_2^2(M)\setminus\{0\}$. In particular, we have $\widetilde{E}_\lambda(\bar{t}(\overline{w}_\lambda,\lambda),\overline{w}_\lambda) > 0$, i.e. $E_\lambda(\bar{t}(\overline{w}_\lambda,\lambda)\overline{w}_\lambda) > 0$. In the specific case $\lambda = \lambda_0$, one has

$$
\begin{aligned}
E_{\lambda_0}(\bar{t}(\overline{w}_{\lambda_0},\lambda_0)\overline{w}_{\lambda_0}) &= \widetilde{E}_{\lambda_0}(\bar{t}(\overline{w}_{\lambda_0},\lambda_0),\overline{w}_{\lambda_0}),\\
&= \inf_{u \in \mathbb{S}} \widetilde{E}_{\lambda_0}(\bar{t}(u,\lambda_0),u),\\
&\leq \widetilde{E}_{\lambda_0}(\bar{t}(u^*,\lambda_0(u^*)),u^*),\\
&= \widetilde{E}_{\lambda_0(u^*)}(t_0(u^*),u^*),\\
&= 0.
\end{aligned}
$$

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This implies that $E_{\lambda_0}(\bar{t}(\overline{w}_{\lambda_0}, \lambda_0)\overline{w}_{\lambda_0}) \leq 0$. Thanks to (17), we have in addition $\widetilde{E}_{\lambda_0}(t_0(u), u) \geq 0$ and $\widetilde{E}_{\lambda_0}(\underline{t}(u, \lambda_0), u) < 0$, for every $u \in H_2^2(M) \setminus \{0\}$. Then, it comes

$$t_0(u) > \underline{t}(u, \lambda_0), \quad \forall u \in H_2^2(M) \setminus \{0\},$$

and

$$\widetilde{E}_{\lambda_0}(\bar{t}(\overline{w}_{\lambda_0}, \lambda_0), \overline{w}_{\lambda_0}) \geq \widetilde{E}_{\lambda_0}(t_0(\overline{w}_{\lambda_0}), \overline{w}_{\lambda_0}) \geq 0.$$

Finally, one gets

$$E_{\lambda_0}(\bar{t}(\overline{w}_{\lambda_0}, \lambda_0)\overline{w}_{\lambda_0}) = 0.$$

Now, assume that $\lambda_0 < \lambda < \widehat{\lambda}$. Since, for every $(t, u) \in]0, +\infty[\times (H_2^2(M) \setminus \{0\})$, the real valued function $\lambda \longmapsto \widetilde{E}_\lambda(t, u)$ is decreasing, it follows that

$$\widetilde{E}_\lambda(t, u) < \widetilde{E}_{\lambda_0}(t, u), \quad \text{for every } t > 0 \text{ and } u \in H_2^2(M) \setminus \{0\}. \tag{20}$$

In addition, we have

$$
\begin{aligned}
\widetilde{E}_\lambda(\bar{t}(\overline{w}_\lambda, \lambda), \overline{w}_\lambda) &= \inf_{u \in \mathbb{S}} \widetilde{E}_\lambda(\bar{t}(u, \lambda), u), \\
&\leq \widetilde{E}_\lambda(\bar{t}(u^*, \lambda), u^*), \\
&< \widetilde{E}_{\lambda_0}(\bar{t}(u^*, \lambda), u^*),
\end{aligned}
$$

where the last inequality follows from (20). Moreover, the real valued function $t \longmapsto \widetilde{E}_{\lambda_0}(t, u^*)$, $(t > 0)$, achieves its global maximum at $t = t_0(u^*)$. Thus, $\widetilde{E}_{\lambda_0}(\bar{t}(u^*, \lambda), u^*) \leq \widetilde{E}_{\lambda_0}(t_0(u^*), u^*) = \widetilde{E}_{\lambda_0(u^*)}(t_0(u^*), u^*) = 0$. Hence $E_\lambda(\bar{t}(\overline{w}_\lambda, \lambda)\overline{w}_\lambda) < 0$, which ends the proof. $\square$

The following reslut shows that the variational character of (5) has a genuine link with the main problem (2).

Theorem 4. *If u is a solution of (19) then $t_0(u)u$ is a solution of (2) for $\lambda = \lambda_0$.*

Proof.
If u is a solution of (19), then $\lambda_0 = \lambda_0(u)$ and for every $\varphi \in H_2^2(M)$, we have

$$
\begin{aligned}
E'_{\lambda_0}(t_0(u)u)(\varphi) &= \frac{1}{2}P'(t_0(u)u)(\varphi) - \frac{\lambda_0}{q}Q'(t_0(u)u)(\varphi) - \frac{1}{r}R'(t_0(u)u)(\varphi), \\
&= \frac{P(u)[t_0(u)]}{2}\left(\frac{P'(u)(\varphi)}{P(u)} - \frac{r-2}{r-q}\frac{Q'(u)(\varphi)}{Q(u)} - \frac{2-q}{r-q}\frac{R'(u)(\varphi)}{R(u)}\right), \\
&= K\left(\frac{r-q}{r-2}\frac{P'(u)(\varphi)}{P(u)} - \frac{Q'(u)(\varphi)}{Q(u)} - \frac{2-q}{r-2}\frac{R'(u)(\varphi)}{R(u)}\right),
\end{aligned}
$$

where

$$K := \frac{r-2}{r-q}\frac{P(u)}{2}[t_0(u)].$$

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By hypothese, one gets $\lambda'_0(u)(\varphi) = 0$, for every $\varphi \in H_2^2(M)$, and

$$\lambda'_0(u)(\varphi) = \lambda_0 \left(\frac{r-q}{r-2} \frac{P'(u)(\varphi)}{P(u)} - \frac{Q'(u)(\varphi)}{Q(u)} - \frac{2-q}{r-2} \frac{R'(u)(\varphi)}{R(u)} \right).$$

We conclude that

$$E'_{\lambda_0}(t_0(u)u)(\varphi) = \frac{K}{\lambda_0} \lambda'_0(u)(\varphi) = 0,$$

for every $\varphi \in H_2^2(M)$, which implies that $t_0(u)u$ is a solution of (2) for $\lambda = \lambda_0$. $\qquad\square$

3   Infinitely many solutions

In this section, we show the existence of infinitely many solutions to (2). More precisely, we carry out two disjoint and infinite sets of solutions to (2): One set consists of solutions with negative energy while the second set contains solutions with arbitrary energy. We briefly recall here some background facts that we use in the sequel [21, 22, 25, 26]. Let

$$\mathcal{A} = \{A \subset \mathbb{S}: \quad A \text{ closed}, \ A = -A\}$$

be the class of closed and symmetric subsets of the complete smooth submanifold $(\mathbb{S}, \|\ \|)$. For every $A \in \mathcal{A}$, $A \neq \emptyset$, let

$$\gamma(A) = \inf \left\{ k \in \mathbb{N}; \ \exists \varphi \in C^0(A, \mathbb{R}^k \setminus \{0\}), \ \forall u \in A, \ \varphi(-u) = -\varphi(u) \right\}$$

be the Krasnoselskii genus [21]. When there does not exist a finite such integer, set $\gamma(A) = +\infty$. Finally, set $\gamma(\emptyset) = 0$. For each positive integer k, let us define

$$\Gamma_k = \{A \in \mathcal{A}; \ A \text{ compact}, \ \gamma(A) \geq k\},$$

$$\underline{c}_k = \inf_{A \in \Gamma_k} \max_{u \in A} \underline{\mathcal{E}}_\lambda(u) \ \text{ and } \ \overline{c}_k = \inf_{A \in \Gamma_k} \max_{u \in A} \overline{\mathcal{E}}_\lambda(u),$$

where

$$\underline{\mathcal{E}}_\lambda(u) = E_\lambda(\underline{t}(u, \lambda)u)$$

and

$$\overline{\mathcal{E}}_\lambda(u) = E_\lambda(\overline{t}(u, \lambda)u).$$

It is well known that $(\underline{c}_k)$ (resp. $(\overline{c}_k)$) is a nondecreasing sequence of critical values of $\underline{\mathcal{E}}_\lambda$ (resp. $\overline{\mathcal{E}}_\lambda$) [26]. Recall that if the sequence $(\underline{c}_k)$ (resp. $(\overline{c}_k)$) is increasing, then $\underline{\mathcal{E}}_\lambda$ (resp. $\overline{\mathcal{E}}_\lambda$) has infinitely many critical points $(\underline{u}_{\lambda,k})$ (resp. $(\overline{u}_{\lambda,k})$) corresponding to the sequence of distinct levels $(\underline{c}_k)$ (resp. $(\overline{c}_k)$). On the other hand, if there are two positive integers j and p such that $\underline{c}_j = \underline{c}_{j+1} = \cdots = \underline{c}_{j+p}$ (resp. $\overline{c}_j = \overline{c}_{j+1} = \cdots = \overline{c}_{j+p}$), then the set of critical points for $\underline{\mathcal{E}}_\lambda$ (resp. $\overline{\mathcal{E}}_\lambda$) corresponding to the level $\underline{c}_j$ (resp. $\overline{c}_j$) is infinite. Hereafter, we set

$$\underline{v}_{\lambda,k} = \underline{t}(\underline{u}_{\lambda,k}, \lambda)\underline{u}_{\lambda,k}$$

and
$$\overline{v}_{\lambda,k} = \overline{t}(\overline{u}_{\lambda,k}, \lambda)\overline{u}_{\lambda,k}.$$

Let us recall that $\underline{v}_{\lambda,k}$ and $\overline{v}_{\lambda,k}$ are solutions of (2), for every $k \in \mathbb{N}^*$.

Theorem 5. *Let* $1 < q < 2 < r < 2^\#$ *and* $0 < \lambda < \widehat{\lambda}$. *Then, there are two disjoint and infinite sets of solutions to (2):* $\{\underline{v}_{\lambda,k}; \ k \in \mathbb{N}^*\}$ *and* $\{\overline{v}_{\lambda,k}; \ k \in \mathbb{N}^*\}$. *In addition, we have*

    a. $\displaystyle\lim_{k \to +\infty} E_\lambda(\overline{v}_{\lambda,k}) = +\infty,$

    b. $E_\lambda(\underline{v}_{\lambda,k}) < 0$ *and* $\displaystyle\lim_{k \to +\infty} E_\lambda(\underline{v}_{\lambda,k}) = 0.$

Proof.

Let us recall that $\partial_{tt}\widetilde{E}_\lambda(\underline{t}(\underline{u}_{\lambda,k}, \lambda), \underline{u}_{\lambda,k}) > 0$ and $\partial_{tt}\widetilde{E}_\lambda(\overline{t}(\overline{u}_{\lambda,k}, \lambda), \overline{u}_{\lambda,k}) < 0$. Then the two sets $\{\underline{v}_{\lambda,k}; \ k \in \mathbb{N}^*\}$ and $\{\overline{v}_{\lambda,k}; \ k \in \mathbb{N}^*\}$ are disjoint.

a. As mentioned above, the sequence $(\overline{c}_k) := \left(\overline{\mathcal{E}}_\lambda(\overline{u}_{\lambda,k})\right) := (E_\lambda(\overline{v}_{\lambda,k}))$ is nondecreasing. Suppose, by contradiction, that

$$\lim_{k \to \infty} \overline{c}_k = \overline{c} < +\infty,$$

and consider the symmetric set

$$K_{\overline{c}} = \left\{ u \in \mathbb{S}; \ \overline{\mathcal{E}}_\lambda(u) = \overline{c} \ \text{and} \ \overline{\mathcal{E}}'_\lambda(u) = 0 \right\}.$$

It is easy to see that
$$\lim_{k \to \infty} E_\lambda(\overline{t}(\overline{u}_{\lambda,k}, \lambda)\overline{u}_{\lambda,k}) \ = \ \overline{c},$$

$$E'_\lambda(\overline{t}(\overline{u}_{\lambda,k}, \lambda)\overline{u}_{\lambda,k}) \ = \ 0.$$

Since E_λ satisfies the Palais-Smale condition, we conclude that $K_{\overline{c}}$ is not empty and compact. Then $\gamma(K_{\overline{c}}) < +\infty$ [25]. Let N be a closed neighborhood of $K_{\overline{c}}$ in $\mathbb{S}$ such that $\gamma(N) = \gamma(K_{\overline{c}})$. The deformation lemma [22, 25, 26, 29], ensures the existence of an odd homeomorphism Φ from $\mathbb{S}$ to $\mathbb{S}$ and $\varepsilon > 0$ such that

$$\Phi\left(A_{\overline{c}+\varepsilon} \setminus \overset{\circ}{N}\right) \subset A_{\overline{c}-\varepsilon}.$$

Using classical properties of the Krasnoselskii genus, one gets

$$\begin{aligned}
\gamma\left(A_{\overline{c}+\varepsilon}\right) &\leq \gamma\left(A_{\overline{c}+\varepsilon} \setminus \overset{\circ}{N}\right) + \gamma(N) \\
&\leq \gamma\left(\Phi\left(A_{\overline{c}+\varepsilon} \setminus \overset{\circ}{N}\right)\right) + \gamma(N) \\
&\leq \gamma\left(A_{\overline{c}-\varepsilon}\right) + \gamma(N).
\end{aligned}$$

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On the other hand, by definition of $\overline{c}$, there is a positive integer j such that $\overline{c} - \varepsilon < c_j \le \overline{c}$. As a consequence, one gets $\gamma\left(A_{\overline{c}-\varepsilon}\right) < j$ and

$$\gamma\left(A_{\overline{c}+\varepsilon}\right) < j + \gamma(K_{\overline{c}}) < +\infty.$$

But this is in contradiction with $\gamma\left(A_{\overline{c}+\varepsilon}\right) = +\infty$. We finally obtain that

$$\lim_{k \to \infty} \overline{c}_k = +\infty,$$

b. It is known from above that $E_\lambda(v_{\lambda,k}) < 0$ for every k. Let us show that $(\underline{c}_k)$ converges to zero as k goes to infinity. Suppose that

$$\lim_{k \to +\infty} \underline{c}_k = \underline{c} < 0.$$

As before, consider the symmetric set

$$K_{\underline{c}} = \left\{u \in \mathbb{S}; \ \ \underline{\mathcal{E}}_\lambda(u) = \underline{c} \ \text{ and } \ \underline{\mathcal{E}}'_\lambda(u) = 0\right\}.$$

Since $\underline{\mathcal{E}}$ satisfies that Palais-Smale condition on $\mathbb{S}$ and $\underline{c} < 0$ then $K_{\underline{c}}$ is not empty, compact wich implies that $\gamma(K_{\underline{c}}) < +\infty$. Let N be a closed neighborhood of $K_{\underline{c}}$ in $\mathbb{S}$ such that $\gamma(N) = \gamma(K_{\underline{c}})$. Applying again the deformation lemma , there are an odd homeomorphism Ψ from $\mathbb{S}$ to $\mathbb{S}$ and $\varepsilon > 0$ such that

$$\Psi\left(A_{\underline{c}+\varepsilon} \setminus \overset{\circ}{N}\right) \subset A_{\underline{c}-\varepsilon}.$$

As mentioned above, we get

$$
\begin{aligned}
\gamma\left(A_{\underline{c}+\varepsilon}\right) &\le \ \gamma\left(A_{\underline{c}+\varepsilon} \setminus \overset{\circ}{N}\right) + \gamma(N) \\
&\le \ \gamma\left(\Psi\left(A_{\underline{c}+\varepsilon} \setminus \overset{\circ}{N}\right)\right) + \gamma(N) \\
&\le \ \gamma\left(A_{\underline{c}-\varepsilon}\right) + \gamma(N).
\end{aligned}
$$

Furthermore, there is a positive integer j such that $\underline{c} - \varepsilon < c_j \le \underline{c}$, then $\gamma\left(A_{\underline{c}-\varepsilon}\right) < j$ and consequently

$$\gamma\left(A_{\underline{c}+\varepsilon}\right) < j + \gamma(K_{\underline{c}}) < +\infty.$$

This contradicts $\gamma\left(A_{\underline{c}+\varepsilon}\right) = +\infty$, which ends the proof. $\qquad\square$

Comment. Consider the specific case where the functions a and b are positive constants. Then the problem (2) possesses two constant and positive solutions $\underline{c}(\lambda)$ and $\overline{c}(\lambda)$. The solutions $\underline{c}(\lambda)$ and $\overline{c}(\lambda)$ realize respectively the (unique) local minimum and the (unique) local maximum of the real valued function

$$\frac{\beta}{2} t^2 - \lambda \frac{a}{q} t^q - \frac{b}{r} t^r, \quad t > 0,$$

where

$$0 < \lambda < \lambda^* := \widehat{C} \frac{\beta^{\frac{r-q}{r-2}}}{ab^{\frac{2-q}{r-2}}}.$$

Notice that $\widehat{\lambda} \leq \lambda^*$. An interesting question is to compar $\underline{\alpha}(\lambda)$ (resp. $\overline{\alpha}(\lambda)$) with $E_\lambda(\underline{c}(\lambda))$ (resp. with $E_\lambda(\overline{c}(\lambda))$). If, for example, $\underline{\alpha}(\lambda) < E_\lambda(\underline{c}(\lambda))$ and $\overline{\alpha}(\lambda) < E_\lambda(\overline{c}(\lambda))$ then (2) possesses four positive solutions.

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