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Competitive Growth in a Life-cycle Model : Existence and Dynamics

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Competitive Growth in a Life-cycle Model : Existence and Dynamics[1]

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Résumé

In this paper, the dynamic behavior of the capital growth rate is analyzed using an overlapping-generations model with continuous trading and finitely lived agents. Assuming a technology satisfying constant returns to capital, the equilibrium growth rate is piecewise-defined by functional differential equations with both delayed and advanced terms. The existence of a solution expressed as a series of exponentials crucially depends on the initial wealth distribution among cohorts. Upon existence, the dynamics of the capital growth rate has a saddle-point trajectory that converges to a unique steady-state. Along the transition path, the growth rate exhibits exponentially decreasing oscillations.

JEL Classification : D50, D90.

Keywords : Overlapping-generations Models, Endogenous Growth, Functionnal Differential Equations of Mixed Type.

As they deal with population or social security issues, applied studies have come to use settings that encompass an arbitrarily large number of overlapping generations of individuals[1] to ease the confrontation with real world annual data. Such models are unfortunately at odds with the benchmark structure of the theoretical literature based upon two periods overlapping generations of individuals. Recent contributions[2] have thus been aimed at developing a fruitful analysis of large-square models[3] and exhibited specific conditions on the existence and dynamic properties of equilibrium paths. However, a limit to their use on a wide spread basis springs from the difficulties that emerge in the course of their analytical resolution. Our paper introduces a simple method that allows for solving models with many overlapping generations and that could be applied to numerous related economic problems ; it also proposes a complete resolution in a specific case which provides new insights on the existence and convergence properties of equilibrium paths.

In models with many overlapping generations, the dynamics of endogenous variables depends on a finite number of their past and future realizations. To comprehend that, suppose, for instance, that the population is composed of individuals that live for T periods. At each date t, the information set that determines the equilibrium of the economy is modified in two ways. First, it is augmented by the extra stream brought by the newborn individuals, namely the expectations for time $t+T-1$. Second, it is conversely reduced by the lost of memory on the information held by those who just died, i.e. the prices for time $t-T-2$.

But it is the very dependency of this current variables on their past and future realizations that creates the analytical difficulty of these models. In discrete time frameworks[4], the analysis of the dynamics requires the study of polynomials, whose order increases with

[1] See Auerbach and Kotlikoff [2] for a deterministic framework and Rios-Rull [27] for a stochastic one.

[2] See notably Kehoe, Levine, Mas-Colell and Woodford [20] and Azariadis, Bullard and Ohanian [3].

[3] Balasko, Cass and Shell [4] showed the equivalence with a two-period lifespan model with many goods.

[4] Samuelson [30] introduced a model with three generations and Gale [16] provided a generalization to many generations.

that this difficulty may however be circumvented using the continuous time framework developed by Cass and Yaari [13]; in a nutshell, it eases the mathematical resolution without affecting the qualitative analysis, the endogenous variables of a discrete time model converging uniformly to their continuous time counterpart when the frequency of trade during a life-span goes to infinity[5]. However, in continuous time frameworks, the dynamics is characterized by a functional differential equation of mixed type (MFDE), i.e. the dynamics is affected by a continuum of discrete delays and advances; as suggested above, advances are generated by the expectations of newborn individuals while delays are associated with the loss of information caused by the recent deaths of the oldest individuals.

We propose an integrated treatment of such a structure in a simple competitive equilibrium case considering an overlapping generations model with production in which preferences are logarithmic and the technology exhibits constant returns to capital. Capital dynamics are generically described by an MFDE that happens to be linear under constant returns to capital. This latter assumption, which constitutes our main departure with Cass and Yaari [13], allows for a global dynamics analysis. We prove that, upon existence of a competitive path, the growth rate has a transitional dynamics and converges to a steady state through exponentially decreasing oscillations.

The existence of an equilibrium is not always ensured in continuous-time overlapping generations models even under rather standard and simple assumptions[6]. Following Rustichini [28], we study the existence of a strictly positive and bounded solution to the MFDE. We show how to use the piecewise definition of the capital dynamics to define the set of initial distributions of wealth that ensure the existence of the equilibrium. We then propose a necessary and sufficient condition for the existence of a competitive equilibrium and provides examples of initial distributions for which there is no solution to the equation. Moreover,

[5] See Demichelis and Polemarchakis [14] in the exchange economy case.

[6] See Burke [11] in a model with a finite number of individuals at each point of time.

studying the roots of the MFDE characteristic equation, we show that the capital either converges to a balanced growth path or diverges : its growth rate assumes therefore a saddle-point structure with a convergent path. Moreover, the converging solution exhibits oscillations that decrease in magnitude and eventually disappear as time goes on. There are no permanent cycles in such economies[8].

These results articulate with the available literature as follows. First note that, while differential equations with delays have been studied in economics since the early beginnings of macro-dynamics[9], very few economic studies have used MFDE[10]. Notably, in an exchange economy by Demichelis and Polemarchakis [14] has recently been find a related class of fluctuations. They appear as resulting from the existence of delays in continuous time environments, as pointed out by Boucekkine, de la Croix and Licandro [8] who understand the overlapping generations framework as a dynamic model with vintage human capital. This also corroborate the results obtained by Azariadis, Bullard and Ohanian [3] : in the framework developed by Kehoe, Levine, Mas-Colell and Woodford [20], they show that the price sequence displays a non monotonic convergence to the steady-state. Azariadis *et al.* conjecture that such an overlapping-generations model with production may hence be used as an argument to explain the short-run fluctuations in the trend-reverting motion of the output. In parallel to this, we show that introducing a demographic structure in a simple one-sector model with constant returns to capital is sufficient to eliminate its most unpleasant conclusion, namely, the absence of transitional dynamics. Our results on equilibrium growth complement in this latter regards those of Boucekkine, Licandro, Puch and del Río [9] on optimal growth. They extend the Rebelo [25]'s AK model by introducing vintage capital and show that the optimal growth rate has an oscillating behavior

[7] For determinacy issues in large-square economies, see Kehoe, Levine, Mas-Colell and Zame [21], Santos and Bona [29], Burke [10] and Laitner [22].

[8] See Aiyagari [1], Préchac [24], Reichlin [26] and Ghiglino and Tvede [17].

[9] See Frisch [15] and Kalecki [19].

[10] For the mathematical literature, see Bellman and Cooke [5], and Hale and Verduyn-Lunel [18] for an introduction and Rustichini [28] and Mallet-Paret and Verduyn-Lunel [23] for further developments in the mixed-type case.

The paper is organized as follows. In section 2, we present the basic framework of the model and determine the functional equations that characterize the capital dynamics at the equilibrium. In section 3, we give an existence condition based on the initial wealth distribution. In section 4, we analyze the capital dynamics. We conclude in section 5.

2 The model

This section develops an overlapping-generations model with continuous trading and finitely-lived individuals. Assuming a technology with constant returns to capital, it characterizes the capital dynamics of such an economy.

Time is continuous and has a finite starting point. Let t denote the time index with $t \geq 0$. Individuals live for an interval of time of length 1. It is assumed that they only derive utility from consumption and that they have logarithmic preferences and no time discount. Let $c(s,t) \geq 0$ denotes the real consumption of an individual who born at time s as of time t. Hence, the intertemporal utility of an individual who born at time $s \in (-1, t]$, denoted $u(s,t)$, is :

$$u(s,t) = \int_t^{s+1} \ln c(s,z)\, dz \qquad (1)$$

During a lifetime, the labor supply is fixed and equal to 1 and $w(t)$, an age-independent labor income is received. Individuals have access to competitive capital markets that yield the risk-free interest rate $r(t)$. Let $a(s,t)$ denotes the real wealth of an individual who born at time s as of time t. The instantaneous budget constraint is therefore for all $t \geq 0$:

$$\frac{\partial a(s,t)}{\partial t} = r(t)\, a(s,t) + w(t) - c(s,t) \qquad (2)$$

Individuals are born with no financial assets except those who are alive when the economy begins which are endowed with a given financial wealth. Therefore, initial conditions writes :

$$\begin{array}{ll} a(s,0) \text{ given} & \text{if } s \in [-1,0] \\ a(s,s) = 0 & \text{if } s > 0 \end{array} \qquad (3)$$

[11]Note that they study a dynamic system that exhibits delayed and advanced terms but use a different resolution method that involves optimal control theory.

$$a(s,s+1) \geq 0 \tag{4}$$

Moreover, assume that :

Assumption 1 $a(s,t)$ and $c(s,t)$ are $\mathcal{C}^1([-1,\infty) \times [0,\infty))$.

Assumption 2 $r(z)$ and $w(z)$ are continuous for all $z \in [t, t+1]$.

Given assumptions 1 and 2, the individual program is to maximize (1) subject to (2), (3) and (4). As the initial conditions differ, it is necessary to distinguish individuals who were alive at time $t = 0$ from those who born at time $t > 0$. The optimal asset accumulation is then expressed as a function of the past and anticipated prices and the initial conditions on asset endowment.

Lemma 1. *The optimal asset accumulation is such that :*

$$a(s,t) = f(a(s,0), w(z), r(z); z \in [0, s+1]) \tag{5}$$

if $(s,t) \in [-1,0] \times [0, s+1]$ *and :*

$$a(s,t) = g(w(z), r(z); z \in [s, s+1]) \tag{6}$$

if $(s,t) \in [0,\infty) \times [s, s+1]$.

Proof. The optimal consumption satisfies for all $t \geq 0$:

$$\frac{\partial c(s,t)}{\partial t} = r(t)\, c(s,t) \tag{7}$$

and :

$$\int_t^{s+1} c(s,z)\, e^{-\int_t^z r(u)du} dz = a(s,t) + \int_t^{s+1} w(z)\, e^{-\int_t^z r(u)du} dz \tag{8}$$

Observe that (7) is the standard Euler condition with no time discount and logarithmic preferences. Condition (8) is the intertemporal budget constraint obtained integrating

Replacing (7) in (8) allows to rewrite the optimal consumption such that :

$$c(s,t) = \frac{a(s,t) + \int_t^{s+1} w(z)\, e^{-\int_t^z r(u)du} dz}{(s+1-t)} \tag{9}$$

For individuals born at time $s \in [-1,0]$ and still alive at time $t \in [0,1]$, the optimal consumption path is obtained using (9) to define $c(s,0)$, then using (7) yields :

$$c(s,t) = \frac{a(s,0)\, e^{\int_0^t r(u)du} + \int_0^{s+1} w(z)\, e^{-\int_t^z r(u)du} dz}{(s+1)} \tag{10}$$

which replaced in (9) gives the optimal asset accumulation along the life cycle such that :

$$\begin{aligned} a(s,t) &= \left(a(s,0)\, e^{\int_0^t r(u)du} + \int_0^{s+1} w(z)\, e^{-\int_t^z r(u)du} dz \right) \frac{(s+1-t)}{(s+1)} \\ & - \int_t^{s+1} w(z)\, e^{-\int_t^z r(u)du} dz \end{aligned} \tag{11}$$

which defines function f. Conversely, for individuals born at time $s > 0$ consumption as of time $t \geq s$ is obtained replacing (3) in (9), then with (7), one has :

$$c(s,t) = \int_s^{s+1} w(z)\, e^{-\int_t^z r(u)du} dz \tag{12}$$

which replaced in (9) yields :

$$\begin{aligned} a(s,t) &= \left(\int_s^{s+1} w(z)\, e^{-\int_t^z r(u)du} dz \right) (s+1-t) \\ & - \int_t^{s+1} w(z)\, e^{-\int_t^z r(u)du} dz \end{aligned} \tag{13}$$

which defines function g.

There is a unique material good, whose price is normalized to 1. It can be used for consumption or for adding to the capital stock. This good is produced by many competitive firms whose aggregate activity is described by a production function with labor $l(t)$ and capital $k(t)$ as inputs. Assume that the production function satisfies :

$$f(k(t), l(t)) = Ak(t)^{\alpha} (e(t)\, l(t))^{1-\alpha} \tag{14}$$

There is no capital depreciation. For all $t \geq 0$, factor prices equal marginal products :

$$r(t) = \alpha A (l(t))^{1-\alpha} \quad (15)$$

$$w(t) = (1-\alpha) A k(t) (l(t))^{-\alpha} \quad (16)$$

The demographic structure is in overlapping generations. Each individual belongs to a cohort whose size is normalized to 1. There is no population growth and, at each point of time, a new cohort enters the economy since the oldest one leaves it. Hence, the aggregate counterpart, $x(t)$, of any individual variable, $x(s,t)$, is obtained by integrating over the birth date, such that : $x(t) = \int_{t-1}^{t} x(s,t)\, ds$.

Definition An equilibrium with perfect expectation is a $k(t)$, $t \geq 0$, that belongs to $\mathbb{R}^+$, is continuous and has bounded variations on $[0,\infty)$ such that (i) individuals maximize their utility subject to the budget constraints, (ii) firms maximize their profits, and (iii) markets clear.

Lemma 2. *The equilibrium $k(t)$ is piecewise-defined by linear functional equations such that :*

$$k(t) = \begin{cases} F(a(s,0); s \in [-1,0],\ k(z); z \in [0,t+1]) & \text{if } t \in (0,1] \\ G(k(z); z \in [t-1,t+1]) & \text{if } t \geq 1 \end{cases} \quad (17)$$

In addition, $a(s,0)$, $s \in [-1,0]$ and $k(0)$ are given.

Proof. Replacing labor market equilibrium condition, $l(t) = 1$, in (15) and (16) yields factor prices : $r(t) = \alpha A$ and $w(t) = (1-\alpha) A k(t)$. Replace them in (11) and (13) and use capital market equilibrium condition $k(t) = \int_{t-1}^{t} a(s,t)\, ds$ to obtain the functional equation in k.

For $t \in (0,1]$, the population is composed of individuals born before and after $t = 0$.

$$\begin{aligned} k(t) &= \int_{t-1}^{0} a(s,0)\, e^{\alpha A t} \frac{(s+1-t)}{(s+1)} ds \\ &+ (1-\alpha) A \left[\int_{t-1}^{0} \left(\int_{0}^{s+1} k(z)\, e^{-\alpha A (z-t)} dz \right) \frac{(s+1-t)}{(s+1)} ds \right. \\ &+ \int_{0}^{t} \left(\left(\int_{s}^{s+1} k(z)\, e^{-\alpha A (z-t)} dz \right) (s+1-t) \right) ds \\ &- \left. \int_{t-1}^{t} \left(\int_{t}^{s+1} k(z)\, e^{-\alpha A (z-t)} dz \right) ds \right] \end{aligned} \tag{18}$$

which defines function F. For $t \geq 1$, there is no individual born before $t = 0$. Hence, use (13) to obtain :

$$\begin{aligned} k(t) &= (1-\alpha) A \left[\int_{t-1}^{t} \left(\int_{s}^{t} k(z)\, e^{-\alpha A (z-t)} dz \right) ds \right. \\ &- \left. \int_{t-1}^{t} (t-s) \left(\int_{s}^{s+1} k(z)\, e^{-\alpha A (z-t)} dz \right) ds \right] \end{aligned} \tag{19}$$

which defines function G.

Remark The dynamics is hence piecewise-defined : as the economy is supposed to have a finite starting point, a first equation characterizes the capital dynamics as long as it exists individuals who were alive at the initial date ; this equation includes the initial distribution of wealth among generations which constitutes the initial condition of our problem ; a second equation then characterizes the dynamics for an infinite future.

Remark The linearity of equations (18) and (19) with respect to k crucially depends on the assumption of constant returns to capital. They would remain linear with a CRRA instantaneous utility function and a strictly positive discount rate since the propensity to consume of same-aged individuals would be time independent.

Differentiating equations (18) and (19) with respect to time yields linear functional differential equations of mixed type (MFDE) : delayed and advanced distributed terms therefore influence the dynamic behavior of $k(t)$. The presence of delays in our model is due to the bounded life-span assumption ; they indeed disappear in overlapping generations models of infinitely-lived individuals as in Weil [31] or of individuals that face an tion is also the cornerstone of the vintage capital literature : Benhabib and Rustichini [6] show that non-constant or one-hoss shay capital depreciation imply delays in the capital dynamics. The capital dynamics also depends on advanced terms that are generated by the individual expectations made by the newcomers in the economy ; alternatively, a model with myopic individuals would produce an equilibrium dynamics characterized by a differential equation with delays only.

3 Existence of a Competitive Equilibrium

Rustichini [28] showed that the Cauchy problem for an MFDE is not well set up because of the advanced terms, and therefore that the differential equation may fail to have a solution. This section shows that the solution of (17) writes as a unique series and gives the shape of the initial distribution of wealth that ensures its existence.

First, the following lemma shows that theorem 6.10 in Bellman and Cooke [5] applies and that solutions are expressed as series of exponential.

Lemma 3. *Suppose* $k(t)$, $t \in [0,2]$ *is given and define* $\tau > 2$. *For all* $t \in [0,\tau]$, *the solution of equation (19) writes :*

$$k(t) = \sum_n \delta_n e^{g_n t} \tag{20}$$

where $g_n \in \mathbb{C}$ *are the roots of equation (19)'s characteristic equation* Q *defined by :*

$$\begin{aligned} Q(g) &= 1-(1-\alpha)A\left[\int_{-1}^{0}\left(\int_{s}^{0} e^{(g-\alpha A)z}dz\right)ds\right. \\ &\quad \left.+\int_{-1}^{0} s\left(\int_{s}^{s+1} e^{(g-\alpha A)z}dz\right)ds\right] \end{aligned} \tag{21}$$

and where $\delta_n \in \mathbb{C}$ *are the residues of a function defined in the proof. Moreover, the* δ_n *are chosen such that* $K(t) > 0$.

Proof. See the appendix.

need to know $k(t)$ for $t \in [0,2]$. However, the economic assumptions only give $a(s,0)$, $s \in [-1,0]$, from which it can deduced $k(0)$. Equation (18) is at this stage crucial to solve the model. It indeed gives a univoque relation between the constants (δ_n) and the initial conditions $a(s,0)$, $s \in [-1,0]$, and eventually defines $k(t)$ for $t \in (0,2]$. The following lemma gives this relation.

Lemma 4. *A solution of (17) shall satisfies for $s \in [-1,0]$:*

$$a(s,0) = \sum_n \delta_n e^{-(\alpha A - g_n)(s+1)} \left[(s+1)\left(g_n^2 - (1+\alpha) A g_n + \alpha A^2\right) + (1-\alpha) A \left((s+1) \int_0^1 e^{-(\alpha A - g_n) z} dz - \int_{-1-s}^{0} e^{-(\alpha A - g_n) z} dz \right) \right] \tag{22}$$

and for $t \geq 0$: $\sum_n \delta_n e^{g_n t} > 0$.

Proof. See the appendix.

Lemma 4 constitutes a necessary and sufficient condition for system (17) to have a solution. Remark with lemma 3 that if it exists a solution, it is unique.

Corollary 5. *It exists $a(s,0)$, $s \in [-1,0]$ such that there is no solution to (17).*

Proof. See the appendix.

Remark Any distribution such that $a(-1,0) > 0$ does not solve (22).

Now turn to the analysis of the dynamic behavior by studying the roots of the characteristic equation (21).

4 The Dynamic Behavior of the Growth Rate

This section proposes a complete study of the capital dynamics through the properties of the characteristic equation. It shows that capital converges to a balanced growth path with exponentially decreasing oscillations if the initial distribution of wealth differs from the one that prevail on the balanced growth path.

Roots of equation (21) are such that :

1) a unique real root denoted $\bar{g}$, such that $\bar{g} < \alpha A$;

2) no complex roots with real part equal to $\bar{g}$, except $\bar{g}$ itself when $\bar{g} > 0$;

3) an infinity of complex roots with real parts $> \bar{g}$;

4) an infinity of complex roots with real parts $< \bar{g}$.

Proof. See the appendix.

The interpretation of these results lies in three corollaries.

Corollary 7. *A balanced growth path exists along which the capital growth rate is lower than the interest rate.*

Proof. On the balanced growth path, the capital growth rate is constant. It is therefore given by the pure real root of the characteristic equation, namely $\bar{g}$.

It is necessary that this growth rate be lower than the interest rate (i.e. αA) to have a positive aggregate capital. To see that point, suppose on the contrary that $\bar{g} > \alpha A$. Then, observe with equation (12) computed at steady-state that $c\,(s, s)$, the initial consumption of an individual who born at time s, is greater than its current wage $w\,(s)$, and with equation (7) that its consumption's growth rate is lower than its wage's growth rate. Since $a\,(s, s) = a\,(s, s+1) = 0$, conclude that the individual is indebted all along his life and therefore that the aggregate capital is negative.

Remark The steady-state growth rate, $\bar{g}$, is not necessarily positive, but it can be easily shown that a $\bar{A} > 1$ exists such that $\bar{g} > 0$ if and only if $A > \bar{A}$. Assume therefore that A is sufficiently large.

Second point of lemma 6 point out that there is no cycle.

Corollary 8. *The capital dynamics either converge to a balanced growth path or diverge.*

$$\begin{aligned}\Gamma_S &= \{g_n,\ \mathrm{Re}(g_n) < \bar{g}\} \\ \Gamma_U &= \{g_n,\ \mathrm{Re}(g_n) > \bar{g}\} \\ \Gamma_C &= \{\bar{g}\}\end{aligned}$$

Now define M_s (resp. M_U, M_C) the subset spanned by the roots that belong to Γ_S (resp. Γ_U, Γ_C). There exists three families of operators :

$$\begin{aligned}T_S(t) &: M_s \longrightarrow M_s \text{ for } t > 0 \\ T_U(t) &: M_U \longrightarrow M_U \text{ for } t < 0 \\ T_C(t) &: M_C \longrightarrow M_C \text{ for } t \in (-\infty, \infty)\end{aligned}$$

and there exists $x > 0$ such that for all $\phi \in M_S$, $\|T_S(t)\phi\| \leq x \|\phi\|$, for all $t > 0$. Therefore elements of $M_s \cup M_c$ define the solutions $k_n(t)$ that converge to the balanced growth path.

Let (p_n, q_n) be such that $g_n = p_n + iq_n$. The converging solutions of (17) writes :

$$k(t) = \delta_1 e^{\bar{g}t} + \sum_{g_n \in \Gamma_S} \delta_n e^{p_n t} \cos(q_n t) \tag{23}$$

Since $p_n < \bar{g}$, the capital dynamics exhibits at the beginning oscillations that decrease in magnitude and finally disappear. Conversely, any solution that includes a $p_n > \bar{g}$ diverges. More importantly, Demichelis and Polemarchakis [14] pointed out that since terms in $\sum_{g_n \in \Gamma_U} \delta_n e^{p_n t} \cos(q_n t)$ exhibit oscillations that increase with time, solutions that include those terms are necessarily incompatible with the positivity constraint on aggregate capital $k(t)$. Hence, as suggested in lemma 3, for all $g_n \in \Gamma_u$, the corresponding δ_n should be set equal to zero.

Burke [10] and Demichelis and Polemarchakis [14] have stressed the importance of the date at which the economy begins. Accordingly, if time extends from an infinite past, the non-negativity constraint on $k(t)$ imposes to also eliminate the exponentially decreasing oscillating solutions. The only valid solution has therefore the following form : $k(t) = \delta_1 e^{\bar{g}t}$.

Remark If the initial distribution of wealth is the one that prevails at the balanced growth path, that is :

$$\begin{aligned}a(s,0) = {} & \delta_1 e^{-(\alpha A - \bar{g})(s+1)} \left[(s+1)\left(\bar{g}^2 - (1+\alpha) A\bar{g} + \alpha A^2\right) \right. \\ & \left. + (1-\alpha) A \left((s+1) \int_0^1 e^{-(\alpha A - \bar{g})z} dz - \int_{-1-s}^{0} e^{-(\alpha A - \bar{g})z} dz \right) \right]\end{aligned} \tag{24}$$

Let $g(t)$ be the growth rate of capital at time t such that :

$$k(t) = k(0)\, e^{\int_0^t g(u)du} \tag{25}$$

Corollary 9. *The growth rate has a saddle-point trajectory. On the stable manifold, it converges with exponentially decreasing oscillations to the steady-state.*

<u>Proof.</u> Observe with (23) that the growth rate of converging solutions satisfy :

$$g(t) = \frac{\delta_1 \bar{g} + \sum_{g_n \in \Gamma_s} \delta_n e^{(p_n - \bar{g})t} \left(p_n \cos(q_n t) - q_n \sin(q_n t)\right)}{\delta_1 + \sum_{g_n \in \Gamma_s} \delta_n e^{(p_n - \bar{g})t} \cos(q_n t)} \tag{26}$$

and that $\lim_{t \to +\infty} g(t) = \bar{g}$. Other solutions diverge.

The presence of bounded life-span generates a transitional dynamics of the growth rate if the economy have a starting point. It indeed creates a persistent dependence of endogenous variables to initial conditions as it is the case in models of vintage capital. This dependance is usually designated as a replacement echoes effect. The simultaneous presence of advanced terms does not change the fact that the endogenous growth rate is no longer a forward variable. It therefore does not jump but converges to its steady-state.

Moreover, the transitional dynamics of the growth rate is oscillating. To understand this, suppose that an economy is on its balanced growth path and faces an unanticipated increase of the total factor productivity and consequently of the interest rate. Each individual has therefore to choose a new and lower level of consumption to be compatible with the exogenous increase of its consumption growth rate. This additional saving induces a strong increase in the capital growth rate that overshoots its new long-run value. Then, generations who born after the shock benefit from an higher human wealth. This relatively reduces their propensity to save and consequently reduces the capital growth rate. This process continues for a long time while it decreases in magnitude. Conversely, if individuals life-span was unbounded, the oversaving from individuals contemporaneous to the shock would no be possible since it would last forever.

In this paper, we have theoretically studied the intertemporal equilibrium path of an overlapping-generations model with finitely-lived individuals and constant returns to capital. We have shown that its existence lies on the initial wealth distribution and that its dynamics to a balanced growth path is governed by exponentially decreasing oscillations. Economies with neoclassical production functions shall exhibit the same properties. Their dynamics are however characterized by a non linear MFDE which limit the theoretical analysis to local dynamics.

Our model suggests that both past and expected values of aggregate variables influence their dynamics. It shall be interesting to test this double dependence on GDP time series.

Proof of lemma 3. In this proof, it is shown that Theorem 6.10 p. 204 in Bellman and Cooke [5] applies for MFDE (19). Differentiating three times (19) yields :

$$\begin{aligned}\frac{d^3k(t)}{dt^3} &= A(1+2\alpha)\frac{d^2k(t)}{dt^2} - \alpha A^2(2+\alpha)\frac{dk(t)}{dt} - (1-\alpha)Ak(t-1)e^{\alpha A} \\ &+ \left[2(1-\alpha)+(\alpha A)^2\right]Ak(t) - (1-\alpha)Ak(t+1)e^{-\alpha A}A1 \end{aligned} \tag{27}$$

Equation (A1) may hence be re-written such that :

$$\begin{aligned}&\left[a_{00}k(t) + a_{10}k(t-1) + a_{11}\frac{dk(t-1)}{dt} + \right.\\ &\left. + a_{12}\frac{d^2k(t-1)}{dt^2} + a_{13}\frac{d^3k(t-1)}{dt^3} + a_{20}k(t-2)\right] = 0\end{aligned} \tag{A2}$$

where the computation of constants a_{ij} is left to the reader. Finally, equation (A1) writes :

$$\sum_{i=0}^{2}\sum_{j=0}^{3} a_{ij}k^{(j)}(t-\omega_i) = 0 \tag{A3}$$

where $\omega_i = i$. Equation (A1) hence writes as equation 6.10.1 p. 197 in Bellman and Cooke. Therefore, theorem 6.10 applies and the general solution of equation (19) is :

$$k(t) = \sum_n \delta_n(t)e^{g_n t} \tag{A4}$$

where g_n is a root of the characteristic equation Q obtained by replacing $k(t)$ by e^{gt} in equation (19) and where $\delta_n(t) \in \mathbb{C}[t]$ is the residue of[12] :

$$e^{st}Q(s)^{-1}p(s) \tag{A5}$$

at the characteristic root $s = g_n$, with :

$$Q(s) = \left(\sum_{i=0}^{2}\sum_{j=0}^{3} a_{ij}s^j e^{-\omega_i s}\right) \tag{A6}$$

and :

$$\begin{aligned} p(s) &= e^{-2s}\sum_{i=0}^{2}\sum_{j=0}^{3} a_{ij}\sum_{\lambda=1}^{j} k^{(j-\lambda)}(2-\omega_i)s^{\lambda-1} \\ &- \sum_{i=0}^{2}\sum_{j=0}^{3} a_{ij}s^j e^{-\omega_i s}\int_{2-\omega_i}^{2} k(z)e^{-sz}dzA7 \end{aligned} \tag{28}$$

[12] see Cartan [12].

$Q'(g) > 0)$, $\delta_n(t) \in \mathbb{C}$. Then, equation (20) follows. Remark that since g_n are conjugate, that $k(t) \in \mathbb{R}$.

<u>Proof of lemma 4.</u> Differentiating twice (18) yields :

$$\begin{aligned} \frac{dk^2(t)}{dt^2} &= A\frac{dk(t)}{dt} + \alpha A\left[\frac{dk(t)}{dt} - Ak(t)\right] + \frac{a(t-1,0)\,e^{\alpha At}}{t} \\ &+ \frac{(1-\alpha)A\left(\int_0^t k(z)\,e^{-\alpha A(z-t)}dz\right)}{t} \\ &- (1-\alpha)A\left(\int_t^{t+1} k(z)\,e^{-\alpha A(z-t)}dz\right) A8 \end{aligned} \tag{29}$$

Hence, replacing (20) and rearranging yields :

$$\begin{aligned} a(t-1,0) &= t\sum_n \delta_n e^{-(\alpha A-g_n)t}\left(g_n^2 - (1+\alpha)Ag_n + \alpha A^2\right) A9 \\ &+ (1-\alpha)A\sum_n \delta_n\left(t\int_t^{t+1} e^{-(\alpha A-g_n)z}dz - \int_0^t e^{-(\alpha A-g_n)z}dz\right) \end{aligned} \tag{30}$$

A change of variable yields (22).

In lemma 4, it is shown that $a(s,0), s \in [-1,0]$ must satisfy (22). Let g_1 be a complex root of the characteristic equation (21) and let $\tilde{g}_1$ be its conjugate. Assume that the initial distribution of wealth writes :

$$\begin{aligned} a(s,0) &= e^{-(\alpha A - g_1)(s+1)} \left[(s+1)\left((g_1)^2 - (1+\alpha) A g_1 + \alpha A^2 \right) \right. \\ &\quad \left. + (1-\alpha) A \left((s+1) \int_0^1 e^{-(\alpha A - g_1) z} dz - \int_{-1-s}^0 e^{-(\alpha A - g_1) z} dz \right) \right] \\ &\quad + e^{-(\alpha A - \tilde{g}_1)(s+1)} \left[(s+1)\left((\tilde{g}_1)^2 - (1+\alpha) A \tilde{g}_1 + \alpha A^2 \right) \right. \\ &\quad \left. + (1-\alpha) A \left((s+1) \int_0^1 e^{-(\alpha A - \tilde{g}_1) z} dz - \int_{-1-s}^0 e^{-(\alpha A - \tilde{g}_1) z} dz \right) \right] \end{aligned}$$

With (22) deduce that their corresponding residues are equal to 1, while all others are equal to zero. With (20) conclude that the solution $k(t)$ writes :

$$k(t) = e^{g_1 t} + e^{\tilde{g}_1 t}$$

which is not positive for all $t \geq 0$.

Proof of lemma 6.

1) Real roots. Observe that $Q'(g) > 0$, that $\lim_{g \to -\infty} Q(g) = -\infty$ and that $Q(\alpha A) = 1$. Since Q is continuous, conclude that a unique $\bar{g} < \alpha A$ such that $Q(\bar{g}) = 0$ exists.

2) No complex roots with real part equal to $\bar{g}$, except $\bar{g}$ itself. As a preliminary, define $P(g)$ such that :

$$P(g) = g - A + (1-\alpha) A \int_{-1}^{0} \left(\int_{s}^{s+1} e^{(g - \alpha A) z} dz \right) ds \tag{A10}$$

and observe that $P(g) = Q(g)(g - \alpha A)$. Now proceed by contradiction. Suppose it exists a complex root with real part $\bar{g}$ denoted $\bar{g} + iq$ that satisfies $P(\bar{g} + iq) = 0$. This writes :

$$\bar{g} - A + (1-\alpha) A \int_{-1}^{0} \left(\int_{s}^{s+1} e^{(\bar{g} - \alpha A) z} \cos(qz)\, dz \right) ds = 0A11 \tag{31}$$

$$q + (1-\alpha) A \int_{-1}^{0} \left(\int_{s}^{s+1} e^{(\bar{g} - \alpha A) z} \sin(qz)\, dz \right) ds = 0A12 \tag{32}$$

Equation (A11) rewrites :

$$A = \bar{g} + (1-\alpha) A \int_{-1}^{0} \left(\int_{s}^{s+1} e^{(\bar{g} - \alpha A) z} \cos(qz)\, dz \right) ds \tag{A13}$$

$$A \leq |\bar{g}| + (1-\alpha) A \int_{-1}^{0} \left(\int_{s}^{s+1} e^{(\bar{g}-\alpha A)z} |\cos(qz)| dz \right) ds \tag{A14}$$

If $|\cos(qz)| < 1$ for some $z \in [-1, 1]$, then

$$A < |\bar{g}| + (1-\alpha) A \int_{-1}^{0} \left(\int_{s}^{s+1} e^{(\bar{g}-\alpha A)z} dz \right) ds \tag{A15}$$

which is not possible, since $|\bar{g}| + (1-\alpha) A \int_{-1}^{0} \left(\int_{s}^{s+1} e^{(\bar{g}-\alpha A)z} dz \right) ds = A$

If $\cos(qz) = 1$ for all $z \in [-1, 1]$; it implies that $q = 0$.

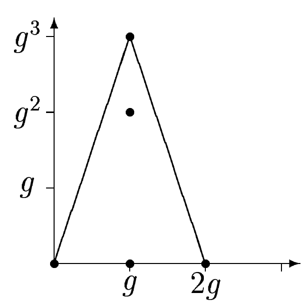
3) and 4) Existence of complex roots with real parts $> \bar{g}$ *and* $< \bar{g}$*.* The roots of large modulus can be localized using the geometrical method proposed by Bellman and Cooke [5] p. 410. Their method is based on the distribution diagram. Compute equation (21) to obtain :

$$Q(g) = 1 + \frac{(1-\alpha) A}{(\alpha A - g)} \left[1 - \frac{e^{g-\alpha A} - 2 + e^{\alpha A - g}}{(\alpha A - g)^2} \right] \tag{A16}$$

Define $\tilde{Q}(g) = (\alpha A - g)^3 e^g Q(g)$ such that :

$$\tilde{Q}(g) = (\alpha A - g)^3 e^g + (1-\alpha) A \left[(\alpha A - g)^2 e^g - e^{2g-\alpha A} + 2e^g - e^{\alpha A} \right] \tag{A17}$$

Hence $\tilde{Q}(\lambda)$ is expressed as $\sum_{i=0}^{3} \sum_{j=0}^{2} g^i e^{jg}$. Then, plot the points x_{ij} with coordinates $(e^{j\lambda}, \lambda^i)$ such that :



Asymptotic roots are those of the convex envelop of the diagram. Here, the distribution diagram is composed of two segments L_1 and L_2 whose slopes are respectively positive and negative. To L_1 and L_2 correspond two locus V_1 and V_2, which belong to $\mathbb{C}$ and

$(1-\alpha)\, Ae^{\alpha A} = 0$, which asymptotically have a negative real part. The zeros of L_2 are those of function : $g^3 e^{-g} - (1-\alpha)\, Ae^{-\alpha A} = 0$, which asymptotically have a positive real part.

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