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Relaxed formulation and existence result of the degenerated elliptic small disturbance model

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Abstract

We consider an elliptic-parabolic PDE, governing the flow of a steady state two-dimensional irrotational compressible fluid in a channel or around a thin profile. The model is formulated in term of a non-coercive variational problem with an integral functional, in a truncated domain. We propose a relaxed formulation in the bounded variation functions space and we prove the existence and the uniqueness of the relaxed optimal solution. We consider also the asymptotic behavior of the solution when the domain growths to infinity.

Key words: Ill-posed variational problem, relaxed problem, degenerated elliptic equation, bounded variation functions.
1991 MSC: 35M10, 35J70, 35R25, 49J45.

1 Introduction

We are interested in the mathematical analysis of the small disturbance model which describes a steady state two-dimensional irrotational compressible flow model around a thin profile.

Theoretical analysis of transonic flows of perfect compressible fluids pose fundamental problems and only partial results concerning the existence and uniqueness of solutions are proved. Indeed, the type of flow is variable, *i.e.* whether the flow is subsonic, sonic or supersonic, the corresponding equation

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is elliptic, parabolic or hyperbolic respectively. The difficulties of the problem come from the non-linearity of the equations and the change of the flow type. The position of the type change (sonic and shock lines) is unknown *a priori*. The presence of jump lines increases the precedent difficulties.

One may find for the elliptic case, in M. Pogu et al [19], some general mathematical modelizations and in M. Amara et al [2] for that particular one.

The physical problem consists in a two-dimensional irrotational compressible flow around a thin profile, in an infinite atmosphere; this flow is assumed to be uniform at infinity. After an asymptotical analysis (cf. M. Amara et al [2]), one restricts the domain to the bounded domain $\Omega = \Omega_R =]-R, R[\times]0, R[$. R is chosen large enough to display the boundary conditions at infinity on the truncated boundary. Therefore, if one denotes by (u, v) the vector valued function describing the perturbation of the velocity of the flow compared to the uniform velocity at infinity, one gets the following non linear elliptic-parabolic-hyperbolic system:

$$\begin{aligned}
&\partial_x f(u) + \partial_y v = 0 && \text{in} && \Omega,\\
&\partial_x v - \partial_y u = 0 && \text{in} && \Omega,\\
&v = z'(x) && \text{on} && \Gamma_p,\\
&u = 0 && \text{on} && \Gamma_2,\\
&v = 0 && \text{on} && \Gamma_0 \cup \Gamma_1,
\end{aligned}$$

where:

$$\begin{aligned}
&\Gamma_p =]-\tfrac{1}{2}, \tfrac{1}{2}[\times\{0\} \text{ (profile boundary)},\\
&\Gamma_2 = \{-R\}\times]0, R[\cup\{R\}\times]0, R[,\\
&\Gamma_0 =]-R, -\tfrac{1}{2}[\times\{0\}\cup]\tfrac{1}{2}, R[\times\{0\},\\
&\Gamma_1 =]-R, R[\times\{R\},
\end{aligned}$$

$$\begin{aligned}
&\forall t \in \mathbb{R}, \quad f(t) = -\frac{\gamma+1}{2}(u_{cr} - t)^2 + \frac{\gamma+1}{2}u_{cr}^2,\\
&\gamma \in \mathbb{R}^{+*}, \quad u_{cr} \in \mathbb{R}^{+*},\\
&z \in W_0^{1,\infty}(]-\frac{1}{2}, \frac{1}{2}[),
\end{aligned}$$

and one assumes (hypothesis of truncated domain) that R is chosen large enough in order to have

$$\frac{\gamma+1}{2}u_{cr}^2 R > ||z||_\infty. \tag{1}$$

One can see that this system is elliptic when $u < u_{cr}$ (subsonic flow), parabolic when $u = u_{cr}$ (sonic flow) and hyperbolic when $u > u_{cr}$ (hypersonic flow).

$\forall U, V \in \mathcal{D}'(\Omega), \quad \forall t \in \mathbb{R}, \quad$ let us note:

- $div(U,V) = \partial_x U + \partial_y V$,
- $t_+ = \max(0,t)$,
- $n = (n_x, n_y)$ the unit outward normal vector to $\Gamma = \partial\Omega$,
- $(U,V).n = U\, n_x + V\, n_y$.

In the sequel, we shall consider subsonic flows (*i.e.* $u \leq u_{cr}$). Thus, thanks to the irrotationality, one can look for $(u,v) = (\partial_x\varphi, \partial_y\varphi)$ where φ is a solution of:

$$div(g(\partial_x\varphi), \partial_y\varphi) = 0 \text{ in } \Omega \tag{2}$$

with, $g(t) = -\frac{\gamma+1}{2}(u_{cr} - t)_+^2 + \frac{\gamma+1}{2}u_{cr}^2$ and with the following boundary conditions:

$$\partial_y\varphi = z'(x) \text{ on } \Gamma_p \quad \text{and} \quad (g(\partial_x\varphi), \partial_y\varphi).n = 0 \text{ on } \partial\Omega\backslash\Gamma_p. \tag{3}$$

Remark 1
i) One may notice that (2) is a degenerated elliptic equation since it is elliptic if $\partial_x\varphi < u_{cr}$ and parabolic if $\partial_x\varphi \geq u_{cr}$.
ii) Note that, if the flow remains subsonic (i.e. $u \leq u_{cr}$), then $g(u) = f(u)$ a.e. in Ω and we get the first physical problem.

The *a priori* natural space associated to Equation (2) is

$$V = \{\xi \in L^2(\Omega) : \partial_x\xi \in L^3(\Omega),\ \partial_y\xi \in L^2(\Omega)\}$$

and one can see φ as an optimal solution of a variational problem, associated with an integral functional:

$$\text{Minimise } F(\xi) \text{ such that } \xi \in V^0 \text{ where:} \tag{4}$$

- $V^0 = \{\xi \in V : \int\limits_\Omega \xi \, dxdy = 0\}$,
- $\forall \xi \in V,\ F(\xi) = \int\limits_\Omega \left[G(\partial_x\xi) + \frac{1}{2}(\partial_y\xi)^2\right] dxdy + \int\limits_{\Gamma_p} z'\xi \, d\sigma \qquad (5)$

with

$$\bullet\, \forall t \in \mathbb{R},\ G(t) = \frac{\gamma+1}{6}(u_{cr}-t)_+^3 - \frac{\gamma+1}{2}u_{cr}^2(u_{cr}-t) + \frac{\gamma+1}{3}u_{cr}^3. \qquad (6)$$

For any real t, we have, $G'(t) = g(t)$ and $G"(t) = (\gamma+1)(u_{cr}-t)_+$, then, the functional F is differentiable and we have formally

$$< F'(\xi), \psi >= \int_\Omega [G'(\partial_x\xi)\partial_x\psi + \partial_y\xi\partial_y\psi]\, dxdy + \int_{\Gamma_p} z'\psi\, d\sigma,$$

$$< F"(\xi), (\psi_1, \psi_2) >= \int_\Omega [G"(\partial_x\xi)\partial_x\psi_1\partial_x\psi_2 + \partial_y\psi_1\partial_y\psi_2]\, dxdy.$$

We note that F is convex and invariant with respect to the constants (so, we use V^0 instead of V), but it is not coercive on these spaces. Therefore, we introduce a less regular space:

$$W = \{\xi \in L^2(\Omega) : \partial_x\xi \in L^1(\Omega),\ \partial_y\xi \in L^2(\Omega),\ \int_\Omega \xi\, dxdy = 0\}.$$

Then, under Hypothesis (1), we prove that F is coercive with respect to the topology of W.

Remark 2 *Let us remark that F is coercive with respect to the topology of V in the convex set $K = \{\xi \in V^0 : \partial_x\xi \leq u_{cr}\}$. Then, there exists a unique φ_0 in K such that $F(\varphi_0) \leq F(\xi)$ for any ξ in K.*

One can find in M. Amara *et al.* [2] a result on the existence of solutions of Problem 4 using perturbation-duality arguments (cf. I. Ekeland *et al.* [13]). Since any minimizing sequence possesses a sub-sequence converging in $BV(\Omega)$, the solution has to be looked for in the space N^0 where

$$N^0 = \{\xi \in N : \int_\Omega \xi\, dxdy = 0\} \quad \text{with} \quad N = \{\xi \in BV(\Omega) : \partial_y\xi \in L^2(\Omega)\}.$$

For any ξ in N, let us note in the sequel:

$$\partial_x\xi = u_\xi\, dxdy + \mu_\xi$$

where u_ξ is the absolutely continuous part of the Radon measure $\partial_x\xi$ with respect to the Lebesgue measure and μ_ξ its singular part.

Therefore, thanks to L. Carbone *et al.* [11], a natural argument of relaxation leads us to consider the new problem:

$$\inf_{\xi \in N^0} \tilde{F}(\xi)$$

where, for any ξ in N,

$$\begin{aligned}\tilde{F}(\xi) = & \int_{\Omega} G(u_\xi)\, dxdy + \frac{1}{2} \int_{\Omega} (\partial_y \xi)^2\, dxdy \\ & + \int_{\Omega} G^\infty(\frac{\mu_\xi}{|\mu_\xi|})\, d|\mu_\xi| + \int_{\Gamma_p} z'\xi\, d\sigma,\end{aligned} \tag{7}$$

G^∞ denotes the recession function of G defined in (9) and $|\mu_\xi| = \mu_\xi^+ + \mu_\xi^-$.

The aim of this paper is to consider this relaxed formulation and extend M. Amara *et al.* [2]'s results.

There is an extensive literature related to the question of relaxation and lower semicontinuity on $BV(\Omega)$, or $W^{1,1}(\Omega)$, for integral functionals of the form

$$I(\xi) = \int_{\Omega} f(x, \xi, \mathbf{u}_\xi)\, dxdy + \int_{\Omega} h(x, \frac{\mu_\xi}{|\mu_\xi|})\, d|\mu_\xi|$$

where $\mathbf{u}_\xi$ is the density of the absolutely continuous part of $\nabla\xi$ with respect to the Lebesgue measure, μ_ξ is the singular part and assuming that there exists p in $[1, +\infty[$ and two positive constants, c and C, such that

$$\forall \varkappa \in \mathbb{R}^n, \quad c(||\varkappa||^p - 1) \leq f(.,.,\varkappa) \leq C(||\varkappa||^p + 1).$$

One may find in D. Kinderlehrer *et al.* [18] (for the semicontinuity) and T. Roubicek [20] (for the relaxation) some approaches based on Young measure theory.

Thanks to the contribution of geometric measures in the theory of bounded variation functions (see L.C. Evans *et al.* [14], E. Guisti [17] or W.P. Ziemer [22]), minimization problems associated with the function $\tilde{F}$ have been considered. See for example the case of image segmentation modelling (see L. Ambrosio *et al.* [4–6]) where one has to take into account the set of "jumps" of ξ. This set, S_ξ, appears in the minimisation formulation since

$$\mu_\xi = (\xi^+ - \xi^-) n\, \mathcal{H}^{n-1}_{\llcorner S_\xi} + C_\xi$$

where $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure on $\mathbb{R}^n$ and C_ξ the Cantor part of the measure $\nabla\xi$.
In L. Ambrosio *et al.* ([4–6]), p belongs to $]1,+\infty[$. Thus, Ambrosio's lower semicontinuity and compactness theorems lead to the study of SBV, the space of special functions with bounded variation, *i.e.*, BV functions ξ such that $C_\xi = 0$. In this context, the fact that $p > 1$ is essential.
Let us also mention similar studies, related to elasticity problems (see C. Baiocchi *et al.* [7]), that lead to the study of $BD(\Omega)$, the space of bounded deformation.
A second example is the analysis of variational problems for phase transitions. In this case (see G. Bouchitté *et al.* [8,9] or I. Fonseca *et al.* [15]), p is assumed to be 1. Thus, one has to consider relaxation and semicontinuity in $BV(\Omega)$.

In our modelling, p has to be 3 for $\frac{\partial\xi}{\partial x} < u_{cr}$, 1 for $\frac{\partial\xi}{\partial x} > u_{cr}$ and 2 for $\frac{\partial\xi}{\partial y}$. Thus, the problem (4) comes out of theses frameworks.

Quote finally L. Carbone *et al.* [11]&[12] where the authors prove that the relaxed functional in the $L^1(\Omega)$ -topology of

$$\xi \in W^{1,\infty}(\Omega) \mapsto \int_\Omega \left[G(u_\xi) + \frac{1}{2}(\partial_y\xi)^2\right] dxdy$$

is equal to

$$\int_\Omega G^{**}(u_\xi)\, dxdy + \frac{1}{2}\int_\Omega (\partial_y\xi)^2\, dxdy + \int_\Omega (G^{**})^\infty(\frac{\mu_\xi}{|\mu_\xi|})\, d|\mu_\xi|,$$

where G^{**} is the convex lower semicontinuous envelope of G (for a general class of function G). Using a density result of $W^{1,\infty}(\Omega)$ in V, our relaxed formulation is a kind of continuous linear perturbation of the above one where $G = G^{**}$.

In fact, thanks to Goffman-Serrin's theorem and to a result of density of V in $BV(\Omega)$, one can prove that

$$\inf_{\xi\in V^0} F(\xi) = \min_{\xi\in N^0} \widetilde{F}(\xi). \tag{8}$$

Therefore, we prove the existence of a unique solution that satisfies a variational formulation in $BV(\Omega)$. Furthermore, we give some information about the asymptotic behavior of the solution when R goes to infinity.

2 Hypothesis and notation

• For any real convex function f, we recall that the recession function of f, denoted by f^∞, is defined by

$$f^\infty(t) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} f(t_0 + \lambda t) \tag{9}$$

where t_0 is any element of $\mathbb{R}$ ($t_0 = 0$ for example).

Therefore, in our context, $G^\infty(t) = \begin{cases} +\infty & if\ t < 0, \\ \frac{\gamma+1}{2} u_{cr}^2\, t & if\ t \geq 0. \end{cases}$

Remark 3
$\tilde{F}(\xi)$ is finite if and only if μ_ξ^- vanishes and $(u_{cr} - u_\xi)^+$ belongs to $L^3(\Omega)$. So, we are naturally led to be interested in the problem

$$\exists \varphi \in K,\ \inf_{\xi \in K} \tilde{F}(\xi) = \tilde{F}(\varphi), \tag{10}$$

where K is the convex cone:

$$K = \{\xi \in N^0 : (u_{cr} - u_\xi)^+ \in L^3(\Omega),\ \mu_\xi^- = 0\}.$$

• For any ξ in N, let us set

$$b(\xi) = \int_{\Gamma_p} z' \xi \, d\sigma$$

and remark that $b(1) = 0$.
Moreover, since $N \hookrightarrow X = L^2(-R, R; H^1(]0, R[))$,

$$b(\xi) = -\int_{\Omega_0} z'(x)(1 - \frac{y}{R}) \partial_y \xi \, dx dy + \frac{1}{R} \int_{\Omega_0} z'(x) \xi \, dx dy, \tag{11}$$

where $\Omega_0 =]-\frac{1}{2}, \frac{1}{2}[\times]0, R[$. In particular, b belongs to X'.

Remark 4
Since $\Omega \subset \mathbb{R}^2$, $BV(\Omega) \hookrightarrow L^2(\Omega)$ and $N = L^2(-R, R; H^1(0, R)) \cap BV(\Omega)$.

• Finally, we denote by c_1, c_2 and c_3 the three nonnegative constants given by:

$$c_1 = ||z'||_{L^2(-\frac{1}{2}\frac{1}{2})}, \quad c_2 = ||z||_\infty u_{cr} + \frac{Rc_1^2}{2}, \quad c_3 = c_2 + (\gamma+1)\, u_{cr}^3 R^2.$$

3 A relaxed problem

3.1 Properties of the relaxed functional

First of all, $\tilde{F}$ has a property of weak lower semicontinuity. Thanks to (11), the convexity of G and the positivity of the singular part of the measure $\partial_x \xi_n$, this result comes mainly from Goffman-Serrin's theorem (cf. J.J. Alibert *et al.* [1] for example).

Proposition 5 *Let* $(\xi_n)_n$ *be a sequence in* K *such that*

$$\xi_n \rightharpoonup \xi \text{ in } L^2(\Omega), \quad \text{and}$$
$$\partial_x \xi_n \rightharpoonup \partial_x \xi \text{ in } M_b(\Omega) - *, \quad \partial_y \xi_n \rightharpoonup \partial_y \xi \text{ in } L^2(\Omega).$$

Then

$$\tilde{F}(\xi) \le \liminf_{n\to+\infty} \tilde{F}(\xi_n).$$

Therefore, in order to obtain (8), let us give a density result in N, inspired by R. Temam [21] in Chap.1, §1.2.
For any positive λ, let us consider the linear homothetic transformation σ_λ defined, for any (x, y) in Ω, by:

$$\sigma_\lambda(x, y) = (\lambda\, x, y) \text{ and } \sigma_\lambda(\Omega) = \{(\lambda x, y) : (x, y) \in \Omega\}.$$

Then, if T is a distribution in Ω, a distribution $\sigma_\lambda T$ can be defined in $\mathcal{D}'(\sigma_\lambda(\Omega))$ by:

$$< \sigma_\lambda T, \xi >_{\sigma_\lambda(\Omega)} = \lambda < T, \sigma_{\frac{1}{\lambda}} \xi >_\Omega, \forall \xi \in \mathcal{D}(\ \sigma_\lambda(\Omega)),$$

where $(\sigma_{\frac{1}{\lambda}}\xi)(x, y) = \xi(\lambda x, y)$. Therefore,

Proposition 6 *For any ξ in N, there exists a sequence $(\xi_n)_{n\geq 1}$ in V such that:*

$$\xi_n \underset{L^2(\Omega)}{\to} \xi\,, \quad |\partial_x \xi_n|(\Omega) \to |\partial_x \xi|(\Omega), \quad \partial_y \xi_n \underset{L^2(\Omega)}{\to} \partial_y \xi \tag{12}$$

and $\lim\limits_{n\to\infty} F(\xi_n) = \widetilde{F}(\xi)$.

Proof. Let us consider ξ in N.
Thanks to R. Temam [21], there exists $\varepsilon_n > 0$ such that if ρ_{ε_n} denotes the classical mollifier sequences in $\mathbb{R}$, $\xi_n = \frac{n+1}{n}(\rho_{\varepsilon_n} * \sigma_{\frac{n+1}{n}}\xi)$ belongs to V, converges to ξ in $L^2(-R,R;H^1(0,R))$ as n tends to $+\infty$ and $\rho_{\varepsilon_n} * \sigma_{\frac{n+1}{n}} u_\xi$ converges towards u_ξ in $L^1(\Omega)$.
Moreover, for any f in $\mathcal{C}_c(\Omega)$ with $\|f\|_\infty \leq 1$,

$$< \rho_{\varepsilon_n} * \sigma_{\frac{n+1}{n}} \mu_\xi, f >_\Omega = \frac{n+1}{n} < \mu_\xi, \sigma_{\frac{n}{n+1}}(\rho_{\varepsilon_n} * f) >_{\sigma_{\frac{n}{n+1}}(\Omega)} \leq \frac{n+1}{n} |\mu_\xi|(\Omega).$$

Thus, since $\rho_{\varepsilon_n} * \sigma_{\frac{n+1}{n}} \mu_\xi$ converges to μ_ξ weakly-$*$ in $M_b(\Omega)$, we get

$$\lim_{n\to+\infty} |\rho_{\varepsilon_n} * \sigma_{\frac{n+1}{n}} \mu_\xi|(\Omega) = |\mu_\xi|(\Omega).$$

Therefore, the result of density is proved since

$$\partial_x \xi_n = \rho_{\varepsilon_n} * \sigma_{\frac{n+1}{n}} u_\xi + \rho_{\varepsilon_n} * \sigma_{\frac{n+1}{n}} \mu_\xi.$$

Moreover, leading from Proposition 5, one has $\tilde{F}(\xi) \leq \liminf_{n\to+\infty} F(\xi_n)$.
First, if ξ does not belong to K, then $\tilde{F}(\xi) = +\infty$ and $\lim_{n\to+\infty} F(\xi_n) = \tilde{F}(\xi)$.
If ξ belongs to K and if we define $u_1 = u_{cr} - (u_{cr} - u_\xi)^+$ and $u_2 = (u_{cr} - u_\xi)^-$, since μ is a nonnegative measure, $u_2 \geq 0$ and $u_1 \leq u_{cr}$, one has

$$\begin{aligned}
\limsup_{n\to+\infty} \int_\Omega [(u_{cr} - \partial_x \xi_n)^+]^3 \, dxdy &\leq \lim_{n\to+\infty} \int_\Omega [(u_{cr} - \rho_{\varepsilon_n} * \sigma_{\frac{n+1}{n}} u_1)^+]^3 \, dxdy \\
&= \lim_{n\to+\infty} \int_\Omega [(u_{cr} - \rho_{\varepsilon_n} * \sigma_{\frac{n+1}{n}} u_1)]^3 \, dxdy \\
= \int_\Omega (u_{cr} - u_1)^3 \, dxdy &= \int_\Omega [(u_{cr} - u_\xi)^+]^3 \, dxdy.
\end{aligned}$$

Moreover,

$$\lim_{n\to+\infty} \int_\Omega (u_{cr} - \partial_x \xi_n) \, dxdy = \int_\Omega (u_{cr} - u_\xi) \, dxdy - \mu_\xi(\Omega),$$

$(\xi_n)_n$ converges to ξ in $L^2(\Omega)$ and $(\partial_y \xi_n)_n$ converges to $\partial_y \xi$ in $L^2(\Omega)$. Then, $b(\xi_n)$ converges towards $b(\xi)$ and (6) leads to $\limsup\limits_{n\to+\infty} F(\xi_n) \leq \tilde{F}(\xi)$.

Corollary 7 $\mathcal{L} := \inf\limits_{\xi\in V^0} F(\xi) = \inf\limits_{\xi\in N^0} \widetilde{F}(\xi)$.

3.2 Existence of a solution

Let us give now a coercivity result. In order to do so, we need the following lemma:

Lemma 8 *For any ξ in K, we have*

$$b(\xi) \geq -c_2 - \frac{1}{4}\|\partial_y \xi\|^2_{L^2(\Omega_0)} + \frac{1}{R}\int\limits_{\Omega_0} z(u_{cr} - u_\xi)\,dxdy - \frac{||z||_\infty}{R}\int\limits_{\Omega_0} d\mu_\xi.$$

Proof. This lemma comes from (11) and the Green formula in $BV(\Omega)$ (see L.C. Evans [14]).

Proposition 9 *For any ξ in K, we have*

$$\begin{aligned}\tilde{F}(\xi) \geq\, & \tfrac{\gamma+1}{12}\|(u_{cr} - u_\xi)^+\|^3_{L^3(\Omega)} + \tfrac{1}{4}\|\partial_y \xi\|^2_{L^2(\Omega)} - c_3 \\ & + (\tfrac{\gamma+1}{2}u_{cr}^2 - \tfrac{1}{R})\left(\|(u_{cr} - u_\xi)^+\|_{L^1(\Omega)} + \mu_\xi(\Omega)\right).\end{aligned}$$

Proof. First, thanks to the previous lemma, one has:

$$\begin{aligned}\tilde{F}(\xi) &= \int\limits_\Omega G(u_\xi)\,dxdy + \frac{1}{2}\int\limits_\Omega (\partial_y \xi)^2\,dxdy + b(\xi) + \frac{\gamma+1}{2}u_{cr}^2 \mu_\xi(\Omega) \\ &\geq \int\limits_\Omega G(u_\xi)\,dxdy + \frac{1}{4}\int\limits_\Omega (\partial_y \xi)^2\,dxdy - c_2 \\ &\quad + \frac{1}{R}\int\limits_{\Omega_0} z(u_{cr} - u_\xi)\,dxdy - \frac{||z||_\infty}{R}\int\limits_{\Omega_0} d\mu_\xi + \frac{\gamma+1}{2}u_{cr}^2\mu_\xi(\Omega), \\ &\geq \int\limits_\Omega G(u_\xi)\,dxdy + \frac{1}{4}\int\limits_\Omega (\partial_y \xi)^2\,dxdy - c_2 \\ &\quad + \frac{1}{R}\int\limits_{\Omega_0} z(u_{cr} - u_\xi)\,dxdy + \left(\frac{\gamma+1}{2}u_{cr}^2 - \frac{||z||_\infty}{R}\right)\mu_\xi(\Omega).\end{aligned}$$

Then, if $\tilde{z}$ denotes the extension of z by zero outside $[-\frac{1}{2}, \frac{1}{2}]$, one gets:

$$
\begin{aligned}
&\int_{\Omega} G(u_\xi)\,dxdy + \frac{1}{R}\int_{\Omega_0} z(u_{cr}-u_\xi)\,dxdy\\
&= \int_{\Omega}\left[\frac{\gamma+1}{6}(u_{cr}-u_\xi)_+^3 - \left(\frac{\gamma+1}{2}u_{cr}^2 - \frac{\widetilde{z}}{R}\right)(u_{cr}-u_\xi) + \frac{\gamma+1}{3}u_{cr}^3\right] dxdy.
\end{aligned}
$$

Since, $\frac{\gamma+1}{2}u_{cr}^2 > \frac{||z||_\infty}{R}$, one has $0 \le \frac{\gamma+1}{2}u_{cr}^2 - \frac{\widetilde{z}}{R} \le (\gamma+1)u_{cr}^2$ and one gets:

$$
\begin{aligned}
&\int_{\Omega} G(u_\xi)\,dxdy + \frac{1}{R}\int_{\Omega_0} z(u_{cr}-u_\xi)\,dxdy\\
&\ge \int_{\Omega}\left[\frac{\gamma+1}{6}(u_{cr}-u_\xi)_+^3 - \left(\frac{\gamma+1}{2}u_{cr}^2 - \frac{\widetilde{z}}{R}\right)(u_{cr}-u_\xi)^+ + \frac{\gamma+1}{3}u_{cr}^3\right] dxdy\\
&\quad + \left(\frac{\gamma+1}{2}u_{cr}^2 - \frac{||z||_\infty}{R}\right)||(u_{cr}-u_\xi)^-||_{L^1(\Omega)},\\
&\ge \int_{\Omega}\left[\frac{\gamma+1}{6}(u_{cr}-u_\xi)_+^3 - (\gamma+1)u_{cr}^2(u_{cr}-u_\xi)^+ + \frac{\gamma+1}{3}u_{cr}^3\right] dxdy\\
&\quad + \left(\frac{\gamma+1}{2}u_{cr}^2 - \frac{||z||_\infty}{R}\right)||(u_{cr}-u_\xi)^-||_{L^1(\Omega)}.
\end{aligned}
$$

Noting that

$$
\forall t \ge 0,\ \frac{\gamma+1}{6}t^3 - (\gamma+1)u_{cr}^2 t + \frac{\gamma+1}{3}u_{cr}^3 \ge \frac{\gamma+1}{12}t^3 - (\gamma+1)u_{cr}^3
$$

one concludes since

$$
\begin{aligned}
\tilde{F}(\xi) \ge &\int_{\Omega}\left[\frac{\gamma+1}{12}(u_{cr}-u_\xi)_+^3 + \left(\frac{\gamma+1}{2}u_{cr}^2 - \frac{||z||_\infty}{R}\right)(u_{cr}-u_\xi)^-\right] dxdy\\
&+\frac{1}{4}\int_{\Omega}(\partial_y\xi)^2\,dxdy + \left(\frac{\gamma+1}{2}u_{cr}^2 - \frac{||z||_\infty}{R}\right)\mu_\xi(\Omega)\\
&-c_2 - (\gamma+1)u_{cr}^3 R^2.
\end{aligned}
$$

Let us conclude this section by proving (8).

Corollary 10 *There exists a function φ in K such that*

$$
\tilde{F}(\varphi) = \mathcal{L},\ i.e.\ \mathcal{L} = \inf_{\xi\in V^0} F(\xi) = \min_{\xi\in N^0} \widetilde{F}(\xi).
$$

Proof. Indeed, for any minimizing sequence $(\xi_n)_n$, Proposition 9 ensures that it is a bounded sequence in K and, after extractions of adequate subsequences, Proposition 5 leads to the conclusion.

3.3 Properties of the solution

Our purpose is to characterize the solution by means of a variational formulation. This last one comes from the G-derivability of $\widetilde{F}$.

Proposition 11 *For any ξ_i in K $(i = 1, 2)$, one has,*

$$\lim_{\theta \to 0^+} \frac{\tilde{F}(\xi_1 + \theta \xi_2) - \tilde{F}(\xi_1)}{\theta} =< \tilde{F}'(\xi_1), \xi_2 >$$

where

$$< \tilde{F}'(\xi_1), \xi_2 > = \int_\Omega g(u_{\xi_1}) u_{\xi_2} \, dxdy + \int_\Omega \partial_y \xi_1 \, \partial_y \xi_2 \, dxdy + b(\xi_2) + \frac{\gamma + 1}{2} u_{cr}^2 \mu_{\xi_2}(\Omega). \tag{13}$$

Proof. Since μ_{ξ_1} and μ_{ξ_2} are positive measures,

$$(\mu_{\xi_1} + \theta \mu_{\xi_2})^+ = \mu_{\xi_1} + \theta \mu_{\xi_2},$$

then one has:

$$\frac{\tilde{F}(\xi_1 + \theta \xi_2) - \tilde{F}(\xi_1)}{\theta} = \int_\Omega \frac{G(u_{\xi_1} + \theta u_{\xi_2}) - G(u_{\xi_1})}{\theta} \, dxdy + \int_\Omega \frac{\theta}{2} (\partial y \xi_2)^2 \, dxdy + b(\xi_2) + \frac{\gamma+1}{2} u_{cr}^2 \mu_{\xi_2}(\Omega) + \int_\Omega \partial y \xi_1 \, \partial y \xi_2 \, dxdy.$$

Since $(\frac{G(u_{\xi_1} + \theta u_{\xi_2}) - G(u_{\xi_1})}{\theta})_\theta$ is a monotone family of $L^1(\Omega)$ functions, Beppo-Levi's monotone theorem leads to the desired conclusion.

Remark 12 *Even if ξ_2 is in $N \backslash K$, $< \tilde{F}'(\xi_1), \xi_2 >$ may exist: if for a given positive value of θ, $\xi_1 + \theta \xi_2$ belongs to K, the same proof holds; if not, $\tilde{F}(\xi_1 + \theta \xi_2)$ is always infinite, as well as $< \tilde{F}'(\xi_1), \xi_2 >$.*

Corollary 13 *The same proof leads to:*

$$\forall \xi_1, \xi_2 \in K, \quad \lim_{\theta \to 0^+} \frac{\tilde{F}((1-\theta)\xi_1 + \theta \xi_2) - \tilde{F}(\xi_1)}{\theta} =< \tilde{F}'(\xi_1), \xi_2 - \xi_1 > .$$

Let us sum up in the following theorem the properties of the solution of (8).

Theorem 14 *There exists a function φ in K such that, if we note $u = u_\varphi$ and $\mu = \mu_\varphi$, one has:*

- $\tilde{F}(\varphi) = \mathcal{L}$,
- $< \tilde{F}'(\varphi), \varphi >= 0 \quad$ *i.e.*

$$\int_\Omega g(u)u\, dxdy + \int_\Omega (\partial_y \varphi)^2\, dxdy + b(\varphi) + \frac{\gamma+1}{2} u_{cr}^2 \mu(\Omega) = 0,$$

- $\forall \psi \in K, < \tilde{F}'(\varphi), \psi > \geq 0 \quad$ *i.e.,*

$$\int_\Omega g(u)u_\psi\, dxdy + \int_\Omega \partial_y \varphi\, \partial_y \psi\, dxdy + b(\psi) + \frac{\gamma+1}{2} u_{cr}^2 \mu_\psi(\Omega) \geq 0.$$

- *if one denotes by $u^* = u_{cr} - (u_{cr} - u)^+$, then*

$$u_* \leq u_{cr} \quad and \quad \mathcal{L} = \int_\Omega \left[G(u^*) - u^* g(u^*) - \frac{1}{2}(\partial_y \varphi)^2 \right] dxdy.$$

Proof. The existence of φ is given in the previous section and, thanks to Proposition 11, points two and three come from the classical results of minimisation in convex cones.
Since $< \tilde{F}'(\varphi), \varphi >= 0$, one has,

$$\begin{aligned} 0 &= \int_\Omega ug(u)\, dxdy + \int_\Omega (\partial_y \varphi)^2\, dxdy + b(\varphi) + \frac{\gamma+1}{2} u_{cr}^2 \mu(\Omega) \\ &= \int_\Omega ug(u)\, dxdy + \frac{1}{2} \int_\Omega (\partial_y \varphi)^2\, dxdy + \tilde{F}(\varphi) - \int_\Omega G(u)\, dxdy. \end{aligned}$$

Therefore, as $\tilde{F}(\varphi) = \mathcal{L}$, one may conclude point four since

$$G(u) - G(u^*) = \frac{\gamma+1}{2} u_{cr}^2 (u_{cr} - u)^- = ug(u) - u^* g(u^*).$$

Remark 15 *Let us note that, for any ψ in $\mathcal{D}(\Omega)$, since ψ and $-\psi$ belong to K, $< \tilde{F}'(\varphi), \psi >= 0$. Therefore,*

$$div(g(u), \partial_y \varphi) = 0 \ in\ \mathcal{D}'(\Omega) \quad and \quad (g(u), \partial_y \varphi).n = z' \mathbb{I}_{\Gamma_p} \ in\ W^{-\frac{2}{3}, \frac{3}{2}}(\partial\Omega)$$

(cf. [16] for the trace) where $\mathbb{I}_{\Gamma_p}$ is the indicatrice function associated to Γ_p.

4 Uniqueness of the relaxed optimal problem

Proposition 16 *Let us consider two solutions to (10) φ_i and note*

$$\partial_x \varphi_i = u_i\, dxdy + \mu_i \quad (i = 1,2)$$

i) If $u_i^* = u_{cr} - (u_{cr} - u_i)^+$ $(i = 1,2)$, we have

$$u_1^* = u_2^* \text{ a.e. in } \Omega \text{ and } \partial_y \varphi_1 = \partial_y \varphi_2 \text{ a.e. in } \Omega.$$

ii) Under the condition on the truncated domain (1), there exists a function f in $BV(-R,R)$ such that:

$$\varphi_1 = \varphi_2 + f; \quad \int_{]-R,R[} f\, dx = 0 \quad \text{and} \quad \int_\Omega (\frac{\gamma+1}{2} u_{cr}^2 - \frac{\widetilde{z}}{R})\, df = 0 \tag{14}$$

where $\widetilde{z}$ denotes the extension of z by zero outside $[-\frac{1}{2}, \frac{1}{2}]$.

Proof. Since φ_1 and φ_2 are two admissible solutions, Corollary 13 leads classically to:

$$< \tilde{F}'(\varphi_1), \varphi_2 - \varphi_1 > + < \tilde{F}'(\varphi_2), \varphi_1 - \varphi_2 > \geq 0.$$

Therefore, one gets

$$\int_\Omega (u_1 - u_2)[g(u_1) - g(u_2)]\, dxdy + \int_\Omega (\partial_y \varphi_1 - \partial_y \varphi_2)^2\, dxdy \leq 0$$

and one concludes i) thanks to the analytic formulation of $g = G'$ and the convexity of G.
Since φ_1 and φ_2 belong to K with $\partial_y \varphi_1 = \partial_y \varphi_2$, there exists a function f in $BV(-R,R)$ such that $\int_{]-R,R[} f\, dx = 0$ satisfying for a.e. (x,y) of Ω,

$$\varphi_1(x,y) = \varphi_2(x,y) + f(x).$$

Let us denote by $A = \{(x,y) \in \Omega;\ u_1 < u_{cr} \textit{ a.e. in } \Omega\}$ and $B = \Omega \backslash A$. Then, since $\partial_y \varphi_1 = \partial_y \varphi_2$ a.e. in Ω and $u_1 = u_2$ a.e. in A, one has: $|df|(A) = 0$ and

$$\tilde{F}(\varphi_1) = \int_A G(u_2)\, dxdy + \int_B G(u_1)\, dxdy + \frac{1}{2} \int_\Omega (\partial_y \varphi_2)^2\, dxdy$$

$$+b(\varphi_1)+\frac{\gamma+1}{2}u_{cr}^2\mu_1(\Omega).$$

Thus, thanks to (6) and denoting by v and ρ respectively the absolutely continuous part of df with respect to the Lebesgue's measure and its singular part, one gets

$$u_1 = u_2 + v, \quad \mu_1 = \mu_2 + \rho \quad \text{and}$$

$$\begin{aligned}\tilde{F}(\varphi_1) = &\int_A G(u_2)\,dxdy - \frac{\gamma+1}{2}u_{cr}^2\int_B (u_{cr}-u_2)\,dxdy \\ &+\frac{\gamma+1}{3}\int_B u_{cr}^3\,dxdy + \frac{1}{2}\int_\Omega (\partial_y\varphi_2)^2\,dxdy + b(\varphi_2) + \frac{\gamma+1}{2}u_{cr}^2\mu_2(\Omega) \\ &+b(f) + \frac{\gamma+1}{2}u_{cr}^2 df(B) \\ &= \tilde{F}(\varphi_2) + \int_\Omega (\frac{\gamma+1}{2}u_{cr}^2 - \frac{\tilde{\tilde{z}}}{R})\,df.\end{aligned}$$

Then, since $\tilde{F}(\varphi_1)=\tilde{F}(\varphi_2)$, ii) is proved.

Theorem 17 *Problem (10) has a unique solution.*

Proof. Let us consider again the above notations and denote by g the $BV(-R,R)$ function such that $\int_{]-R,R[} g\,dx = 0$ and $dg = [df]^-$.
Therefore, $dg = v^- dx + \rho^-$ and one has $dg(A)=0$. Moreover, $u_2 - v^- \geq 0$ and $\mu_2 - \rho^- \geq 0$.
Then,

$$\begin{aligned}\tilde{F}(\varphi_2 - g) = &\tilde{F}(\varphi_2) + \int_B \Big[G(u_2 - v^-) - G(u_2)\Big]\,dxdy \\ &- b(g) - \frac{\gamma+1}{2}u_{cr}^2\rho^-(\Omega) \\ &= \tilde{F}(\varphi_2) - \frac{\gamma+1}{2}u_{cr}^2\int_B v^-\,dxdy - \int_{\Gamma_p} z' g\,d\sigma - \frac{\gamma+1}{2}u_{cr}^2\rho^-(\Omega) \\ &= \tilde{F}(\varphi_2) + \int_{\Gamma_p} z\,dg - \frac{\gamma+1}{2}u_{cr}^2 R dg(-R,R) \\ &\leq \tilde{F}(\varphi_2) + \Big[||z||_\infty - \frac{\gamma+1}{2}u_{cr}^2 R\Big]\,dg(-R,R).\end{aligned}$$

As φ_2 is a solution to (10), hypothesis (1) leads to $dg=0$ and to $g=0$ since $\int_{]-R,R[} g\,dx = 0$.

So, f is a non decreasing function and (14) and (1) prove that $f = 0$. Thus, $\varphi_1 = \varphi_2$ and the solution is unique.

5 Towards the unbounded domain

Consider $R_2 > R_1 > R_0 = \frac{2\|z\|_\infty}{(\gamma+1)u_{cr}^2}$. For any $i = 1, 2$, let us note:

• $\Omega_{R_i} =]-R_i, R_i[\times]0, R_i[$, $\Omega_\infty = \mathbb{R}\times\mathbb{R}^+$,

$$N_{R_i} = \{\varphi \in BV(\Omega_{R_i}) : \partial_y\varphi \in L^2(\Omega_{R_i})\}$$

and $\widetilde{F}_{R_i}$ the corresponding functional on N_{R_i}.

• φ_{R_i} the solution of $\widetilde{F}_{R_i}(\varphi_i) = \min\limits_{\xi\in N_{R_i}} \widetilde{F}_{R_i}(\xi) = \mathcal{L}_{R_i}$ with $\partial_x\varphi_{R_i} = u_{R_i}\,dxdy + \mu_{R_i}$ and $u^*_{R_i} = u_{cr} - (u_{cr} - u_{R_i})^+$.

Theorem 18 *There exists $\mathcal{L}$ in $\mathbb{R}$, $U \in L^2(\Omega_\infty)\cap L^3(\Omega_\infty)$, $V \in L^2(\Omega_\infty)$, $\varphi \in BV_{loc}(\Omega_\infty)$ and ρ a non negative measure on Ω_∞ such that:*

$$\begin{aligned}
&\bullet\ \mathcal{L}_R \underset{R\to+\infty}{\to} \mathcal{L} &&= \textstyle\int_{\Omega_\infty}\left[G(U) - Ug(U) - \frac{1}{2}(V)^2\right] dxdy,\\
&\bullet\ U \le u_{cr},\quad \partial_x\varphi = U\,dxdy + \rho,\quad \partial_y\varphi = V,\\
&\bullet\ div(g(U), V) &&= 0 \quad in \quad \mathcal{D}'(\Omega_\infty),\\
&\bullet\ (g(U), V).n &&= \widetilde{z} \quad in \quad \mathcal{D}'(]-R, R[),\ \forall R > 0.
\end{aligned}$$

First, let us give a lemma.

Lemma 19 *For any real s and t, one has:*

$$\begin{aligned}
G(s) - G(t) &- (s-t)G'(t)\\
&= \tfrac{\gamma+1}{6}\left[(u_{cr}-s)^+ - (u_{cr}-t)^+\right]^2 \left[(u_{cr}-s)^+ + 2(u_{cr}-t)^+\right.\\
&\qquad\qquad\qquad\qquad \left. + 3(u_{cr}-s)^-\right].
\end{aligned}$$

In order to prove the theorem, note that

$$\varphi \in N_{R_2} \Rightarrow \varphi_{|\Omega_{R_1}} \in N_{R_1}$$

and that

$$\widetilde{F}_{R_1}(\varphi_{|\Omega_{R_1}}) \le \widetilde{F}_{R_2}(\varphi),$$

thus $\mathcal{L}_{R_1} \le \mathcal{L}_{R_2}$. Since $\widetilde{F}_{R_i}(0) = 0$, $(\mathcal{L}_R)_{R>R_0}$ is bounded and non decreasing. Therefore, there exists a real $\mathcal{L}$ such that $\mathcal{L}_R$ converges to $\mathcal{L}$ when R goes to infinity.
Since $\mathcal{L}_{R_1} = \int_{\Omega_{R_1}} \left[G(u^*_{R_1}) - u^*_{R_1} g(u^*_{R_1}) - \frac{1}{2}(\partial_y \varphi_{R_1})^2\right] dxdy$, one has:

$$\begin{aligned}
\widetilde{F}_{R_1}(\varphi_{R_2}) = & \mathcal{L}_{R_1} + \int\limits_{\Omega_{R_1}} \left[G(u_{R_2}) - G(u^*_{R_1}) + u^*_{R_1} g(u^*_{R_1})\right] dxdy + b(\varphi_{R_2}) \\
& + \frac{1}{2} \int\limits_{\Omega_{R_1}} \left[(\partial_y \varphi_{R_2})^2 + (\partial_y \varphi_{R_1})^2\right] dxdy + \frac{\gamma+1}{2} u^2_{cr} \mu_{R_2}(\Omega_{R_1}) \\
= & \mathcal{L}_{R_1} + \int\limits_{\Omega_{R_1}} \left[G(u_{R_2}) - G(u^*_{R_1}) - (u_{R_2} - u^*_{R_1}) g(u^*_{R_1})\right] dxdy \\
& + \int\limits_{\Omega_{R_1}} u_{R_2} g(u^*_{R_1})\, dxdy + \int\limits_{\Omega_{R_1}} \partial_y \varphi_{R_2} \partial_y \varphi_{R_1}\, dxdy + b(\varphi_{R_2}) \\
& + \frac{1}{2} \int\limits_{\Omega_{R_1}} (\partial_y \varphi_{R_2} - \partial_y \varphi_{R_1})^2\, dxdy + \frac{\gamma+1}{2} u^2_{cr} \mu_{R_2}(\Omega_{R_1}).
\end{aligned}$$

Since, $\varphi_{R_2|\Omega_{R_1}} \in N_{R_1}$,

$$\int\limits_{\Omega_{R_1}} g(u^*_{R_1}) u_{R_2}\, dxdy + \int\limits_{\Omega_{R_1}} \partial_y \varphi_{R_2} \partial_y \varphi_{R_1}\, dxdy + b(\varphi_{R_2}) + \frac{\gamma+1}{2} u^2_{cr} \mu_{R_2}(\Omega_{R_1}) = 0$$

and, as

$$\begin{aligned}
& \int\limits_{\Omega_{R_1}} \left[G(u_{R_2}) - G(u^*_{R_1}) - (u_{R_2} - u^*_{R_1}) g(u^*_{R_1})\right] dxdy \\
= & \int\limits_{\Omega_{R_1}} \left[G(u^*_{R_2}) - G(u^*_{R_1}) - (u^*_{R_2} - u^*_{R_1}) g(u^*_{R_1})\right] dxdy \\
& + \int\limits_{\Omega_{R_1}} \left[G(u_{R_2}) - G(u^*_{R_2}) - (u_{R_2} - u^*_{R_2}) g(u^*_{R_1})\right] dxdy
\end{aligned}$$

with

$$\begin{aligned}
& \int\limits_{\Omega_{R_1}} \left[G(u_{R_2}) - G(u^*_{R_2}) - (u_{R_2} - u^*_{R_2}) g(u^*_{R_1})\right] dxdy \\
= & \int\limits_{\Omega_{R_1}} [g(u^*_{R_2}) - g(u^*_{R_1})](u_{R_2} - u^*_{R_2})\, dxdy \ge 0
\end{aligned}$$

one has:

$$\widetilde{F}_{R_1}(\varphi_{R_2}) \geq \mathcal{L}_{R_1} + \int\limits_{\Omega_{R_1}} \left[G(u^*_{R_2}) - G(u^*_{R_1}) - (u^*_{R_2} - u^*_{R_1})g(u^*_{R_1})\right] dxdy$$
$$+\frac{1}{2}\int\limits_{\Omega_{R_1}} (\partial_y\varphi_{R_2} - \partial_y\varphi_{R_1})^2\, dxdy$$

and

$$\begin{aligned}\mathcal{L}_{R_2} = \widetilde{F}_{R_2}(\varphi_{R_2}) &\geq \mathcal{L}_{R_1} \\ &+ \int\limits_{\Omega_{R_1}} \left[G(u_{R_2}) - G(u^*_{R_1}) - (u_{R_2} - u^*_{R_1})g(u^*_{R_1})\right] dxdy \\ &+\frac{1}{2}\int\limits_{\Omega_{R_1}} (\partial_y\varphi_{R_2} - \partial_y\varphi_{R_1})^2\, dxdy + \frac{\gamma+1}{2}u_{cr}^2\mu_{R_2}(\Omega_{R_2}\backslash\Omega_{R_1}) \\ &+ \int\limits_{\Omega_{R_2}\backslash\Omega_{R_1}} \left[G(u_{R_2}) + \frac{1}{2}(\partial_y\varphi_{R_2})^2\right] dxdy.\end{aligned}$$

Therefore, thanks to lemma 19,

$$\begin{aligned}&G(u^*_{R_2}) - G(u^*_{R_1}) - (u^*_{R_2} - u^*_{R_1})g(u^*_{R_1}) \\ \geq\, &\frac{\gamma+1}{6}|u^*_{R_2} - u^*_{R_1}|^2\left[|u^*_{R_2} - u^*_{R_1}| + 3(u_{cr} - \max(u^*_{R_1}, u^*_{R_2}))\right],\end{aligned}$$

one gets:

$$\begin{aligned}\mathcal{L}_{R_2} - \mathcal{L}_{R_1} \geq &\frac{\gamma+1}{6}\int\limits_{\Omega_{R_1}} |u^*_{R_2} - u^*_{R_1}|^3\, dxdy + \frac{1}{2}\int\limits_{\Omega_{R_1}} (\partial_y\varphi_{R_2} - \partial_y\varphi_{R_1})^2\, dxdy \\ &+ \int\limits_{\Omega_{R_2}\backslash\Omega_{R_1}} \left[G(u^*_{R_2}) + \frac{1}{2}(\partial_y\varphi_{R_2})^2\right] dxdy.\end{aligned}$$

Note that for every $t \leq u_{cr}$, one has $G(t) = \dfrac{\gamma+1}{6}t^2(3u_{cr} - t)$, so

$$G(t) \geq \frac{\gamma+1}{6}u_{cr}t^2 + \frac{\gamma+1}{6}|t|^3$$

and

$$\mathcal{L}_{R_2} - \mathcal{L}_{R_1} \geq \frac{\gamma+1}{6}\int\limits_{\Omega_{R_1}} |u^*_{R_2} - u^*_{R_1}|^3\, dxdy + \frac{1}{2}\int\limits_{\Omega_{R_1}} (\partial_y\varphi_{R_2} - \partial_y\varphi_{R_1})^2\, dxdy$$

$$+\int\limits_{\Omega_{R_2}\setminus\Omega_{R_1}} \left[\frac{\gamma+1}{6}|u^*_{R_2}|^3 + \frac{1}{2}(\partial_y\varphi_{R_2})^2\right] dxdy.$$

On the other hand, since $u^*_{R_i} \leq u_{cr}$, lemma 19 leads to

$$\begin{aligned}
&G(u^*_{R_2}) - G(u^*_{R_1}) - (u^*_{R_2} - u^*_{R_1})g(u^*_{R_1}) \\
\geq &\frac{\gamma+1}{2}|u^*_{R_2} - u^*_{R_1}|^2 \left[u_{cr} - \max(u^*_{R_1}, u^*_{R_2})\right]
\end{aligned}$$

and one has:

$$\begin{aligned}
&\frac{\gamma+1}{2}\left(\int\limits_{\Omega_{R_1}} |u^*_{R_2} - u^*_{R_1}|^3\, dxdy\right)^{\frac{2}{3}} \left(\int\limits_{\Omega_{R_1}} |u^*_{R_1}|^3\, dxdy\right)^{\frac{1}{3}} + \mathcal{L}_{R_2} - \mathcal{L}_{R_1} \\
\geq &\frac{\gamma+1}{2}\int\limits_{\Omega_{R_1}} |u^*_{R_2} - u^*_{R_1}|^2\, dxdy \\
&+\int\limits_{\Omega_{R_2}\setminus\Omega_{R_1}} \left[\frac{\gamma+1}{6}u_{cr}(u^*_{R_2})^2 + \frac{1}{2}(\partial_y\varphi_{R_2})^2\right] dxdy.
\end{aligned}$$

Let us note $U_R = u^*_R\mathbb{I}_{\Omega_R}$, $V_R = \partial_y\varphi^*_R\mathbb{I}_{\Omega_R}$. As $\mathcal{L}_R$ converges, it satisfies Cauchy's property and the above inequalities imply that $\{(U_R, V_R)\}_{R\geq R_0}$ satisfies Cauchy's property in $[L^3(\Omega_\infty)\cap L^2(\Omega_\infty)]\times L^2(\Omega_\infty)$. Therefore, there exists (U, V), limit in $[L^3(\Omega_\infty)\cap L^2(\Omega_\infty)]\times L^2(\Omega_\infty)$ of sequence $\{(U_R, V_R)\}_{R\geq R_0}$ such that $U \leq u_{cr}$,

$$\mathcal{L} = \int\limits_{\Omega_\infty} \left[G(U) - Ug(U) - \frac{1}{2}(V)^2\right] dxdy$$

and $div(g(U), V) = 0$ in $D'(\Omega_\infty)$.

Let us consider the increasing sequence $(\Omega_m)_{m\in\mathbb{N}_0}$ of sets where $\mathbb{N}_0 = \{m \in \mathbb{N},\ m > R_0\}$ and note that for any fixed m in $\mathbb{N}_0$, $(\varphi_n)_{n\geq m}$ is a bounded sequence in $BV(\Omega_m)$. Therefore, by a diagonal argument of subsequence extractions, there exists φ in $BV_{loc}(\Omega_\infty)$ such that $\partial_x\varphi = U + \rho$ (ρ is a non negative measure) and $\partial_y\varphi = V$.

6 Conclusion and open problems

• We have proved the existence and uniqueness of the solution of Problem (10) in a bounded domain. It is not a solution of the initial partial differential

equation since it is a solution to $div(g(u), \partial_y\varphi) = 0$ in the distribution sense in Ω and not to $div(g(\partial_x\varphi), \partial_y\varphi) = 0$. The degenerated behavior of the problem is illustrated by the singular part μ.
We have $\partial_x\varphi = u\, dxdy + \mu$; when $u \leq u_{cr}$ (*i.e.* the problem remains elliptic), then we have $\mu = 0$ and we get the solution of the initial problem and the PDE. But, when the property $u \leq u_{cr}$ a.e. in Ω is not satisfied, the degeneraticity of the problem appears in the parabolic zones (i.e. where $u \geq u_{cr}$). Let us give informations about the singular part. Following L.C. Evans [14], denotes by λ and μ respectively the approximative limit-sup and the approximative limit-inf of φ. Then, since φ is a bounded variation function, if $S_\varphi = \{(x, y) \in \Omega,\ \lambda < \mu\}$ denotes the set of approximative discontinuity, one has $\partial_x\varphi_{\llcorner S_\varphi} = (\varphi^+ - \varphi^-)\, n_x\, d\mathcal{H}^1_{\llcorner S_\varphi}$ and φ is $\mathcal{H}^1$-a.e. approximatively continuous in $\Omega\backslash S_\varphi$.
Moreover, since φ belongs to N, for a.e. x in $]-R, R[$, the function $y \mapsto \varphi(x, y)$ belongs to $H^1(0, R)$. Then, there exists a countably family of segments such that $S_\varphi = \bigcup_{n\in\mathbb{N}} \{x_n\} \times [a_{x_n}, b_{x_n}]$ $\mathcal{H}^1$-a.e.

• We have proved the convergence of the sequences $\{(U_R, V_R)\}_{R\geq R_0}$ and $(\mathcal{L}_R)$. We have now to study the physical problem in the unbounded domain $\mathbb{R}\times\mathbb{R}^+$, *i.e.* consider the optimal problem in this unbounded domain and compare its value to $\mathcal{L}$. The major difficulty is then to obtain Poincaré's inequalities.

References

[1] Alibert J.J. & Bouchitte G., Non-uniform integrability and generalized Young measures. *Journal of Convex Analysis*, Vol 4, **1** (1997) 129-147.

[2] Amara M., Obeid A. & Vallet G., *Existence results for degenerated nonlinear elliptic partial differential equation.* Prépublication de l'Université de Pau et des Pays de l'Adour, 2001/30 (2001).

[3] Amara M., Obeid A. & Vallet G., *A relaxed formulation for a degenerated elliptic equation.* Prépublication de l'Université de Pau et des Pays de l'Adour, 2001/31 (2001).

[4] Ambrosio L., Existence theory for a new class of variational problems. *Arch. Ration. Mech. Anal.*, 111, **4**, (1990) 291-322.

[5] Ambrosio L., On the lower semicontinuity of quasiconvex integrals in $SBV(\Omega, \mathbb{R}^k)$. *Nonlinear Anal., Theory Methods Appl.*, 23, **3** (1994) 405-425.

[6] Ambrosio L., Braides A. & Garroni A., Special functions with bounded variation and with weakly differentiable traces on the jump set. *NoDEA*, 5, **2** (1998) 219-243.

[7] Baiocchi C., Buttazzo G., Gastaldi F. & Tomarelli F., General existence theorems for unilateral problems in continum mechanics. *Arch. Rational Mech. Anal.*, **100** (1988) 149-189.

[8] Bouchitte G. & Dal Maso G., Integral representation and relaxation of convex local functionals on $BV(\Omega)$. *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, IV. Ser. 20, **4** (1993) 483-533.

[9] Bouchitte G., Fonseca I. & Mascarenhas L.,
1) A global method for relaxation. *Arch. Ration. Mech. Anal.*, 145, **1** (1998) 51-98.
2) Relaxation of variational problems under trace constraints. *Nonlinear Analysis*, **49** (2002) 221-246.

[10] Braides A. & Chiadò Piat V., A derivation formula for convex integral functionals defined on $BV(\Omega)$. *Journal of convex Analysis,* 2, **1/2** (1995) 69-85.

[11] Carbone L. & De Arcangelis R., On the relaxation of some classes of unbounded integral functional, *Matematiche Catania*, 51, **2** (1996) 221-252.

[12] Carbone L. & De Arcangelis R., *Unbounded functionals in the calculus of variations. Representations, relaxation, and homogenization.* Chapman & Hall, Research Notes in Mathematics (2002).

[13] Ekeland I. & Temam R., *Convex analysis and variational problems.* North-Holland, Amsterdam (1976).

[14] Evans L.C. & Gariepy R.F., *Mesure theory and fine properties of functions.* Studies in advanced mathematics, CRC Press, London (1992).

[15] Fonseca I. & Mueller S., Relaxation of quasiconvex functionals in $BV(\Omega, \mathbb{R}^p)$ for integrands $f(x, u, \nabla u)$. *Arch. Ration. Mech. Anal.,* 123, **1** (1993) 1-49.

[16] Galdi G. P., An introduction to the mathematical theory of the Navier-Stokes equations. Vol.1 Linearized steady problems, Springer Verlag, Berlin (1994).

[17] Giusti E., *Minimal surfaces and functions of bounded variation.* Birkhäuser, Basel (1983).

[18] Kinderlehrer D. & Pedregal P., Weak convergence of integrands and the Young measure representation. *SIAM J. Math. Anal.,* 23, **1** (1992) 1-19.

[19] Pogu M. & Tournemine G., *Modélisation et résolution d'équations de la mécanique des milieux continus.* Ellipses (1992).

[20] Roubicek T., Relaxation of vectorial variational problems. *Math. Bohem*, 120, **4** (1995) 411-430.

[21] Temam R., *Navier-Stockes equations.* North Holland, Amsterdam (1979).

[22] Ziemer W.P., *Weakly differentiable functions : Sobolev spaces and functions of bounded variation.* Springer-Verlag, Berlin (1989).

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