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Université de La Rochelle
Avenue Michel Crépeau
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Asymptotic Behavior for a Reaction Diffusion System with Unbounded Coefficients

Mokhtar Kirane et Nasser-eddine Tatar

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Mokhtar KIRANE
Labotatoire de Mathématiques, Pôle Sciences et Technologie
Université de La Rochelle
Avenue Michel Crépeau, 17042 La Rochelle Cedex, France
e-mail: mokhtar.kirane@univ-lr.fr

and

Nasser-eddine TATAR
King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Dhahran, 31261, Saudi Arabia
e-mail: tatarn@kfupm.edu.sa

Abstract

We consider a system which may serve as a model for a ferment catalytic reaction in chemistry; the model consists of a system of reaction diffusion equations with unbounded time dependent coefficients and different polynomial reaction terms. An exponential decay of the globally bounded solutions is proved. The key tool of the proofs are properties of analytic semigroups and some inequalities.

AMS Subject Classification: 35K57

Key Words: Sectorial operator, Analytic semi-group, reaction-diffusion system, exponential decay.

1 Introduction

Our goal in this paper is to study the problem

$$\begin{cases} u_t - (d_1\Delta - b_1)u = a_1(t)w^m - a_2(t)u^n v^k, \\ v_t - d_2\Delta v = (a_1(t) + a_3(t))\, w^m - a_2(t)u^n v^k, \\ w_t - (d_3\Delta - b_2)w = -\,(a_1(t) + a_3(t))\, w^m - a_4(t)w + a_2(t)u^n v^k, \\ \text{ in } \ \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \ \text{ on } \ \partial\Omega \times (0,\infty) \\ (u,v,w)(x,0) = (u_0(x), v_0(x), w_0(x)), \ \text{ in } \ \Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^N$, $N \geq 1$ is a bounded domain with smooth boundary $\partial\Omega$ and $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$. The diffusion coefficients d_i, $i = 1, 2, 3$

are assumed positive constants and u_0, v_0 and w_0 are given nonnegative bounded initial functions. The coefficients $a_i(t)$ are of the form $t^{\sigma_i}h_i(t)$, where $\sigma_i \geq 0$, $i = 1, 2, 3$ and $\sigma_4 = 0$. The functions $h_i(t)$ are nonnegative continuous functions. The powers m, n and k are assumed greater than one.

Systems with time dependent nonlinearities were considered in [5] by Kahane. Specifically, a system of the form

$$\begin{cases} -u_t + Lu = f(x,t,u,v) \text{ in } \Omega \times (0,\infty) \\ -v_t + Mv = g(x,t,u,v) \text{ in } \Omega \times (0,\infty) \end{cases}$$

where L and M are uniformly elliptic operators, with boundary conditions of Robin type is studied. He proved that the solution converges to the stationary state, that is the solution of the limiting elliptic problem. To this end he assumed that

$$f(x,t,u,v) \to \bar{f}(x,u,v)$$

and

$$g(x,t,u,v) \to \bar{g}(x,u,v)$$

uniformly in Ω and (u,v) in any bounded subset of the first quadrant in $\mathbf{R}^2$. The matrix formed by the partial derivatives $\bar{f}_u, \bar{f}_v, \bar{g}_u$ and $\bar{g}_v$ satisfies a column diagonal dominance type condition. In the present work we do not make such restrictions.

In the case $m = n = k = 1$, $b_1 = b_2 = 0$ and $a_i(t) \equiv a_i$ are constants, this problem has been studied by Wang [14]. If moreover $a_3 = a_4 = 0$, then this problem may also serve as a model for sugar transporting into red blood cells (see Ruan [13], Rothe [12], Feng [1], Morgan [10] and references therein). In particular, Wang proved a convergence result and an exponential decay result in $C^\mu(\bar{\Omega})$, $\mu \in [0,2)$. Unfortunately, his methods seems to be not valid for the present problem (1) because of the unboundedness of the coefficients.

The well posedness and the boundedness of the sought solutions are deduced from the paper of Morgan [10]. It suffices to consider, in Morgan's notation $H(z) = z_1 + z_2 + 2z_3$. The boundedness of the solutions is guaranteed by his Proposition 1.2. If the initial data are nonnegative then so are the corresponding solutions (see [8]).

It is our task here to prove a convergence and an exponential decay result for problem (1). To this end we shall adapt the methods used in Hoshino [3] for the former result and the methods used in Kirane and Tatar [7] for the latter goal.

2 Preliminaries

In this section we present the notation that will be used in this paper and prepare some material which will be useful in our proofs.

By $W^{l,p}(\Omega)$ we denote the usual Sobolev space of order $l \geq 0$ for $1 \leq p \leq \infty$. The space $C^{\mu}(\bar{\Omega})$, $\mu \geq 0$ is the Banach space of $[\mu]-$times continuously differentiable functions in $\bar{\Omega}$ whose $[\mu]-$th order derivatives are Hölder continuous with exponent $\mu - [\mu]$.

Definition 1. *For $p \in (1, \infty)$, we define*

$$D(A_p) = D(B_p) = D(G_p) = \left\{ y \in W^{2,p}(\Omega) : \partial y / \partial \nu \left|_{\partial\Omega} = 0 \right. \right\},$$

$$A_p y = -(d_1 \Delta - b_1) y, \; B_p y = -d_2 \Delta y, \quad G_p y = -(d_3 \Delta - b_2) y.$$

The operators $-A_p$, $-B_p$ and $-G_p$ defined in this way are sectorial operators (see Henry [2]) and generate analytic semigroups $\left\{e^{-tA_p}\right\}_{t\geq 0}$, $\left\{e^{-tB_p}\right\}_{t\geq 0}$ and $\left\{e^{-tG_p}\right\}_{t\geq 0}$, respectively.

Definition 2. *By Q_0 and Q_+ we designate the following projection operators, for $y \in L^p(\Omega)$, $p \in (1, \infty)$*

$$Q_0 y = |\Omega|^{-1} \int_{\Omega} y(x) dx \;\text{ and }\; Q_+ y = y - Q_0 y$$

where $|\Omega|$ is the volume of Ω.

Definition 3. *The restriction of B_p onto $Q_+ L^p(\Omega)$ will be denoted by B_{p+} i.e. $B_{p+} = B_p \left|_{Q_+ L^p(\Omega)}\right.$.*

For $p \in (1, \infty)$, the operator B_{p+} generates an analytic semigroup denoted by $\left\{e^{-tB_{p+}}\right\}_{t\geq 0}$ in $Q_+ L^p(\Omega)$.

We define the fractional powers of the above operators in the usual way (see Henry [2]).

Lemma 1. *Let A be a sectorial operator in $X = L^p(\Omega)$, $1 \leq p < \infty$ with $D(A) = X^1 \subset W^{m,p}(\Omega)$ for some $m \geq 1$. Then, for $0 \leq \alpha \leq 1$, we have $X^{\alpha} \subset C^{\mu}$ when $0 \leq \mu < m\alpha - \frac{N}{p}$.*

Lemma 2. *Let λ denote the least positive eigenvalue of the Laplacian with homogeneous Neumann boundary condition. Let $p \in (1, \infty)$. For every $\alpha \in [0, 1)$ there exist positive constants C_i, $i = 1, 2, 3$ such that for $t > 0$ and $y \in L^p(\Omega)$*

$$\left\|A_p^{\alpha} e^{-tA_p} y\right\|_p \leq C_1 t^{-\alpha} e^{-b_1 t} \left\|y\right\|_p,$$

$$\left\|B_{p+}^{\alpha} e^{-tB_{p+}} Q_+ y\right\|_p \leq C_2 t^{-\alpha} e^{-d_2 \lambda t} \left\|Q_+ y\right\|_p,$$

$$\left\|G_p^{\alpha} e^{-tG_p} y\right\|_p \leq C_3 t^{-\alpha} e^{-b_2 t} \left\|y\right\|_p.$$

Lemma 3. *Let $p \in (1,\infty)$, $l \geq 1$ and $\alpha \in [0,1)$. Then there exists a positive constant C such that*

$$\|y\|_{pl} \leq C \left\|A_p^{\alpha} y\right\|_p^{\theta} \cdot \|y\|_p^{1-\theta}$$

where θ satisfies

$$\frac{N(l-1)}{2pl\alpha} < \theta < 1.$$

See Henry [2] for the proofs of these three lemmas.

Lemma 4. *Let $\alpha \in [0,1)$ and $\beta \in \mathbf{R}$. There exists a positive constant $C = C(\alpha,\beta)$ such that*

$$\int_0^t s^{-\alpha} e^{\beta s} ds \leq \begin{cases} Ce^{\beta t} \text{ if } \beta > 0 \\ C(t+1) \text{ if } \beta = 0 \\ C \text{ if } \beta < 0. \end{cases}$$

See Hoshino and Yamada [4], for instance, for the proof of this lemma.

Lemma 5. *Let μ, ν, τ and $z > 0$, then*

$$z^{1-\nu} \int_0^z (z-\xi)^{\nu-1} \xi^{\mu-1} e^{-\tau\xi} d\xi \leq C\tau^{-\mu}$$

where C is a constant independent of z.

See Michalski [9] or Kirane and Tatar [6] for the proof of this lemma.

Let $y(t)$ satisfy

$$y(t) \leq C + \sum_{i=1}^{p} \int_0^t \lambda_i(s) g_i(y(s)) ds, \; t \in [a,b] \tag{2}$$

where the functions g_i, $1 \leq i \leq p$ are continuous and nondecreasing on $[0,\infty)$ and positive on $(0,\infty)$ such that g_{i+1}/g_i, $1 \leq i \leq p-1$ are nondecreasing on $(0,\infty)$.

We define

$$G_k(y) = \int_{y_k}^{y} \frac{ds}{g_k(s)}, \; y > 0, \; y_k > 0, \; 1 \leq k \leq p$$

and G_k^{-1} the inverse function of G_k.

For $b \geq \tilde{b} \geq a$ and $\lambda_i : [a,b] \to [0,\infty)$, $1 \leq i \leq p$ integrable functions, we define the functions $\varphi_0(y) = y$ and

$$\varphi_k = \psi_k \circ \psi_{k-1} \circ ... \circ \psi_1,$$

$$\psi_k = G_k^{-1}\left[G_k(y) + \alpha_k(a,\tilde{b})\right],$$

where $\alpha_k(a,\tilde{b}) = \int\limits_a^{\tilde{b}} \lambda_k(s)ds$.

Lemma 6. *(Pinto [11])*

Assume that the functions g_i, $1 \le i \le p$ are as above, the functions y and λ_i, $1 \le i \le p$ are continuous and nonnegative on $[a,b]$ and the constant C is positive. If (2) holds, then for $t \in [a,\tilde{b})$

$$y(t) \le G_p^{-1}\left[G_p(\varphi_{p-1}(C)) + \int\limits_a^t \lambda_p(s)ds\right]$$

where $\tilde{b} \in [a,b]$ is a number such that

$$\alpha_k(a,\tilde{b}) = \int\limits_a^{\tilde{b}} \lambda_k(s)ds \le \int\limits_{\varphi_{k-1}}^{\infty} \frac{ds}{g_k(s)},\ 1 \le k \le p.$$

3 Convergence

In this section we state a convergence result of the solutions in the space $C^\mu(\Omega)$, $\mu \in [0,2)$. This result will be needed in the sequel. Without loss of generality, we shall assume that $b_1 = b_2 = b$.

Let q and q^* be such that $\frac{1}{q} + \frac{1}{q^*} = 1$ and

$$q^* = \begin{cases} \frac{2r+1}{r}, & \text{if } r \le 1 \\ 2, & \text{if } r > 1 \end{cases}$$

with $r = \frac{1-\alpha}{\alpha}$.

Theorem 1. *Suppose that $1 - q\alpha > 0$, $1 + q(\sigma_i - \alpha l) > 0$, $i = 1,2,3$ for some l such that $1 < l < \min\{m,n,k\}$, $h_i \in L^{q^*}(0,\infty)$, $i = 1,2,3,4$ and $a_1(t) \le Ca_3(t)$ for some positive constant C, $\forall t > 0$. Then for every $\mu \in [0,2)$*

$$u(t) \to 0, v(t) \to v_\infty \ \text{and}\ \ w(t) \to 0 \ \text{in}\ C^\mu(\Omega)\ \ \text{as}\ \ t \to \infty$$

where $v_\infty = |\Omega|^{-1}\left\{\int\limits_\Omega w_0 dx - \int\limits_0^\infty \int\limits_\Omega (h_4(s) + b)w(s)dxds\right\}$.

The proof is similar to the proof of Theorem 2.2 in [3] with minor modifications that will be clear from the proof of our next and main result. It is therefore omitted.

4 Rates of convergence

In this section we suppose that N and p satisfy

(H) $2\alpha > \frac{N}{p}$ and $\max\left\{1, \frac{N(m-1)}{2p\alpha}, \frac{N(n-1)}{2p\alpha}\right\} < \min\{m,n\}$

Theorem 2. *Assume that the hypotheses of Theorem 10 and **(H)** are fulfilled. If*

$$(l-1)\int_0^\infty h(s)ds < \log\left(\frac{1+C_0^{l-1}}{C_0}\right)$$

for some constant l such that $\max\left\{1, \frac{N(m-1)}{2p\alpha}, \frac{N(n-1)}{2p\alpha}\right\} < l < \min\{m,n\}$ and a constant $C_0 = C_0(\|u_0\|_p, \|w_0\|_p)$ to be determined in the proof, then for $\mu \in [0,2)$ and N, p such that $0 \le \mu < 2\alpha - \frac{N}{p}$ we have

(a) $\|u\|_{C^\mu(\bar\Omega)}, \ \|w\|_{C^\mu(\bar\Omega)} \le Ce^{-(b-\varepsilon)t}\left(\|u_0\|_p + \|w_0\|_p\right)$, where $0 < \varepsilon < b$, as $t \to \infty$.

(b) (i) If $d_2\lambda < lb$,

$$\|v - v_\infty\|_{C^\mu(\bar\Omega)} \le Ce^{-\min\{(b-\varepsilon), d_2\lambda\}t} \text{ as } t \to \infty.$$

(ii) If $d_2\lambda \ge lb$ and for $i = 1,2,3$

$$\int_0^t e^{q^*\rho s} h_i^{q^*}(s)ds = O(e^{q^*\tilde\rho t}) \text{ as } t \to \infty$$

for some $\rho > d_2\lambda - l(b-\varepsilon)$ and $\tilde\rho < d_2\lambda$, then

$$\|v - v_\infty\|_{C^\mu(\bar\Omega)} \le Ce^{-\min\{(b-\varepsilon), d_2\lambda-\tilde\rho\}t} \text{ as } t \to \infty.$$

Proof. **A. The decay rate of $\|u\|_{C^\mu(\bar\Omega)}$ and $\|w\|_{C^\mu(\bar\Omega)}$:**

Clearly we have the integral equations associated with the first and third equation of (1)

$$u(t) = e^{-tA_p}u(0) + \int_0^t e^{-(t-s)A_p}\left\{a_1(s)w^m - a_2(s)u^n v^k\right\} ds \tag{3}$$

$$w(t) = e^{-tG_p}w(0)$$
$$+\int_0^t e^{-(t-s)G_p}\left\{-(a_1(s)+a_3(s))w^m - a_4(s)w + a_2(s)u^n v^k\right\} ds. \tag{4}$$

Applying A_p^α and G_p^α, $0<\alpha<1$ to both sides of (3) and (4), respectively, we infer from Lemma 5 that

$$\begin{aligned}\left\|A_p^\alpha u\right\|_p \leq C_1 t^{-\alpha}e^{-bt}\left\|u_0\right\|_p + C_1\int_0^t (t-s)^{-\alpha}e^{-b(t-s)}s^{\sigma_1}h_1(s)\left\|w^m\right\|_p ds \\ +C_1\int_0^t (t-s)^{-\alpha}e^{-b(t-s)}s^{\sigma_2}h_2(s)\left\|u^n v^k\right\|_p ds\end{aligned}$$

and

$$\begin{aligned}\left\|G_p^\alpha w\right\|_p \leq\ & C_3 t^{-\alpha}e^{-bt}\left\|w_0\right\|_p \\ & +C_3\int_0^t (t-s)^{-\alpha}e^{-b(t-s)}(s^{\sigma_1}h_1(s)+s^{\sigma_3}h_3(s))\left\|w^m\right\|_p ds \\ & +C_3\int_0^t (t-s)^{-\alpha}e^{-b(t-s)}h_4(s)\left\|w\right\|_p ds \\ & +C_3\int_0^t (t-s)^{-\alpha}e^{-b(t-s)}a_2(s)\left\|u^n v^k\right\|_p ds.\end{aligned}$$

As $0\leq v\leq M$, for some $M>0$, we have for all $t\geq\delta>0$

$$\begin{aligned}e^{bt}\left\|A_p^\alpha u\right\|_p \leq C_1\delta^{-\alpha}\left\|u_0\right\|_p + C_1\int_0^t (t-s)^{-\alpha}e^{bs}s^{\sigma_1}h_1(s)\left\|w^m\right\|_p ds \\ +C_1 M^k\int_0^t (t-s)^{-\alpha}e^{bs}s^{\sigma_2}h_2(s)\left\|u^n\right\|_p ds,\end{aligned} \tag{5}$$

and

$$\begin{aligned}e^{bt}\left\|G_p^\alpha w\right\|_p \leq C_3\delta^{-\alpha}\left\|w_0\right\|_p + C_3\int_0^t (t-s)^{-\alpha}e^{bs}h_4(s)\left\|w\right\|_p ds \\ +C_3\int_0^t (t-s)^{-\alpha}e^{bs}(s^{\sigma_1}h_1+s^{\sigma_3}h_3)(s)\left\|w^m\right\|_p ds \\ +C_3 M^k\int_0^t (t-s)^{-\alpha}e^{bs}s^{\sigma_2}h_2(s)\left\|u^n\right\|_p ds.\end{aligned} \tag{6}$$

Lemma 6 and the uniform boundedness of w allow us to write

$$\left\|w^m\right\|_p = \left\|w\right\|_{mp}^m \leq C\left\|G_p^\alpha w\right\|_p^{m\theta}.\left\|w\right\|_p^{m(1-\theta)} \leq C_4\left\|G_p^\alpha w\right\|_p^{m\theta} \tag{7}$$

with $\frac{N(m-1)}{2pm\alpha}<\theta<1$.

In the same way we have

$$\|u^n\|_p \le C_5 \left\|A_p^\alpha u\right\|_p^{n\theta} \quad \text{with} \quad \frac{N(n-1)}{2pn\alpha} < \theta < 1. \tag{8}$$

Let us choose l such that

$$\max\left\{1, \frac{N(m-1)}{2p\alpha}, \frac{N(n-1)}{2p\alpha}\right\} < l < \min\{m,n\}$$

and $\theta = \frac{l}{m}$ in (7) and $\theta = \frac{l}{n}$ in (8). Then

$$\|w^m\|_p \le C_4 \left\|G_p^\alpha w\right\|_p^l \quad \textit{and} \quad \|u^n\|_p \le C_5 \left\|A_p^\alpha u\right\|_p^l. \tag{9}$$

Taking (9) into account in (5) and (6), we obtain

$$\begin{aligned} e^{bt}\left\|A_p^\alpha u\right\|_p &\le C_1\delta^{-\alpha}\|u_0\|_p \\ +C_6 &\int_0^t (t-s)^{-\alpha} e^{b(1-l)s} s^{\sigma_1} h_1(s) \left(e^{bs}\left\|G_p^\alpha w\right\|_p\right)^l ds \\ +C_7 &\int_0^t (t-s)^{-\alpha} e^{b(1-l)s} s^{\sigma_2} h_2(s) \left(e^{bs}\left\|A_p^\alpha u\right\|_p\right)^l ds. \end{aligned} \tag{10}$$

and

$$\begin{aligned} e^{bt}\left\|G_p^\alpha w\right\|_p &\le C_3\delta^{-\alpha}\|w_0\|_p + C_3\int_0^t (t-s)^{-\alpha} h_4(s) e^{bs}\|w\|_p\, ds \\ +C_8 &\int_0^t (t-s)^{-\alpha} e^{b(1-l)s} (s^{\sigma_1}h_1 + s^{\sigma_3}h_3)(s) \left(e^{bs}\left\|G_p^\alpha w\right\|_p\right)^l ds \\ +C_9 &\int_0^t (t-s)^{-\alpha} e^{b(1-l)s} s^{\sigma_2} h_2(s) \left(e^{bs}\left\|A_p^\alpha u\right\|_p\right)^l ds. \end{aligned} \tag{11}$$

The second term in the right hand side of (11) may be estimated in the following manner, for $0 < \varepsilon < b$

$$\begin{aligned} \int_0^t (t-s)^{-\alpha} h_4(s) e^{bs}\|w\|_p\, ds &= \int_0^t (t-s)^{-\alpha} e^{\varepsilon s} e^{(b-\varepsilon)s} h_4(s)\|w\|_p\, ds \\ &\le \left(\int_0^t (t-s)^{-\alpha q} e^{\varepsilon q s} ds\right)^{\frac{1}{q}} \left(\int_0^t h_4^{q^*}(s)\left(e^{(b-\varepsilon)s}\|w\|_p\right)^{q^*} ds\right)^{\frac{1}{q^*}} \\ &\le C_{10} e^{\varepsilon t}\left(\int_0^t h_4^{q^*}(s)\left(e^{(b-\varepsilon)s}\|w\|_p\right)^{q^*} ds\right)^{\frac{1}{q^*}}. \end{aligned} \tag{12}$$

We have used the Hölder inequality, Lemma 7 and the embedding $D(G_p^\alpha) \subset L^p$ to derive the last inequalities in (12). Multiplying (10) and (11) by $e^{-\varepsilon t}$, setting

$$U(t) = e^{(b-\varepsilon)t} \left\|A_p^\alpha u\right\|_p, W(t) = e^{(b-\varepsilon)t} \left\|G_p^\alpha w\right\|_p$$

and taking into account (12) we find for $t \geq \delta > 0$

$$U(t) \leq C_1 \delta^{-\alpha} \|u_0\|_p + C_6 \int_0^t (t-s)^{-\alpha} e^{-(b-\varepsilon)(l-1)s} s^{\sigma_1} h_1(s) W(s)^l ds$$
$$+C_7 \int_0^t (t-s)^{-\alpha} e^{-(b-\varepsilon)(l-1)s} s^{\sigma_2} h_2(s) U(s)^l ds,$$

and

$$W(t) \leq C_3 \delta^{-\alpha} \|w_0\|_p + C_{10} \left(\int_0^t h_4^{q^*}(s) W(s)^{q^*} ds \right)^{\frac{1}{q^*}}$$
$$+C_8 \int_0^t (t-s)^{-\alpha} e^{-(b-\varepsilon)(l-1)s} (s^{\sigma_1} h_1 + s^{\sigma_3} h_3)(s) W(s)^l ds$$
$$+C_9 \int_0^t (t-s)^{-\alpha} e^{-(b-\varepsilon)(l-1)s} s^{\sigma_2} h_2(s) U(s)^l ds.$$

Next, as $1 - q\alpha > 0$, then using the Hölder inequality and Lemma 8, we obtain for $t \geq \delta > 0$

$$U(t) \leq C_1 \delta^{-\alpha} \|u_0\|_p + C_{11} \delta^{-\alpha} \left(\int_0^t h_1^{q^*}(s) W(s)^{lq^*} ds \right)^{\frac{1}{q^*}}$$
$$+C_{12} \delta^{-\alpha} \left(\int_0^t h_2^{q^*}(s) U(s)^{lq^*} ds \right)^{\frac{1}{q^*}}$$

and

$$W(t) \leq C_3 \delta^{-\alpha} \|w_0\|_p + C_{10} \left(\int_0^t h_4^{q^*}(s) W(s)^{q^*} ds \right)^{\frac{1}{q^*}}$$
$$+C_{13} \delta^{-\alpha} \left(\int_0^t h_1^{q^*}(s) W(s)^{lq^*} ds \right)^{\frac{1}{q^*}} + C_{14} \delta^{-\alpha} \left(\int_0^t h_3^{q^*}(s) W(s)^{lq^*} ds \right)^{\frac{1}{q^*}}$$
$$+C_{15} \delta^{-\alpha} \left(\int_0^t h_2^{q^*}(s) U(s)^{lq^*} ds \right)^{\frac{1}{q^*}}.$$

Now we use the inequality

$$(x_1+x_2+...+x_n)^r \le n^{r-1}(x_1^r+x_2^r+...+x_n^r)$$

to get

$$\begin{aligned} U(t)^{q^*} \le\ & 3^{q^*-1}\left(C_1\delta^{-\alpha}\left\|u_0\right\|_p\right)^{q^*} + 3^{q^*-1}\left(C_{11}\delta^{-\alpha}\right)^{q^*}\int_0^t h_1^{q^*}(s)W(s)^{lq^*}ds \\ & +3^{q^*-1}\left(C_{12}\delta^{-\alpha}\right)^{q^*}\int_0^t h_2^{q^*}(s)U(s)^{lq^*}ds, \end{aligned} \tag{13}$$

and

$$\begin{aligned} W(t)^{q^*} \le\ & 5^{q^*-1}\left(C_3\delta^{-\alpha}\left\|w_0\right\|_p\right)^{q^*} + 5^{q^*-1}C_{10}^{q^*}\int_0^t h_4^{q^*}(s)W(s)^{q^*}ds \\ & +5^{q^*-1}\left(C_{13}\delta^{-\alpha}\right)^{q^*}\int_0^t h_1^{q^*}(s)W(s)^{lq^*}ds + 5^{q^*-1}\left(C_{14}\delta^{-\alpha}\right)^{q^*}\int_0^t h_3^{q^*}(s)W(s)^{lq^*}ds \\ & +5^{q^*-1}\left(C_{15}\delta^{-\alpha}\right)^{q^*}\int_0^t h_2^{q^*}(s)U(s)^{lq^*}ds. \end{aligned} \tag{14}$$

Putting $F(t)=U(t)^{q^*}+W(t)^{q^*}$, we infer from (13) and (14) that

$$F(t) \le C_0\left(\left\|u_0\right\|_p, \left\|w_0\right\|_p\right) + \int_0^t h(s)(F(s)+F(s)^l)ds,\ t\ge\delta>0$$

where $C_0\left(\left\|u_0\right\|_p, \left\|w_0\right\|_p\right) = 3^{q^*-1}\left(C_1\delta^{-\alpha}\left\|u_0\right\|_p\right)^{q^*} + 5^{q^*-1}\left(C_3\delta^{-\alpha}\left\|w_0\right\|_p\right)^{q^*}$ and $h(s) = \max\left\{(C_{17}+C_{19})h_1^{q^*}(s), C_{20}h_3^{q^*}(s), (C_{18}+C_{21})h_2^{q^*}(s), C_{22}h_4^{q^*}(s)\right\}$ where C_i, $i=17,18,...,22$ are the coefficients of the integral terms in (13) and (14) in the order.

Let $G(z)=\int_{z_0}^{z}\frac{dy}{y+y^l}$. Then, by Lemma 9 we may conclude that

$$\begin{aligned} F(t) &\le G^{-1}\left[G\left(C_0\left(\left\|u_0\right\|_p, \left\|w_0\right\|_p\right)\right) + \int_0^t h(s)ds\right] \\ &\le C_0\left(1+C_0^{l-1}\right)^{\frac{1}{1-l}} e^{\int_0^t h(s)ds} \cdot \left[1-\frac{C_0}{1+C_0^{l-1}}e^{(l-1)\int_0^t h(s)ds}\right]^{\frac{1}{1-l}}. \end{aligned}$$

From our assumptions on $\|u_0\|_p$, $\|w_0\|_p$ and $h(t)$ we deduce that

$$\left\|A_p^\alpha u\right\|_p \leq Ce^{-(b-\varepsilon)t}\left(\|u_0\|_p + \|w_0\|_p\right) \tag{15}$$

and

$$\left\|G_p^\alpha w\right\|_p \leq Ce^{-(b-\varepsilon)t}\left(\|u_0\|_p + \|w_0\|_p\right). \tag{16}$$

The decay rates in $C^\mu(\bar{\Omega})$, $\mu \in [0,2)$ follow from Lemma 4.

B. The decay rate of $\|v - v_\infty\|_{C^\mu(\bar{\Omega})}$:

As in Hoshino [3], let us write

$$v - v_\infty = (Q_0 v(t) - v_\infty) + Q_+ v(t),$$

and estimate the terms in the right hand side separately.

a. The estimation of $Q_0 v(t) - v_\infty$:

Integrating the second equation in (1) over $(0,t)\times\Omega$, we have

$$\int_\Omega v(x,t)dx + \int_0^t \int_\Omega \left\{(a_1(s)+a_3(s))w^m - a_2(s)u^n v^k\right\} dxds = \int_\Omega v_0(x)dx.$$

It appears then that

$$|Q_0 v(t) - v_\infty| = |\Omega|^{-1}\left|\int_t^\infty \int_\Omega \left\{(a_1(s)+a_3(s))w^m - a_2(s)u^n v^k\right\} dxds\right|.$$

In the rest of the proof, C will denote a generic positive constant which may be different at different occurrences.

Using (9), (15), (16) and the Hölder inequality we see that

$$\begin{aligned}
&|Q_0 v(t) - v_\infty| \\
&\leq C\int_t^\infty \left\{(s^{\sigma_1}h_1 + s^{\sigma_3}h_3)(s)e^{-l(b-\varepsilon)s} + s^{\sigma_2}h_2(s)M^k e^{-l(b-\varepsilon)s}\right\} ds \\
&\leq C\int_t^\infty \left\{\sum_{i=1}^3 s^{\sigma_i}h_i(s)\right\} e^{-l(b-\varepsilon)s}ds \\
&\leq C\int_t^\infty \left\{\sum_{i=1}^3 s^{\sigma_i}h_i(s)\right\} e^{-(b-\varepsilon)s}.e^{-(l-1)(b-\varepsilon)s}ds \\
&\leq Ce^{-(b-\varepsilon)t}\int_t^\infty \left\{\sum_{i=1}^3 s^{\sigma_i}h_i(s)\right\} e^{-(l-1)(b-\varepsilon)s}ds \\
&\leq Ce^{-(b-\varepsilon)t}.
\end{aligned} \tag{17}$$

We also used Lemma 7 in the last inequality.

b. The estimation of $Q_+v(t)$:

In order to estimate $Q_+v(t)$ let us apply $B^{\alpha}_{p+}Q_+$ to the integral equation associated with the second equation in (1). We find for all $t \geq \delta > 0$

$$B^{\alpha}_{p+}Q_+v(t) = B^{\alpha}_{p+}e^{-(t-\delta)B_{p+}}Q_+v(\delta)$$
$$+\int_{\delta}^{t} B^{\alpha}_{p+}e^{-(t-s)B_{p+}}Q_+\left[(a_1(s)+a_3(s))w^m - a_2(s)u^n v^k\right] ds.$$

Taking the L^p-norm and using the second inequality in Lemma 5, we obtain

$$\left\|B^{\alpha}_{p+}Q_+v(t)\right\|_p \leq C_2(t-s)^{-\alpha}e^{-d_2(t-s)\lambda}\left\|Q_+v(\delta)\right\|_p + C_2\left\|Q_+\right\|_{L^p(\Omega)\to L^p(\Omega)}$$
$$\int_{\delta}^{t}(t-s)^{-\alpha}e^{-d_2\lambda(t-s)}\left\{(s^{\sigma_1}h_1 + s^{\sigma_3}h_3)\left\|w^m\right\| + s^{\sigma_2}h_2M^k\left\|u^n\right\|\right\} ds.$$

Next, using (9), (15) and (16) we see that for all $t \geq \delta + T$

$$\begin{aligned}\left\|B^{\alpha}_{p+}Q_+v(t)\right\|_p &\leq Ce^{-d_2\lambda t}\left\|Q_+v(\delta)\right\|_p \\ &+Ce^{-d_2\lambda t}\int_{\delta}^{t}(t-s)^{-\alpha}e^{d_2\lambda s}e^{-l(b-\varepsilon)s}\left(\sum_{i=1}^{3}s^{\sigma_i}h_i(s)\right) ds \\ &\leq Ce^{-d_2\lambda t}\left\{\left\|Q_+v(\delta)\right\|_p + \int_{0}^{t-\delta}(t-\delta-s)^{-\alpha}\right. \\ &\left.\times e^{-[l(b-\varepsilon)-d_2\lambda](s+\delta)}\left(\sum_{i=1}^{3}(s+\delta)^{\sigma_i}h_i(s+\delta)\right) ds\right\}\end{aligned} \tag{18}$$

(i) If $d_2\lambda < lb$, then choose ε such that $0 < l\varepsilon < lb - d_2\lambda$. Hence we may apply Lemma 8, together with the Hölder inequality to get

$$\left\|B^{\alpha}_{p+}Q_+v(t)\right\|_p \leq Ce^{-d_2\lambda t} \; for\ all\ t \geq \delta + T. \tag{19}$$

(ii) If $d_2\lambda \geq lb$, then $l(b-\varepsilon) - d_2\lambda < 0$. Multiplying by $e^{\rho(s+\delta)}.e^{-\rho(s+\delta)}$, with $\rho > d_2\lambda - l(b-\varepsilon)$, the integrand in (18), using the Hölder inequality and Lemma 8 we see that

$$\begin{aligned}\left\|B^{\alpha}_{p+}Q_+v(t)\right\|_p &\leq Ce^{-d_2\lambda t}\left\{\left\|Q_+v(\delta)\right\|_p + e^{-\tilde{\rho}t}\right\} \\ &\leq Ce^{-(d_2\lambda-\tilde{\rho})t}.\end{aligned} \tag{20}$$

The conclusion follows from (17), (19) and (20) □

Remark 1. *The condition $a_1(t) \leq Ca_3(t)$ needed in Theorem 10 has not been used in Theorem 11. So, the decay rates hold provided one may prove a convergence result without this condition.*

Remark 2. *The assumption* $\max\left\{1, \frac{N(m-1)}{2p\alpha}, \frac{N(n-1)}{2p\alpha}\right\} < \min\{m,n\}$ *may be relaxed somewhat using different* l_1 *and* l_2 *and applying Lemma 9 with* $p = 2$.

Remark 3. *We also have exponential decay in (b) (ii) without the growth condition on* h_i, $i = 1, 2, 3$ *in case* $\sigma_i = 0$, $i = 1, 2, 3$.

Remark 4. *It is possible to obtain sharper estimates using comparison results by replacing the bounds* M^k *with* $v_\infty^k - Ce^{-(b-\varepsilon)t}$ *in (5) and* $v_\infty^k + Ce^{-(b-\varepsilon)t}$ *in (6), for large values of* t.

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References

[1] W. Feng, Coupled system of reaction-diffusion equations and applications in Carrier faciliated diffusion, *Nonlinear Anal. T.M.A.*, Vol. **17**, No. **3** (1991), 285-311.

[2] D. Henry, "Geometric Theory of Semilinear Parabolic Equations", Lecture Notes in Math. 840, Springer-Verlag, Berlin, 1981.

[3] H. Hoshino, On the convergence properties of global solutions for some reaction diffusion systems under Neumann boundary conditions, *Diff. and Integ. Eqs.*, Vol. **9**, No. **4** (1996), 761-778.

[4] H. Hoshino and Y. Yamada, Solvability and smoothing effect for semilinear parabolic equations, *Funkc. Ekvac.*, **34** (1994), 475-494.

[5] C. S. Kahane, On the asymptotic behavior of solutions of nonlinear parabolic systems under Robin type boundary conditions, *Funkc. Ekvac.*, **26** (1983), 51-78.

[6] M. Kirane and N.-E. Tatar, Global existence and stability of some semilinear problems, *Arch. Math. Brno*, Tomus **36** (2000), 1-12.

[7] M. Kirane and N.-E. Tatar, Convergence rates for a reaction diffusion system, *Zeit. Anal. Anw. (J. Anal.Math.)*, **20** (2001) 2, 347-357.

[8] R. H. Martin and M. Pierre, Nonlinear reaction-diffusion systems, *in Nonlinear Equations in the Applied Sciences*, *Math. Sci. Engrg.*, 185 W. F. Ames and C. Rogers, Editors, Academic Press, Boston, MA 1992, pp. 363-398.

[9] M. W. Michalski, Derivatives of non integer order and their applications, *Dissertationes Mathematicae, Polska Akademia Nauk, Instytut Matematyczny*, Warszawa, (1993).

[10] J. Morgan, Global existence for semilinear parabolic systems, *SIAM J. Math. Anal.*, Vol. **20**, No. **5** (1989), 1128-1144.

[11] M. Pinto, Integral inequalities of Bihari-type and applications, *Funkc. Ekvac.*, **33**, No. **3** (1990), 387-404.

[12] F. Rothe, "Global solutions of reaction-diffusion systems", Lecture Notes in Math. 1072, Springer-Verlag, Berlin, New York, 1983.

[13] W. Ruan, Bounded solutions for reaction diffusion systems with nonlinear boundary conditions, *Nonlinear Anal. T.M.A.*, Vol. **14**, No. **12** (1990), 1051-1077.

[14] M. X. Wang, Asymptotic behavior of solutions to some reaction diffusion systems, *Chinese J. Contemp. Math.*, Vol. **18**, No. **3** (1997), 249-260.

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