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Université de La Rochelle
Avenue Michel Crépeau
17042 La Rochelle Cedex 1
http://www.univ-lr.fr/labo/lmca

Third order differential equations with fixed critical points

Sadjia Aït-Mokhtar

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Sadjia Aït-Mokhtar
Université de La Rochelle
Laboratoire de Mathématiques et Applications
Avenue Michel Crépeau
17042 La Rochelle Cedex 1, FRANCE
E-mail : schettab@univ-lr.fr

Abstract : We consider a class of third order homogenous differential equation whose coefficients are analytic functions of the complex variable. We find necessary and sufficient conditions about the coefficients in order that the only movable singularities of the solutions may be poles.

1 Introduction

Let us consider the differential equation :

$$\omega'''^2 + a_0\omega\omega''' + a_1\omega'\omega''' + a_2\omega''\omega''' + a_3\omega''^2 + a_4\omega'\omega'' + a_5\omega\omega'' + a_6\omega'^2 + a_7\omega\omega' + a_8\omega^2 = 0 \tag{1}$$

where $a_i\,(i = 0, 1, \ldots, 8)$ are analytic functions of the complex variable x. The object of this article is to find necessary and sufficient conditions about the coefficients $a_i\,(i = 0, 1, \ldots, 8)$ in order that the only movable singularities of solutions of (1) may be poles.

Definition 1 *: We say that a differential equation belong to the class $\mathcal{M}$ if and only if the only movable singularities of her solutions are poles.*

Notation : We will denote by $\mathcal{T}(\lambda, \mu, \varphi)$ a transformation defined by :

$$\begin{cases} v(x) = \lambda(x)u(t) + \mu(x) \\ t = \varphi(x) \end{cases} \tag{2}$$

where λ, μ, and φ are analytic functions of the complex variable x.

2 Setting

The change of variables

$$\omega' = v\omega \tag{3}$$

transforms equation (1) into the form

$$v''^2 + 2A(x,v,v')v'' + B(x,v,v') = 0 \tag{4}$$

where

$$2A(x,v,v') = (6v + a_2)v' + 2v^3 + a_2v^2 + a_1v + a_0$$

and

$$B(x,v,v') = (9v^2 + 3a_2v + a_3)v'^2 + (6v^4 + 4a_2v^3 + 3a_1v^2 + 2a_3v^2 + a_4v + 3a_0v + a_5)v' + v^6 + +a_2v^5 + (a_1 + a_3)v^4 + (a_0 + a_4)v^3 + (a_5 + a_6)v^2 + a_7v + a_8$$

Lemma 2 *Equation (1) belongs to the class $\mathcal{M}$ if and only if the parametric singularities of any solution of (4) are simple poles with integer residu.*

Proof: Suppose that a solution v of (4) admits a movable singularity (critical or not critical) which is'nt a pole. The relation (3) implies that this singularity is not a pole of ω.

Suppose now that x_0 is a pole of order k of v. Then in a neighberhood of x_0, v is written in the form

$$v(x) = \frac{b_{-k}}{(x-x_0)^k} + \frac{b_{-k+1}}{(x-x_0)^{k-1}} + \cdots + \frac{b_{-1}}{x-x_0} + \sum_{n\geq 0} b_n(x-x_0)^n. \tag{5}$$

After substitution of (5) in (3) and integration, one obtain

$$\omega(x) = C\exp\left(\int_*^x \left(\frac{b_{-k}}{(s-x_0)^k} + \frac{b_{-k+1}}{(s-x_0)^{k-1}} + \cdots + \frac{b_{-1}}{s-x_0} + \sum_{n\geq 0} b_n(s-x_0)^n\right) ds\right). \tag{6}$$

If $k > 1$, then x_0 is essential singularity of ω.
If $k = 1$, then x_0 is a pole of ω if and only if b_{-1} is an integer

Conversely, if x_0 is a pole of ω, then the relation (3) implies that x_0 is a simple pole of v and his residu is 1. ∎

According to **Lemma 2**, it is therefore sufficient to study the type of the parametric singularities of the solutions of (9). The organization of this article is as follows. We first find necessary conditions for the absence of parametric critical points of the solutions of (9). In a second time, we prove that these conditions are sufficient also. Finally, we prove that the movable singularities of the solutions are poles with integer residus.

3 Main results and proofs

The resolution of equation (4) with respect to v'' is given by

$$v'' = P_1(x,v)v' + P_3(x,v) \pm \sqrt{A(x,v,v')^2 - B(x,v,v')} \qquad (7)$$

where

$$\begin{cases} P_1(x,v) := -3v - a_2/2 = \alpha_1 v + \alpha_0 \\ P_3(x,v) := -v^3 - a_2/2v^2 - a_1/2v - a_0/2 = \beta_3 v^3 + \beta_2 v^2 + \beta_1 v + \beta_0 \end{cases} \qquad (8)$$

and

$$4(A(x,v,v')^2 - B(x,v,v')) = (a_2^2 - 4a_3)v'^2 + 2\left[(a_2^2 - 4a_3)v^2 + (a_1a_2 - 2a_4)v + (a_0a_2 - a_5)v\right]v'$$
$$+ (a_2^2 - 4a_3)v^4 + 2(a_1a_2 - 2a_4)v^3 + (a_1^2 + 2a_0a_2 - 4a_5 - 4a_6)v^2 + 2(a_0a_1 - 2a_7)v + (a_0^2 - 4a_8).$$

We will distinguish two cases according to that $a_2^2 - 4a_3 \neq 0$ or $a_2^2 - 4a_3 = 0$
Case1 : Suppose that $a_2^2 - 4a_3 \neq 0$
In this case equation (7) can be written

$$v'' = P_1(x,v)v' + P_3(x,v) + b\sqrt{v'^2 - Q_2(x,v)v' - Q_4(x,v)} \qquad (9)$$

where

$$\begin{cases} b^2 = a_2^2 - 4a_3 \\ Q_2(x,v) := 2(\delta_2 v^2 + \delta_1 v + \delta_0) \\ Q_4(x,v) := \eta_4 v^4 + 2\eta_3 v^3 + 2\eta_2 v^2 + 2\eta_1 v + 2\eta_0 \end{cases} \qquad (10)$$

$$\begin{cases} \delta_2 = -1 \quad \delta_1 = -\frac{a_1a_2 - 2a_4}{b^2} \quad \delta_0 = -\frac{a_0a_2 - 2a_5}{b^2} \\ \eta_4 = -1 \quad \eta_3 = -\frac{a_1a_2 - 2a_4}{b^2} \quad \eta_2 = -\frac{a_1^2 + 2a_0a_2 - 4a_5 - 4a_6}{b^2} \\ \eta_0 = -\frac{a_0^2 - 4a_8}{b^2} \end{cases} \qquad (11)$$

By a transformation $\mathcal{T}(\lambda, \mu, \varphi)$ which satisfy

$$\begin{cases} 2\mu' = -2\mu^2 + \delta_1 \mu + \delta_0 \\ 2\lambda' = -4\mu\lambda + \delta_1 \lambda \\ \varphi' = -\lambda \end{cases} \qquad (12)$$

one can suppose

$$\delta_1 = \delta_0 = 0. \tag{13}$$

The identities above are equivalent to

$$a_1 a_2 = 2a_4 \quad \text{and} \quad a_0 a_2 = 2a_5.$$

Taking into account (13), equation (9) becomes

$$v'' = P_1(x,v)v' + P_3(x,v) + b\sqrt{(v'-v^2)^2 + 2\eta_2 v^2 - 2\eta_1 v - 2\eta_0}. \tag{14}$$

Let us denote

$$Q(x,v) = \eta_2 v^2 + \eta_1 v + \eta_0.$$

Théorème 3 *The solutions of (14) are free from movable critical points if and only if the conditions*

$$\begin{cases} \dfrac{a_1^2 - 4a_6}{b^2} = \dfrac{a_1}{4} \\ \dfrac{a_0 a_1 - 4a_7}{b^2} = \dfrac{a_0}{4} \\ a_1' = -2a_0 - 2a_4 \\ a_0' = -4\dfrac{a_0^2 - 4a_8}{b^2} - a_0 a_2 \\ \left(\dfrac{a_0^2 - 4a_8}{b^2}\right)' = -a_2 \dfrac{a_0^2 - 4a_8}{b^2}. \end{cases} \tag{15}$$

are satisfied

Proof According to [1], a necessary condition which makes all solutions of (14) with fixed critical points is that the solutions of

$$(v' - v^2)^2 - 2Q(x,v)$$

may be singular solutions of (14).

We then obtain the following identities

$$\begin{cases} 2v^3 + \frac{\partial Q}{\partial v} = -P_1(x,v)v^2 + P_3(x,v) \\ -v^2 P_3(x,v) + 2v^5 - 4vQ(x,v) + \frac{\partial Q}{\partial x} = -P_1(x,v)v^4 + 2P_1(x,v)Q(x,v). \end{cases} \tag{16}$$

If we identify the coefficients of the polynomials taking into account (13) we get

$$\begin{cases} \beta_1 = 2\eta_2 \\ \beta_0 = 2\eta_1 \\ \eta_2' = 2\alpha_0\eta_2 - \eta_1 \\ \eta_1' = 2\alpha_0\eta_1 - 2\eta_0 \\ \eta_2' = 2\alpha_0\eta_0 \end{cases} \tag{17}$$

and

$$\begin{cases} \beta_3 - \alpha_1 - 2 = 0 \\ \beta_2 - \alpha_0 = 0 \\ \beta_1 + 2\alpha_1\eta_2 + 4\eta_2 = 0 \\ \beta_0 + 2\alpha_0\eta_2 + 2\alpha_1\eta_1 + 4\eta_1 = \eta_2' \\ 2\alpha_1\eta_0 + 2\alpha_1\eta_0 + 4\eta_0 = \eta_1' \\ 2\alpha_0\eta_0 = \eta_0' \end{cases} \tag{18}$$

After simplification, the identities above gives (15). This proves that these conditions are necessary.

The next step, is to show that the conditions (15) are sufficient.
Suppose $\alpha_0 \neq 0$ and set $\alpha_0 = \dfrac{g'}{g}$, where g is an analytic function.
Set also

$$\begin{cases} \eta_2 = g^2 D_2 \\ \eta_1 = g^2 D_1 \\ \eta_0 = g^2 D_0. \end{cases} \tag{19}$$

where $D_{1,2,3}$ are analytic functions.
Taking into account the identities (17) and (18) one obtain

$$\begin{cases} D_0(x) = K \\ D_1 = -2Kx + K_1 \\ D_2(x) = Kx^2 - Kx + K_2 \end{cases} \tag{20}$$

where K, K_1 and K_2 are constants.
Denote

$$G(v) = D_2 v^2 + D_1 v + D_0 \tag{21}$$

we then can write

$$Q(v) = g^2 G(v) \tag{22}$$

and equation (14) becomes

$$v'' + 2vv' = \frac{g'}{g}(v' - v^2) - v(v' + v^2) + g^2 \frac{\partial G}{\partial v} + b\sqrt{(v' + v^2)^2 - 2g^2 G(v)}. \tag{23}$$

Equation (23) is equivalent to the system

$$\begin{cases} v' - v^2 = gy \\ y' = yv + g\dfrac{\partial G}{\partial v} + b\sqrt{y^2 - G(v)}. \end{cases} \tag{24}$$

Setting

$$\begin{cases} V = y + \sqrt{y^2 - 2G(v)} \\ U = y - \sqrt{y^2 - 2G(v)}, \end{cases} \tag{25}$$

one obtain

$$\begin{cases} VU = 2G \\ y = \dfrac{V}{2} + \dfrac{G}{V} \\ \sqrt{y^2 - 2G(v)} = \dfrac{V}{2} - \dfrac{G}{V}. \end{cases} \tag{26}$$

Denoting

$$\begin{cases} D = gD_2 \\ d = gD_1 \end{cases} \tag{27}$$

we get

$$v = \frac{V' - BV - d}{V + 2D}. \tag{28}$$

In order to obtain the equation satisfied by V, one eliminate v between (28) and $v' = v^2 + g(\frac{V}{2} + \frac{G}{2})$. This gives

$$V'' = \frac{1}{2}(\frac{1}{V} + \frac{3}{V+2D})V'^2 + B(x,V)V' + C(x,V) \tag{29}$$

where

$$B(x,V) = \frac{-2BV - 2d + 2D'}{V+2D},$$

and

$$C(x,V) = \frac{g^2}{2}V^2 + (gD + B' + B^2)V + gK + d' + \frac{2DK - d^2/2}{V} + \frac{(BD - 2BD' - B^2)V + d^2/2 - 2dD'}{V+2D}.$$

Proposition 4 *The solutions of (29) have no parametric critical points.*

Corollary 5 *The solutions of (23) have no parametric critical points.*

(provide from (28) and **Proposition 4**.)

Proof (Proposition 4) Setting

$$W = \frac{V+2D}{D} \tag{30}$$

one obtain

$$\frac{V'}{V+2D} = -\frac{W'}{W(W-1)} + \frac{D'}{DW}. \tag{31}$$

Consequently

$$v = -\frac{W' + (W-1)\left(\frac{dW}{2D} - \frac{D'}{D} + B - \frac{d}{2D}\right)}{W(W-1)}. \tag{32}$$

For abreviation, let us denote

$$\begin{cases} L = \frac{D'}{D} - B + \frac{d}{2D} \\ M = \frac{d}{2D}. \end{cases} \tag{33}$$

The equation satisfied by W is then given by

$$W'' = \frac{3}{2}\frac{W'^2}{W} + 2\left(\frac{L}{W} - M\right)W' - gW\left(D + \frac{K(W-1)^2}{2D}\right)$$
$$- (W-1)\left((M'W - L') + \frac{1}{2}(MW - L)(MW + L - 3M + \frac{L}{M})\right). \qquad (34)$$

The change of variable

$$W = z^{-1} \qquad (35)$$

transforms equation (34) into

$$z'' - \frac{1}{2}\frac{z'^2}{z} = 2(Lz - M)z' + \frac{1}{2}(z-1)(M - Lz)\left(Lz + L - 3M + \frac{M}{z}\right)$$
$$+ (z-1)(L'z - M') + g\left(Dz + \frac{K}{2Dz}(z-1)^2\right). \qquad (36)$$

The two equations (34) and (36) are equivalent in the sense that if the solutions of one of them have no movable critical points the solutions of the second one also will be free from movable critical points.

For equation (36), two cases are considered separately, namely

1. $L = 0$
2. $L \neq 0$.

1. *Case* $L = 0$, equation (36) become

$$z'' - \frac{1}{2}\frac{z'^2}{z} = -\frac{D_1}{D_2}z' + \left(\frac{K}{2D_2} - \frac{D_1^2}{8D_2^2}\right)\frac{1}{z} + \left(g^2 D_2 + \frac{D_1^2}{8D_2^2} - \frac{K}{2D_2}\right)z. \qquad (37)$$

Denote

$$\begin{cases} \alpha = -\frac{D_1}{D_2} \\ \beta = g^2 D_2 + \frac{D_1^2}{8D_2^2} - \frac{K}{2D_2} \\ \delta = \frac{K}{2D_2} - \frac{D_1^2}{8D_2^2}. \end{cases} \qquad (38)$$

A transformation

$$\begin{cases} z(x) = \lambda(x) y(t) \\ t = \varphi(x) \end{cases} \tag{39}$$

which satisfy

$$\begin{cases} \lambda'' - \frac{1}{2}\frac{\lambda'^2}{\lambda} = \alpha\lambda' + \beta\lambda \\ \varphi'^2 = -\frac{2\delta}{\lambda^2} \end{cases} \tag{40}$$

brings equation (37) into the form

$$y'' = \frac{1}{2}\frac{y'^2}{y} - \frac{1}{2y}. \tag{41}$$

Equation (41) can be integrated using the method of variation of parameters by setting

$$y'^2 = hy \tag{42}$$

where h is analytic function. Derive (42) and substitute the result into (41) gives

$$h = \frac{1}{y} + 2k$$

where k is constant. Thus

$$y'' = k,$$

which proves **Proposition 4** in case $L = 0$.

2. *Case* $L \neq 0$. In this case, equation (36) become

$$\begin{aligned} z'' - \frac{1}{2}\frac{z'^2}{z} &= 2Lzz' - \frac{L^2}{2}z^3 + \frac{1}{z}\left(\frac{K}{2D_2} - \frac{D_1^2}{8D_2^2}\right) - \frac{D_1}{D_2}z' + \left(L' + L\frac{D_1}{D_2}\right)z^2 \\ &+ \left(\frac{L^2}{2} - \frac{D_1}{D_2}L + \frac{D_1^2}{8D_2^2} - \frac{K}{2D_2} + g^2 D_2\right)z. \end{aligned} \tag{43}$$

The transformation

$$\begin{cases} z(x) = \lambda(x) y(t) \\ t = \varphi(x) \end{cases} \tag{44}$$

which satisfy

$$\begin{cases} \lambda^4 = \frac{K}{8L^2 D_2} - \frac{D_1^2}{32L^2 D_2^2} \\ \\ \varphi' = -L\lambda \end{cases} \tag{45}$$

transform (43) into

$$y'' - \frac{1}{2}\frac{y'^2}{y} = -2yy' - \frac{1}{2}y^3 - \frac{1}{2}\frac{1}{y} + fy \tag{46}$$

where f is analytic function.
The solutions of (46) are determined by the two Riccati equations

$$y' = -y^2 + 1 + 2hy \tag{47}$$

$$h' = -h^2 + \frac{1}{2}(f-1). \tag{48}$$

Setting $y = \frac{H'}{H}$ in (47), taking into account (48) we get for H the equation

$$H^{(4)} - 2fH^{(2)} - f'H' - H = 0 \tag{49}$$

and the **Proposition 4** follows in this case. ∎

The solutions v of (14) satisfy

$$v(x) = \frac{\varphi'(x)z'(\varphi(x))}{z(\varphi(x)) - 1} - \lambda(x)\frac{D_1}{2D_2} + \mu(x); \tag{50}$$

where λ, φ and μ are those of the transformation (2) and z is solution of (37) for $L = 0$ and solution of (43) for $L \neq 0$.
Consequently the only movable singularities of v are those of $\dfrac{\varphi'(x)z'(\varphi(x))}{z(\varphi(x)) - 1}$ which are poles with residu equal to 1. ∎

Case2 : Suppose that $a_2^2 - 4a_3 = 0$
In this case equation (7) becomes

$$v'' = 2P_1(x,v)v' + 2P_3(x,v) + \sqrt{Q_1(x,v)v' + Q_3(x,v)} \tag{51}$$

where

$$\begin{cases} 2P_1(x,v) := -3v - \frac{a_2}{2} \\ 2P_3(x,v) := -v^3 - \frac{a_2}{2}v^2 - \frac{a_1}{2}v - \frac{a_0}{2} \\ Q_1(x,v) := 2(a_1a_2 - 2a_4)v + 2(a_0a_2 - 2a_5) \\ Q_3(x,v) := 2(a_1a_2 - 2a_4)v^3 + (a_1^2 + 2a_0a_2 - 4a_5 - 4a_6)v^2 + 2(a_0a_1 - a_7)v + (a_0^2 - 4a_8) \end{cases} \tag{52}$$

For simplification, let us denote

$$\begin{cases} P_1(x,v) := \alpha_1 v + \alpha_0 \\ P_3(x,v) := \beta_3 v^3 + \beta_2 v^2 + \beta_1 v + \beta_0 \\ Q_1(x,v) := \delta_1 v + \delta_0 \\ Q_3(x,v) := \eta_3 v^3 + \eta_2 v^2 + \eta_1 v + \eta_0 \end{cases} \tag{53}$$

Théorème 6 *The solutions of (51) are not free from movable critical points.*

Proof We consider again four cases, namely

a) $\delta_1 \neq 0$

b) $\delta_1 = 0$ and $\delta_0 = 0$

c) $\delta_1 = 0$, $\delta_0 \neq 0$ and $\eta_2 = 0$

d) $\delta_1 = 0$, $\delta_0 \neq 0$ and $\eta_2 \neq 0$

a) $\delta_1 \neq 0$
According to [1], a necessary condition which makes all solutions of (51) with fixed critical points is that the solutions of

$$Q_1(x,v)v' + Q_3(x,v) = 0 \tag{54}$$

may be singular solutions of

$$v'' = 2P_1(x,v)v' + 2P_3(x,v). \tag{55}$$

Dividing equation (54) by v and deriving it, one find a second order equation which can not be identified to (55). This proves that the solutions of (54) are not singular solutions of (55). Thus the solutions of (51) are not free from movable critical singularities in this case.

b) $\delta_1 = 0$ **and** $\delta_0 = 0$
In this case, equation (51) takes the form

$$v'' = 2P_1(x,v)v' + 2P_3(x,v) + \sqrt{\eta_2 v^2 + \eta_1 v + \eta_0}. \tag{56}$$

A necessary condition which makes all solutions of (56) free from movable critical singularities is that the solutions of

$$\eta_2 v^2 + \eta_1 v + \eta_0 = 0 \tag{57}$$

may be singular solutions of

$$v'' = 2P_1(x,v)v' + 2P_3(x,v) = 0 \tag{58}$$

A derivation (2 times) of (57) gives a second order equation which can not be identified to (58). Thus the solutions of (56) are not free from movable critical singularities.

c) $\delta_1 = 0$**,** $\delta_0 \neq 0$ **and** $\eta_2 = 0$
In this case, equation (51) takes the form

$$v'' = 2P_1(x,v)v' + 2P_3(x,v) + \frac{\sqrt{\delta_0}}{2}\sqrt{v' + \frac{\eta_1}{\delta_0}v + \frac{\eta_0}{\delta_0}}. \tag{59}$$

By a transformation (2) which satisfy

$$\begin{cases} \mu' = \frac{\eta_1}{\delta_0}\mu + \frac{\eta_0}{\delta_0} \\ \lambda' = \frac{\eta_1}{\delta_0}\lambda \end{cases} \tag{60}$$

one can suppose that

$$\eta_1 = \eta_0 = 0.$$

Then, equation (59) becomes

$$v'' = 2P_1(x,v)v' + 2P_3(x,v) + \frac{\sqrt{\delta_0}}{2}\sqrt{v'}. \tag{61}$$

A necessary condition for which the solutions of (61) are free from movable critical points is that the solutions of

$$v' = 0$$

may be singular solutions of (61). This implies that

$$P_3 \equiv 0$$

which is impossible since the coefficient of v^3 in P_3 is different from 0. Hence, the solutions of (61) are not free from movable singularities.

d) $\delta_1 = 0$, $\delta_0 \neq 0$ **and** $\eta_2 \neq 0$
In this case, equation (51) takes the form

$$v'' = 2P_1(x,v)v' + 2P_3(x,v) + \frac{\sqrt{\delta_0}}{2}\sqrt{v' + \frac{\eta_2}{\delta_0}v^2 + \frac{\eta_1}{\delta_0}v + \frac{\eta_0}{\delta_0}}. \tag{62}$$

By a transformation (2) which satisfy

$$\begin{cases} \mu' = \frac{\eta_2}{\delta_0}\mu^2 + \frac{\eta_1}{\delta_0}\mu + \frac{\eta_0}{\delta_0} \\ \\ \lambda' = \frac{\eta_1}{\delta_0}\lambda + 2\frac{\eta_2}{\delta_0}\lambda\mu \\ \\ \varphi' = \frac{\eta_2}{\delta_0}\lambda \end{cases} \tag{63}$$

one can suppose that

$$\eta_1 = \eta_0 = 0.$$

Then, equation (59) becomes

$$v'' = 2P_1(x,v)v' + 2P_3(x,v) + \frac{\sqrt{\delta_0}}{2}\sqrt{v' + v^2}. \tag{64}$$

According to [1], a necessary condition for absence of parametric critical points is that the solutions of

$$v' + v^2 = 0$$

will be singular solutions of (64). This implies that

$$P_3(x,v) = v^3 + P_1(x,v)v^2. \tag{65}$$

An identification of the coefficients of the polynomials gives the identities

$$\begin{cases} \beta_3 = 1 + \alpha_1 \\ \\ \beta_2 = \alpha_0 \\ \\ \beta_1 = \beta_0 = 0 \end{cases} \tag{66}$$

In (66) the first and the second identities are identically satisfied. The third one implies that

$$a_1 = a_0 = 0.$$

A substitution of (65) into the equation (64) transforms it into

$$v'' = 2P_1(x,v)(v' + v^2) + 2v^3 + \frac{\sqrt{\delta_0}}{2}\sqrt{v' + v^2}. \tag{67}$$

Let us set

$$v' + v^2 = y^2.$$

The equation (67) is then equivalent to the system

$$\begin{cases} v' = y^2 - v^2 \\ y' = -\frac{1}{2}vy + \alpha_0 y + c \end{cases} \tag{68}$$

where $c = \dfrac{\sqrt{\delta_0}}{4}$.

So, v is given by

$$v = -2\frac{y' - \alpha_0 y - c}{y}$$

where y satisfy the equation

$$y'' - 3\frac{y'^2}{y} = -5c\frac{y'}{y} - 4\alpha_0 y' + \frac{2c^2}{y} + (\alpha_0' + 2\alpha_0^2)y + c' + 4\alpha_0 c. \tag{69}$$

Hence, the solutions of (67) have no parametric critical points if the solutions of (69) have no parametric critical points.

According to [3], a necessary condition which makes the solutions of (69) free from parametric critical points is that the coefficient of $\dfrac{y'^2}{y}$ will be of the form $1 - \frac{1}{n}$ where n is an integer. As the coefficient of $\dfrac{y'^2}{y}$ in (69) isn't of the needed form, the solutions are not free from parametric critical points. ∎

References

[1] F. J. Bureau. Equations différentielles du second ordre en y et du second degré en y'' dont l'integrale générale est à points critiques fixes, Annali di Mathematica Pura et Applicata, 4^{e}s, t. **XVI**, (1972), 163-281.

[2] F. J. Bureau. Sur la recherche des équations différentielles du second ordre dont l'integrale générale est à points critiques fixes, Bulletin de la classe des sciences de l'Académie Royale de Belgique, 5^{e}s, t. **XXV**, (1939), 51-68.

[3] F. J. Bureau. Differential equations with fixed critical points, Annali di Mathematica Pura et Applicata, (1964), IV, **64**, 229-364; (1964), IV, **66**, 1-116.

[4] E. L. Ince, Ordinary differential equations, Dover Publications, New-York (1926).

[5] N. S. Kolesnikova et N. A. Lukachevitch, Sur certaines classes d'équations différentielles du troisième ordre à points critiques fixes, Differential Equations, t. **8**, n11, November 1972, 2082-2086.

Liste des prépublications

99-1 Monique Jeanblanc et Nicolas Privault. A complete market model with Poisson and Brownian components. A paraître dans *Proceedings of the Seminar on Stochastic Analysis, Random Fields and Applications*, Ascona, 1999.

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00-02 Nicolas Privault. Hypothesis testing and Skorokhod stochastic integration. *Journal of Applied Probability*, **37** (2000) 560-574.

00-03 Changgui Zhang. La fonction théta de Jacobi et la sommabilité des séries entières q-Gevrey, I. *C. R. Acad. Sci. Paris, Série I* **331** (2000) 31-34.

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