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Existence and Asymptotic Behavior for a Convection Problem

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Abstract

We prove global existence and exponential decay of solutions for a system which arise in thermal convection flow. For sufficiently small initial data, these results improve previous ones in [8]. Further, we investigate the behavior of solutions for arbitrarily large initial data. In particular, we show that the length of the interval on which we have existence and exponential decay is inverse proportional to the size of the initial data.

Key Words and Phrases: Asymptotic behavior, exponential decay, global existence, Henry-Gronwall-Bihari type inequality, nonstationary convection problem, semi-group theory.

AMS Subject Classifications: 76D03, 76E06, 35B40

1 Introduction

We consider the following initial value problem which appears in thermal convection flow

$$\begin{cases} \partial_t v + (v.\nabla)v = \Delta v - \tau g + h - \nabla\pi, \ x \in \Omega, \ t > 0, \\ \nabla.v = 0, \ x \in \Omega, \ t > 0, \end{cases}$$
$$\partial_t \tau + (v.\nabla)\tau = \Delta\tau, \ x \in \Omega, \ t > 0,$$
$$v(x,t) = 0, \ \tau(x,t) = \xi(x,t), \ x \in \Gamma, \ t > 0,$$
$$v(x,0) = v_0(x), \ \tau(x,0) = \tau_0(x), \ x \in \Omega,$$

where Ω is a bounded region in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary Γ and

$v = (v^1(x,t), \dots, v^N(x,t))$ denotes the velocity field of the fluid,

$\pi = \pi(x,t)$ is the pressure,

$\tau = \tau(x,t)$ is the temperature,

$h = (h^1(x,t), \dots, h^N(x,t))$ represents the external force,

$v_0 = (v_0^1(x), \dots, v_0^N(x))$ is the initial velocity,

$\tau_0 = \tau_0(x)$ is the initial temperature,

$\xi = \xi(x,t)$ is the boundary temperature and

$g = (g^1(x), \dots, g^N(x))$ is the gravitational vector.

This model is derived in [2] and [9]. In [8], Hishida studied this problem after reducing it to an abstract Cauchy problem of the form

$$\begin{cases} \frac{dv}{dt} + A_p v = F(v,\theta), \ t > 0, \ v(0) = v_0 \\ \frac{d\theta}{dt} + B_q\theta = G(v,\theta), \ t > 0, \ \theta(0) = \theta_0 \end{cases} \tag{1}$$

with

$$\begin{cases} F(v,\theta) = -\mathbf{P}_p(v.\nabla)v - \mathbf{P}_p\theta g, \\ G(v,\theta) = -(v.\nabla)v - (v.\nabla)\phi. \end{cases}$$

Here $\mathbf{P}_p$ is the projection from $L^p(\Omega)^N$ onto $L^p_\sigma(\Omega)$ = the completion of $C^\infty_{0,\sigma}(\Omega) = \{\varphi \in C_0^\infty(\Omega)^N, \ \nabla.\varphi = 0\}$ in $L^p(\Omega)^N$, $1 < p < \infty$ via the Helmholz decomposition $L^p(\Omega)^N = L^p_\sigma(\Omega) \oplus G_p(\Omega)$ with $G_p(\Omega) = \{\nabla\pi, \ \pi \in W^{1,p}(\Omega)\}$ (see [3]). The spaces $W^{l,p}(\Omega)$, $1 \leq p \leq \infty$, $l \geq 0$ are the usual Sobolev spaces with $W^{0,p}(\Omega) = L^p(\Omega)$.

The function $\phi = \phi(x,t)$ is solution of

$$\begin{cases} \partial_t\phi = \Delta\phi, \ x \in \Omega, \ t > 0, \\ \phi(x,t) = \xi(x,t), \ x \in \Gamma, \ t > 0, \\ \phi(x,0) = \phi_0(x), \ x \in \Omega \end{cases}$$

where $\phi_0 = \phi_0(x)$ is defined by

$$\begin{cases} \Delta\phi_0 = 0, \ in\ \Omega \\ \phi_0(x) = \xi(x,0), \ on\ \Gamma. \end{cases}$$

The operators B_q and A_p are defined as follows:

$$B_q = -\Delta \text{ with domain } D(B_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$$

and

$$A_p = -\mathbf{P}_p\Delta \text{ with domain } D(A_p) = D(B_q)^N \cap L^p_\sigma(\Omega).$$

$-B_q$ and $-A_p$ generate then bounded analytic semigroups $\{\exp(-tB_q),\ t \geq 0\}$ on $L^q(\Omega)$ and $\{\exp(-tA_p),\ t \geq 0\}$ on $L^p_\sigma(\Omega)$ respectively. See [7,16] and [4,5]. The fractional powers B_q^β and A_p^α are defined in the usual way. We will occasionally drop the subscripts p and q in the sequel.

Existence, uniqueness (see Theorem 1 below) and regularity results have been established in [8] for sufficiently small initial data. The local existence result is quite general. However, to prove global existence and exponential decay of solutions, the author was forced to assume that

$$\|\nabla\phi\|_\infty = O(e^{-\omega t}) \ \text{ with } \ \omega > 0.$$

In this paper we would like to improve these results. Namely, we will enlarge the set of functions ϕ to all functions ϕ such that

$$\|\nabla\phi\|_\infty = O(t^{-\omega}) \ \text{ with } \ \omega \geq 0.$$

Observe that for $w = 0$, this will extend the result in [8] to functions ϕ satisfying

$$\|\nabla\phi\|_\infty = O(e^{-\omega t}) \ \text{ with } \ \omega \geq 0.$$

This was not possible in [8] because of the nature of the argument used there. We will also discuss the case of functions ϕ satisfying $\|\nabla\phi\|_\infty = O(t^\omega)$, $\omega \geq 0$ and give an upper bound for w assuring our goals. The key tool in our proof is the inequality in Lemma 1 below.

Furthermore, we will establish existence and decay results of solutions to problem (1) for not necessarily *small* data. It will be proved that solutions exist for arbitrarily large initial data, however, the existence is not global. Solutions are extended to some "maximal" finite interval on which they decay exponentially. In fact, the length of the interval on which we have existence and exponential decay is found to be "inverse proportional" to the size of the initial data. The proofs rely on a generalized Gronwall inequality (Lemma 2 below) as well as the inequality in Lemma 1. They are in fact based on a desingularization argument which can be found in [11-14]. we end this work by showing that if $\|\nabla\phi\|_\infty = O((t+1)^{-\omega})$ for some $\omega \geq 0$ (without any other condition on ω), then we have a polynomial decay of the same type but with an arbitrary large power.

The plan of the paper is as follows: In the next section we prepare some material needed in the sequel. Then, we present the existence (local and global) and asymptotic behavior results of Hishida [8]. In Section 3, we discuss how to improve these last results by appropriately modifying Hishida's arguments. Section 4 is devoted to the analysis of solutions with arbitrarily large initial data.

2 Preliminaries

Below we collect some results from [8] concerning, in particular, local and global existence as well as an exponential decay result for solutions of this problem. Moreover, we present some lemmas which will be useful to prove our results. For convenience, we shall keep the same notation as in [8].

Lemma 1. *If μ, ν, $\tau > 0$ and $z > 0$, then*

$$z^{1-\nu} \int_0^z (z-\xi)^{\nu-1} \xi^{\mu-1} e^{-\tau\xi} d\xi \leq K(\nu, \mu, \tau)$$

where $K(\nu, \mu, \tau)$ is a positive constant.

See [15] or [11] for the proof.

Lemma 2. *Let $a(t)$, $b(t)$, $K(t)$, $\psi(t)$ be nonnegative, continuous functions on the interval $I = (0, T)$ $(0 < T \leq \infty)$, $\Phi : (0, \infty) \to \mathbb{R}$ be a continuous, nonnegative and nondecreasing function, $\Phi(0) = 0$, $\Phi(u) > 0$ for $u > 0$ and let $A(t) = \max_{0 \leq s \leq t} a(s)$, $B(t) = \max_{0 \leq s \leq t} b(s)$. Assume that*

$$\psi(t) \leq a(t) + b(t) \int_0^t K(s) \Phi(\psi(s)) ds, \ t \in I.$$

Then

$$\psi(t) \leq W^{-1} \left[W(A(t)) + B(t) \int_0^t K(s) ds \right], \ t \in (0, T_1),$$

where $W(v) = \displaystyle\int_{v_0}^{v} \frac{d\sigma}{\Phi(\sigma)}$, $v \geq v_0 > 0$, W^{-1} is the inverse of W and $T_1 > 0$ is such that $W(A(t)) + B(t) \displaystyle\int_0^t K(s) ds \in D(W^{-1})$ for all $t \in (0, T_1)$.

See [1] for the proof.

Our problem (1) will be studied via the associated system of integral equations

$$\begin{cases} v(t) = e^{-tA_p} v_0 + \displaystyle\int_0^t e^{-(t-s)A_p} F(v, \theta)(s) ds, \\ \theta(t) = e^{-tB_q} \theta_0 + \displaystyle\int_0^t e^{-(t-s)B_q} G(v, \theta)(s) ds. \end{cases} \tag{2}$$

That is, we will consider weak solutions of (1). Then it is possible to show that these solutions are strong (see Theorems 3 and 4 in [8]).

Lemma 3. *Let $1 < p, q < \infty$.*

(i) Assume

$$\begin{cases} 0 \leq \delta < 1/2 + N(1 - 1/p)/2, & \delta + 2\mu \geq N/2p + 1/2 \\ \delta + \mu > 1/2 \text{ and } \mu > 0. \end{cases}$$

Then

$$\left\| A^{-\delta} \mathbf{P}(v.\nabla)w \right\|_p \leq C_1 \left\| A^\mu v \right\|_p \left\| A^\mu w \right\|_p \text{ for } v, w \in D(A_p^\mu),$$

with some positive constant $C_1 = C_1(\delta, \mu)$.

(ii) Assume

$$\begin{cases} 0 \leq \hat{\delta} < 1/2 + N(1 - 1/q)/2, & \hat{\delta} + \nu + \hat{\nu} \geq N/2p + 1/2, \\ \hat{\delta} + \hat{\nu} > 1/2, & \nu > N(1/p - 1/q)/2 \text{ and } \nu, \hat{\nu} \geq 0. \end{cases}$$

Then

$$\left\| B^{-\hat{\delta}}(v.\nabla)\theta \right\|_q \leq C_2 \left\| A^\nu v \right\|_p \left\| B^{\hat{\nu}} \theta \right\|_q \text{ for } v \in D(A_p^\nu), \ \theta \in D(B_q^{\hat{\nu}}),$$

with some positive constant $C_2 = C_2(\hat{\delta}, v, \hat{\nu})$.

(iii) Let $\hat{\mu} \geq 0$ satisfy

$$N/q - 2\hat{\mu} \leq N/p.$$

Then

$$\left\| \mathbf{P}\theta g \right\|_p \leq C_3 \left\| B^{\hat{\mu}} \theta \right\|_q \left\| g \right\|_\infty \text{ for } \theta \in D(B_q^{\hat{\mu}}),$$

with some positive constant $C_3 = C_3(\hat{\mu})$.

(iv) Let $\nu \geq 0$ satisfy $N/p - 2\nu \leq N/q$. Then

$$\left\| (v.\nabla)\phi \right\|_q \leq C_4 \left\| A^\nu v \right\|_p \left\| \nabla \phi \right\|_\infty \text{ for } v \in D(A_p^\nu),$$

with some positive constant $C_4 = C_4(\nu)$.

See Lemma 3.3 in [8].

Let Λ_1 denote the least eigenvalue of the Laplace operator with zero Dirichlet boundary condition.

Lemma 4. *For each $\lambda_1 \in (0, \Lambda_1)$, $\alpha \geq 0$ and $\beta \geq 0$, we have*

$$\left\| A^\alpha e^{-tA} v \right\|_p \leq C_{\alpha,\lambda_1} t^{-\alpha} e^{-\lambda_1 t} \left\| v \right\|_p \text{ for } v \in L_\sigma^p(\Omega)$$

and

$$\left\| B^\beta e^{-tB} \theta \right\|_p \leq \hat{C}_{\beta,\lambda_1} t^{-\beta} e^{-\lambda_1 t} \left\| \theta \right\|_q \text{ for } \theta \in L^q(\Omega)$$

with some positive constants C_{α,λ_1} and $\hat{C}_{\beta,\lambda_1}$.

Theorem 1. *Assume that* $g \in C(\bar{\Omega})^N$ *and* $\nabla\phi \in Lip([0,T];C(\bar{\Omega})^N)$ *for every* $T > 0$. *Let* p *and* q *satisfy*

$$\max\{N/3, 1\} < p < \infty,\ 1 < q < \infty \text{ and } |1/p - 1/q| < 2/N.$$

If $\{v_0, \theta_0\} \in D(A_p^\gamma) \times D(B_q^{\hat{\gamma}})$ *with*

$$\begin{cases} \{N/2p - 1/2\}^+ \le \gamma < 1,\ 0 \le \hat{\gamma} < 1 \\ \text{and } \gamma - \hat{\gamma} - \frac{N}{2}\left(\frac{1}{p} - \frac{1}{q}\right) \in (-1, 1], \end{cases}$$

then the following assertions hold:

(a) there exists a positive constant T_* *depending on* $\left\|A_p^\gamma v_0\right\|_p$ *and* $\left\|B_q^{\hat{\gamma}}\theta_0\right\|_q$ *such that problem (1), (2) has a unique mild solution* $\{v, \theta\}$ *on* $[0, T_*]$ *in* $C\left([0,T_*]; D(A_p^\gamma) \times D(B_q^{\hat{\gamma}})\right)$,

(b) suppose moreover that $\|\nabla\phi(t)\|_\infty = O(e^{-\omega t})$, $\omega > 0$ *as* $t \to \infty$, *then for a fixed* $\lambda \in (0, \Lambda_1)$, *there exists a positive constant* $\varepsilon = \varepsilon(\lambda, N, p, q, \gamma, \hat{\gamma}, \omega)$ *such that if*

$$\left\|A_p^\gamma v_0\right\|_p + \left\|B_q^{\hat{\gamma}}\theta_0\right\|_q + \sup_{0<t<\infty} e^{\omega t}\|\nabla\phi(t)\|_\infty \le \varepsilon,$$

then the solution $\{v, \theta\}$ *exists globally in time and satisfies for each* $\{\alpha, \beta\} \in [\gamma, 1) \times [\hat{\gamma}, 1)$

$$\left\|A_p^\gamma v(t)\right\|_p \le C(\alpha, \lambda) t^{\gamma-\alpha} e^{-\lambda t}\left(\left\|A_p^\gamma v_0\right\|_p + \left\|B_q^{\hat{\gamma}}\theta_0\right\|_q\right),$$
$$\left\|B_q^{\hat{\gamma}}\theta(t)\right\|_q \le \hat{C}(\beta, \lambda) t^{\hat{\gamma}-\beta} e^{-\lambda t}\left(\left\|A_p^\gamma v_0\right\|_p + \left\|B_q^{\hat{\gamma}}\theta_0\right\|_q\right),$$

for $0 < t < \infty$ *with some positive constants* $C(\alpha, \lambda)$ *and* $\hat{C}(\beta, \lambda)$.

For the proof of this result see [8, Theorems 1 and 2]. The local existence proof relies on the choice of $\mu, \nu \in (\gamma, 1 - \delta)$ and $\hat{\mu}, \hat{\nu} \in (\hat{\gamma}, 1 - \hat{\delta})$ such that

$$1 + \gamma \ge \delta + 2\mu,\ 1 + \gamma \ge \hat{\delta} + \nu + \hat{\nu},\ \nu < 1 + \gamma - \hat{\gamma}$$

where $\delta = 0$ (resp. $\hat{\delta} = 0$) if $\gamma \in (0,1)$ (resp. $\hat{\gamma} \in (0,1)$) and $\delta > 0$ (resp. $\hat{\delta} > 0$) is arbitrary small if $\gamma = 0$ (resp. $\hat{\gamma} = 0$). In fact it is divided into two parts:

Case (a): If $\left|\gamma - \hat{\gamma} - \frac{N}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\right| < 1$, we choose $\hat{\mu} < 1 + \hat{\gamma} - \gamma$.

Case (b): If $\gamma - \hat{\gamma} - \frac{N}{2}\left(\frac{1}{p} - \frac{1}{q}\right) = 1$, we take $\hat{\mu} = 1 + \hat{\gamma} - \gamma$.

In the sequel we shall make use of these constants. Their existence is taken for granted from [8]. Finally, we will also make use of the following lemma

Lemma 5. *If* $0 \le \alpha < 1$ *and* τ, μ, $\sigma > 0$, *then*

$$\int_0^t q(t-s)^{-\alpha} e^{-\tau(t-s)}(\sigma s + 1)^{-\mu} ds \le L(\alpha, \tau, \mu, \sigma)(\sigma t + 1)^{-\mu}$$

where $L(\alpha, \tau, \mu, \sigma)$ *is a positive constant and* $q(t) = \min\{1, t\}$.

See Step 2 of Section 6 in [10].

3 Extensions of Theorem 1

In this section we state and prove some theorems which extend the results in Theorem 1 to functions ϕ whose gradients are not necessarily exponentially decaying to zero in the L^∞-norm. We obtain similar results for functions whose gradients are polynomially decaying to zero (in the L^∞-norm), uniformly bounded or even with a polynomial growth (up to a certain order). Let $\lambda \in (0, \Lambda_1)$. By C we will denote a generic positive constant which may change from line to line. Let us define the range of ω as follows:

Case (a): $1 - \beta < \omega < 1$.

Case (b): $1 + \gamma - \hat{\gamma} - \nu \leq \omega < 1 + \gamma - \nu$.

Theorem 2. *Assume that the hypotheses of Theorem 1 hold. Then, for α sufficiently close to 1 and $\beta \in (\hat{\gamma}, 1)$, we have global existence and exponential decay of solutions (away from zero) for any ϕ such that*

$$\|\nabla\phi(t)\|_\infty = O(t^{-\omega}),$$

where $\omega \geq 0$ is in the ranges mentioned above.

Proof. To prove global existence it is sufficient to establish a priori estimates for the solution. It is well known that uniform boundedness of solutions (or simply boundedness by a continuous function) allows us to extend them for all $t > 0$. Let λ_1 be any value between λ and Λ_1, for instance $\lambda_1 = \frac{\lambda+\Lambda_1}{2}$. Using the integral equations (2) associated to the differential equations in (1) and Lemma 4, we obtain

$$\begin{aligned} \|A^\alpha v(t)\|_p \leq \; & C\Big\{ t^{-(\alpha-\gamma)} e^{-\lambda_1 t} \|A^\gamma v_0\|_p \\ & + \int_0^t e^{-\lambda_1(t-s)} (t-s)^{-\alpha-\delta} \|A^\mu v(s)\|_p^2 \, ds \\ & + \|g\|_\infty \int_0^t e^{-\lambda_1(t-s)} (t-s)^{-\alpha} \left\|B^{\hat{\mu}}\theta(s)\right\|_q ds \Big\}, \end{aligned} \tag{3}$$

for $\alpha \in (\gamma, 1-\delta)$ and some $C > 0$. We also have

$$\begin{aligned} \left\|B^\beta\theta(t)\right\|_q \leq \; & C\Big\{ t^{-(\beta-\hat{\gamma})} e^{-\lambda_1 t} \left\|B^{\hat{\gamma}}\theta_0\right\|_q \\ & + \int_0^t e^{-\lambda_1(t-s)} (t-s)^{-\beta-\hat{\delta}} \|A^\nu v(s)\|_p \left\|B^{\hat{\nu}}\theta(s)\right\|_q ds \\ & + t^{\beta-\hat{\gamma}} \int_0^t e^{-\lambda_1(t-s)} (t-s)^{-\beta} \|A^\nu v(s)\|_p \|\nabla\phi(s)\|_\infty \, ds \Big\} \end{aligned} \tag{4}$$

for $\beta \in (\hat{\gamma}, 1-\hat{\delta})$. The relations (3) and (4) will be our reference inequalities in the sequel. Our proof will be divided into two parts.

Case (a) $\hat{\mu} < 1 + \hat{\gamma} - \gamma$:

This case corresponds to case (a) in the proof of the local existence part in [8]. To fix ideas suppose that $1 - \beta < \omega < 1$. We can pick σ such that $\hat{\gamma} < \sigma < \min(\hat{\mu}, \beta)$ and $\sigma - \hat{\gamma}$ $(< 1/2)$ so small that

$$1 - \beta + (\sigma - \hat{\gamma}) \leq \omega. \tag{5}$$

Since $1 + \hat{\gamma} - \sigma < 1$, choose $\hat{\gamma}, \sigma$ and α such that $1 + \hat{\gamma} - \sigma < \alpha$ (observe that this might be a condition on α if one cannot find $\hat{\gamma}$ and σ such that the inequality holds for all $\alpha \in (\gamma, 1)$). Then select κ such that $\gamma < \kappa < \alpha$ and $\kappa - \gamma$ $(< 1/2)$ is so small that

$$1 + \hat{\gamma} - \sigma + (\kappa - \gamma) \leq \alpha \textit{ and } \omega + (\kappa - \gamma) < 1. \tag{6}$$

Clearly, from (5) and the fact that $\kappa > \gamma$, we have

$$1 + \gamma - \beta - \kappa + (\sigma - \hat{\gamma}) < \omega. \tag{7}$$

Next, multiplying (3) by $t^{\kappa-\gamma}e^{\lambda t}$ we obtain

$$\begin{aligned} t^{\kappa-\gamma}e^{\lambda t} \left\| A^{\alpha} v(t) \right\|_p &\leq C e^{(\lambda - \lambda_1)t} \\ \times \Bigg\{ \left\| A^{\gamma} v_0 \right\|_p &+ t^{\kappa-\gamma} E_{\mu}(t)^2 \int_0^t e^{(\lambda_1 - 2\lambda)s} (t-s)^{-\alpha-\delta} s^{2(\gamma-\kappa)} ds \\ &+ \|g\|_{\infty} t^{\kappa-\gamma} \hat{E}_{\hat{\mu}}(t) \int_0^t e^{(\lambda_1 - \lambda)s} (t-s)^{-\alpha} s^{\hat{\gamma}-\sigma} ds \Bigg\}, \end{aligned} \tag{8}$$

for $t \geq a > 0$. We set

$$E_{\alpha}(t) := E_{\alpha,\kappa}(t) := \sup_{0 \leq s \leq t} s^{\kappa-\gamma} e^{\lambda s} \left\| A^{\alpha} v(s) \right\|_p, \tag{9}$$

and

$$\hat{E}_{\beta}(t) := \hat{E}_{\beta,\sigma}(t) := \sup_{0 \leq s \leq t} s^{\sigma-\hat{\gamma}} e^{\lambda s} \left\| B^{\beta} \theta(s) \right\|_q. \tag{10}$$

Since $1 - \alpha - \delta > 0$, $1 + 2(\gamma - \kappa) > 0$ and $1 + \hat{\gamma} - \sigma > 0$, then clearly from (6), (8) and the definitions (9) and (10), we can write

$$E_{\alpha}(t) \leq C \left\{ \left\| A^{\gamma} v_0 \right\|_p + E_{\mu}(t)^2 + \hat{E}_{\hat{\mu}}(t) \right\} \tag{11}$$

for $t \geq a > 0$. Similarly, multiplying (4) by $t^{\sigma-\hat{\gamma}}e^{\lambda t}$, we get

$$\begin{aligned} t^{\sigma-\hat{\gamma}}e^{\lambda t} \left\| B^{\beta} \theta(t) \right\|_q &\leq C e^{(\lambda - \lambda_1)t} \\ \times \Bigg\{ \left\| B^{\hat{\gamma}} \theta_0 \right\|_q &+ t^{\sigma-\hat{\gamma}} E_{\nu}(t) \hat{E}_{\hat{\nu}}(t) \int_0^t e^{(\lambda_1 - 2\lambda)s} (t-s)^{-\beta-\hat{\delta}} s^{\gamma+\hat{\gamma}-\kappa-\sigma} ds \\ &+ t^{\sigma-\hat{\gamma}} E_{\nu}(t) \int_0^t e^{(\lambda_1 - \lambda)s} (t-s)^{-\beta} s^{\gamma-\kappa} \left\| \nabla \phi(s) \right\|_{\infty} ds \Bigg\}. \end{aligned} \tag{12}$$

for $t \geq a > 0$. As $\sigma - \hat{\gamma} < 1/2$ and $\kappa - \gamma < 1/2$, we have $1 - (\kappa - \gamma) - (\sigma - \hat{\gamma}) > 0$. Using the assumption on ϕ and (7), we entail from (12) that

$$\hat{E}_{\beta}(t) \leq C \left\{ \left\| B^{\hat{\gamma}} \theta_0 \right\|_q + E_{\nu}(t) \hat{E}_{\hat{\nu}}(t) + k_{\infty} E_{\nu}(t) \right\} \tag{13}$$

where $k_\infty = \sup_{0 \le t \le \infty} t^\omega \|\nabla\phi(t)\|_\infty$. Let

$$E(t) := \max\left\{E_\mu(t), E_\nu(t), \hat{E}_{\hat{\mu}}(t), \hat{E}_{\hat{\nu}}(t)\right\}. \tag{14}$$

Note that as α is close to 1, we may consider $\mu, \nu \le \alpha$. Also from the choices $1+\gamma \ge \hat{\delta}+\nu+\hat{\nu}$ and $\hat{\mu} < 1+\hat{\gamma}-\gamma$ (already adopted in the local existence proof in [8]) we may consider $\hat{\nu}, \hat{\mu} \le \beta$ (or else take smaller parameters $\hat{\nu}$ and $\hat{\mu}$). Therefore, by the embedding properties of $D(A^\alpha)$ and $D(B^\beta)$, it appears from (11) and (13) that

$$E(t) \le C\left\{\|A^\gamma v_0\|_p + \left\|B^{\hat{\gamma}}\theta_0\right\|_q + E^2(t) + k_\infty E(t)\right\} \tag{15}$$

for all $t \ge a > 0$.

If $\|A^\gamma v_0\|_p$, $\left\|B^{\hat{\gamma}}\theta_0\right\|_q$ and k_∞ are sufficiently small, then we infer from (15) that

$$E(t) \le C\left(\|A^\gamma v_0\|_p + \left\|B^{\hat{\gamma}}\theta_0\right\|_q\right)$$

for all $t \ge a > 0$. By virtue of the definitions (9), (10) and (14) we see from (15) that

$$\begin{cases} \|A^\alpha v(t)\|_p \le Ct^{-(\kappa-\gamma)}e^{-\lambda t}\left(\|A^\gamma v_0\|_p + \left\|B^{\hat{\gamma}}\theta_0\right\|_q\right) \\ \left\|B^\beta\theta(t)\right\|_q \le Ct^{-(\sigma-\hat{\gamma})}e^{-\lambda t}\left(\|A^\gamma v_0\|_p + \left\|B^{\hat{\gamma}}\theta_0\right\|_q\right) \end{cases}$$

for every $\{\alpha, \beta\} \in (\gamma, 1-\delta) \times (\hat{\gamma}, 1-\hat{\delta})$ with $\alpha > 1+\hat{\gamma}-\sigma$.

Case (b): $\hat{\mu} = 1+\hat{\gamma}-\gamma$:

Here we pick $E_\alpha(t) := E_{\alpha,\alpha}(t)$ and $\hat{E}_\beta(t) := \hat{E}_{\beta,\beta}(t)$ (see definitions (9) and (10)). Multiplying (3) and (4) by $t^{\alpha-\gamma}e^{\lambda t}$ and $t^{\beta-\hat{\gamma}}e^{\lambda t}$ respectively and proceeding as in the proof of part (a), we get

$$\begin{aligned} t^{\alpha-\gamma}e^{\lambda t}\|A^\alpha v(t)\|_p &\le Ce^{(\lambda-\lambda_1)t} \\ \times\Bigg\{\|A^\gamma v_0\|_p &+ t^{\alpha-\gamma}E_\mu(t)^2\int_0^t e^{(\lambda_1-2\lambda)s}(t-s)^{-\alpha-\delta}s^{2(\gamma-\mu)}ds \\ &+ \|g\|_\infty t^{\alpha-\gamma}\hat{E}_{\hat{\mu}}(t)\int_0^t e^{(\lambda_1-\lambda)s}(t-s)^{-\alpha}s^{\hat{\gamma}-\hat{\mu}}ds\Bigg\}, \end{aligned}$$

and

$$\begin{aligned} t^{\beta-\hat{\gamma}}e^{\lambda t}\left\|B^\beta\theta(t)\right\|_q &\le Ce^{(\lambda-\lambda_1)t} \\ \times\Bigg\{\left\|B^{\hat{\gamma}}\theta_0\right\|_q &+ t^{\beta-\hat{\gamma}}E_\nu(t)\hat{E}_{\hat{\nu}}(t)\int_0^t e^{(\lambda_1-2\lambda)s}(t-s)^{-\beta-\hat{\delta}}s^{\gamma+\hat{\gamma}-\nu-\hat{\nu}}ds \\ &+ k_\infty t^{\beta-\hat{\gamma}}E_\nu(t)\int_0^t e^{(\lambda_1-\lambda)s}(t-s)^{-\beta}s^{\gamma-\nu-\omega}ds\Bigg\} \end{aligned}$$

with the new definitions of $E_\mu(t)$, $E_\nu(t)$, $\hat{E}_{\hat{\mu}}(t)$ and $\hat{E}_{\hat{\nu}}(t)$. Clearly from the assumptions on the exponents we have $1+2(\gamma-\mu) = (1+\gamma-\delta-2\mu)+(\gamma+\delta) > 0$,

$1+\hat{\gamma}-\hat{\mu}>\hat{\gamma}+\hat{\delta}>0$ and $1+\gamma+\hat{\gamma}-\nu-\hat{\nu}=(1+\gamma-\hat{\delta}-\nu-\hat{\nu})+\hat{\gamma}+\hat{\delta}>0$. If we assume that

$$0<1+\gamma-\hat{\gamma}-\nu\leq\omega<1+\gamma-\nu,$$

then we obtain global existence and exponential decay of solutions. $\square$

If $\omega=0$, then, according to the last proof, it is possible to choose κ such that (6) is valid and $1-\beta<\kappa-\gamma$ provided that β is close enough to 1. Therefore, Theorem 1 and Theorem 2 may be extended to $0\leq\omega<1$ provided that α and β are close enough to 1.

Corollary 1. *If in addition to the hypotheses of Theorem 2, β is sufficiently close to 1, then the results in Theorem 1 and Theorem 2 may be extended to $\omega=0$.*

From the previous proof it can also be seen that we may even consider functions ϕ such that

$$\|\nabla\phi(t)\|_{\infty}=O(t^{\tau}),\tau\geq 0$$

with sufficiently small τ. Therefore, we may state the following corollary

Corollary 2. *Assume the same hypotheses as in Theorem 1. Then, the results of Theorem 2 hold for all ϕ satisfying $\|\nabla\phi(t)\|_{\infty}=O(t^{\tau})$ with sufficiently small τ.*

4 Large initial data

In this section we treat the case of arbitrarily large initial data (not necessarily small as was assumed in [8] and in Section 3). To this end we use an argument combining Lemma 1 and Lemma 2. The latter lemma is sometimes referred to as the Henry-Gronwall-Bihari type inequality.

Theorem 3. *Let the hypotheses of Theorem 1 hold with $\|\nabla\phi(t)\|_{\infty}=O(t^{-w})$ as $t\to\infty$ for some positive constant ω to be determined. Assume further that $\mu-\gamma<1/2r$, $\hat{\mu}-\hat{\gamma}<1/r$ and $\nu+\hat{\nu}-\gamma-\hat{\gamma}<1/r$ where r is such that $\frac{1}{r}+\frac{1}{r^*}=1$ and*

$$r^*=\begin{cases}\frac{2\xi+1}{\xi}, & \xi\leq 1,\\ 2, & \xi>1,\end{cases}$$

with $\xi=\min\{y,z\}$, $y=\frac{1-(\alpha+\delta)}{\alpha+\delta}$, $z=\frac{1-(\beta+\hat{\delta})}{\beta+\hat{\delta}}$. Then the mild solution of (1), (3) exists on some time interval $[a,T]$, $T>0$ on which, for each $\{\alpha,\beta\}\in[\gamma,1)\times[\hat{\gamma},1)$

$$\begin{aligned}\|A^{\alpha}v(t)\|_{p}&\leq Ct^{\gamma-\alpha}e^{-\lambda t}\left(\|A^{\gamma}v_0\|_p^{r^*}+\left\|B^{\hat{\gamma}}\theta_0\right\|_q^{r^*}\right),\\ \left\|B^{\beta}\theta(t)\right\|_q&\leq\hat{C}t^{\hat{\gamma}-\beta}e^{-\lambda t}\left(\|A^{\gamma}v_0\|_p^{r^*}+\left\|B^{\hat{\gamma}}\theta_0\right\|_q^{r^*}\right).\end{aligned}$$

Proof. Let $\lambda_1=\frac{\lambda+\Lambda_1}{2}$. Applying the operators A^{α}, $\gamma\leq\alpha<1-\delta$ and B^{β}, $\hat{\gamma}\leq\beta<1-\delta$ to both sides of

$$\Phi(v,\theta)(t)=\int_0^t e^{-(t-s)A_p}F(v,\theta)(s)ds$$

and

$$\Psi(v,\theta)(t)=\int_0^t e^{-(t-s)B_q}G(v,\theta)(s)ds$$

and taking the L^p and L^q norms respectively, we obtain using Lemma 3 and Lemma 4

$$\begin{aligned}\|A^\alpha\Phi(v,\theta)(t)\|_p &\le C_{\alpha+\delta,\lambda_1}C_1\int_0^t e^{-\lambda_1(t-s)}(t-s)^{-\alpha-\delta}\|A^\mu v(s)\|_p^2\,ds\\ &+C_{\alpha,\lambda_1}C_3\|g\|_\infty\int_0^t e^{-\lambda_1(t-s)}(t-s)^{-\alpha}\left\|B^{\hat\mu}\theta(s)\right\|_q ds\\ &= C_{\alpha+\delta,\lambda_1}C_1I_1+C_{\alpha,\lambda_1}C_3\|g\|_\infty I_2\end{aligned}$$

and

$$\begin{aligned}&\left\|B^\beta\Psi(v,\theta)(t)\right\|_q\\ &\le \hat C_{\beta+\hat\delta,\lambda_1}C_2\int_0^t e^{-\lambda_1(t-s)}(t-s)^{-\beta-\hat\delta}\|A^\nu v(s)\|_p\left\|B^{\hat\nu}\theta(s)\right\|_q ds\\ &+\hat C_{\beta,\lambda_1}C_4\int_0^t e^{-\lambda_1(t-s)}(t-s)^{-\beta}\|A^\nu v(s)\|_p\|\nabla\phi(s)\|_\infty ds\\ &= \hat C_{\beta+\hat\delta,\lambda_1}C_2I_3+\hat C_{\beta,\lambda_1}C_4I.\end{aligned}$$

Therefore, from (2) we have

$$\|A^\alpha v(t)\|_p\le C_{\alpha-\gamma,\lambda_1}t^{\gamma-\alpha}e^{-\lambda_1t}\|A^\gamma v_0\|_p+C_{\alpha+\delta,\lambda_1}C_1I_1+C_{\alpha,\lambda_1}C_3\|g\|_\infty I_2 \tag{16}$$

and

$$\left\|B^\beta\theta(t)\right\|_q\le \hat C_{\beta-\hat\gamma,\lambda_1}t^{\hat\gamma-\beta}e^{-\lambda_1t}\left\|B^{\hat\gamma}\theta_0\right\|_q+\hat C_{\beta+\hat\delta,\lambda_1}C_2I_3+\hat C_{\beta,\lambda_1}C_4I. \tag{17}$$

Here, we set

$$E_\alpha(t)=t^{\alpha-\gamma}e^{\lambda t}\|A^\alpha v(t)\|_p, \tag{18}$$

and

$$\hat E_\beta(t)=t^{\beta-\hat\gamma}e^{\lambda t}\left\|B^\beta\theta(t)\right\|_q. \tag{19}$$

Let us estimate I_i, $i=1,2,3,4$ separately.

(a) Assume that $\alpha+\delta<1/2$ and $\beta+\hat\delta<1/2$.

Estimate of I_1 :

Clearly by the definition of $E_\mu(t)$ (see (18)) we can write

$$I_1=e^{-\lambda_1t}\int_0^t(t-s)^{-(\alpha+\delta)}s^{2\gamma-2\mu}e^{(\lambda_1-2\lambda)s}E_\mu^2(s)ds$$

and by the Cauchy-Schwarz inequality

$$I_1\le e^{-\lambda_1t}\left(\int_0^t(t-s)^{-2(\alpha+\delta)}s^{4(\gamma-\mu)}e^{2(\lambda_1-2\lambda)s}ds\right)^{\frac12}\left(\int_0^t E_\mu^4(s)ds\right)^{\frac12}. \tag{20}$$

At this stage two cases have to be discussed.

(i) If $\lambda_1 - 2\lambda < 0$ and $\mu - \gamma < 1/4$, then Lemma 1 and (20) imply that

$$I_1 \leq e^{-\lambda_1 t} K_1^{\frac{1}{2}} t^{-(\alpha+\delta)} \left(\int_0^t E_\mu^4(s) ds \right)^{\frac{1}{2}}$$

where $K_1 = K(1 - 2(\alpha + \delta), 1 + 2(k_1 + 2\gamma - 2\mu), 2\lambda - \lambda_1)$. The abbreviation is for space convenience.

(ii) If $\lambda_1 - 2\lambda \geq 0$ and $\mu - \gamma < 1/4$, then

$$\begin{aligned} I_1 &\leq e^{-\lambda_1 t} e^{(\lambda_1 - 2\lambda)t} \left(\int_0^t (t-s)^{-2(\alpha+\delta)} s^{4(\gamma-\mu)} ds \right)^{\frac{1}{2}} \left(\int_0^t E_\mu^4(s) ds \right)^{\frac{1}{2}} \\ &\leq \mathbf{B}_1^{\frac{1}{2}} e^{-2\lambda t} \left(t^{1-2(\alpha+\delta)+4(\gamma-\mu)} \right)^{\frac{1}{2}} \left(\int_0^t E_\mu^4(s) ds \right)^{\frac{1}{2}}, \end{aligned}$$

where the constant $\mathbf{B}_1 = \mathbf{B}(1 - 2(\alpha + \delta), 1 + 4(\gamma - \mu))$ and $\mathbf{B}$ is the well known beta function.

Estimate of I_2 :

According to the definition of I_2 and (19), we see that

$$I_2 = e^{-\lambda_1 t} \int_0^t (t-s)^{-\alpha} s^{\hat{\gamma}-\hat{\mu}} e^{(\lambda_1 - \lambda)s} \hat{E}_{\hat{\mu}}(s) ds.$$

As $\lambda_1 - \lambda > 0$, if $\hat{\mu} - \hat{\gamma} < 1/2$, we have by virtue of the Cauchy-Schwarz inequality and the definition of the beta function

$$\begin{aligned} I_2 &\leq e^{-\lambda_1 t} e^{(\lambda_1 - \lambda)t} \left(\int_0^t (t-s)^{-2\alpha} s^{2(\hat{\gamma}-\hat{\mu})} ds \right)^{\frac{1}{2}} \left(\int_0^t \hat{E}_{\hat{\mu}}^2(s) ds \right)^{\frac{1}{2}} \\ &\leq e^{-\lambda t} \mathbf{B}^{\frac{1}{2}} \left(1 - 2\alpha, 1 + 2(\hat{\gamma} - \hat{\mu})\right) \left(t^{1-2\alpha+2(\hat{\gamma}-\hat{\mu})} \right)^{\frac{1}{2}} \left(\int_0^t \hat{E}_{\hat{\mu}}^2(s) ds \right)^{\frac{1}{2}} \\ &\leq \mathbf{B}_2^{\frac{1}{2}} e^{-\lambda t} \left(t^{1-2\alpha+2(\hat{\gamma}-\hat{\mu})} \right)^{\frac{1}{2}} \left(\int_0^t \hat{E}_{\hat{\mu}}^2(s) ds \right)^{\frac{1}{2}}. \end{aligned}$$

Estimate of I_3 :

In account of the definitions (18) and (19) we may write

$$I_3 = e^{-\lambda_1 t} \int_0^t (t-s)^{-\beta-\hat{\delta}} s^{\gamma+\hat{\gamma}-\nu-\hat{\nu}} e^{(\lambda_1 - 2\lambda)s} \hat{E}_{\hat{\nu}}(s) E_\nu(s) ds.$$

(iii) If $\lambda_1 - 2\lambda < 0$ and $1 + 2(\gamma + \hat{\gamma} - \nu - \hat{\nu}) > 0$, then we may apply Lemma 1 to get

$$I_3 \leq e^{-\lambda_1 t} K^{\frac{1}{2}}(1-2(\beta+\hat{\delta}), 1+2(\gamma+\hat{\gamma}-\nu-\hat{\nu}), 2\lambda-\lambda_1)$$
$$\times t^{-(\beta+\hat{\delta})} \left(\int_0^t \hat{E}_{\hat{\nu}}^2(s) E_\nu^2(s) ds\right)^{\frac{1}{2}}$$
$$\leq K_3^{\frac{1}{2}} e^{-\lambda_1 t} t^{-(\beta+\hat{\delta})} \left(\int_0^t \hat{E}_{\hat{\nu}}^2(s) E_\nu^2(s) ds\right)^{\frac{1}{2}}.$$

(iv) If $\lambda_1 - 2\lambda \geq 0$ and $1+2(\gamma+\hat{\gamma}-\nu-\hat{\nu}) > 0$, then by the definition of the beta function

$$I_3 \leq e^{-\lambda_1 t} e^{(\lambda_1-2\lambda)t} \left(\int_0^t (t-s)^{-2(\beta+\hat{\delta})} s^{2(\gamma+\hat{\gamma}-\nu-\hat{\nu})} ds\right)^{\frac{1}{2}} \left(\int_0^t \hat{E}_{\hat{\nu}}^2(s) E_\nu^2(s) ds\right)^{\frac{1}{2}}$$
$$\leq \mathbf{B}_3^{\frac{1}{2}} e^{-2\lambda t} \left(t^{1-2(\beta+\hat{\delta})+2(\gamma+\hat{\gamma}-\nu-\hat{\nu})}\right)^{\frac{1}{2}} \left(\int_0^t \hat{E}_{\hat{\nu}}^2(s) E_\nu^2(s) ds\right)^{\frac{1}{2}}$$

where $\mathbf{B}_3 = \mathbf{B}\left(1-2(\beta+\hat{\delta}), 1+2(\gamma+\hat{\gamma}-\nu-\hat{\nu})\right)$.

Estimate of I_4 :

We have

$$I_4 = e^{-\lambda_1 t} \int_0^t (t-s)^{-\beta} s^{\gamma-\nu} e^{(\lambda_1-\lambda)s} E_\nu(s) \left\|\nabla\phi(s)\right\|_\infty ds.$$

If $\omega < \frac{1}{2}+\gamma-\nu$ (note here that $\beta < 1/2$ since we assumed $\beta+\hat{\delta} < 1/2$) then

$$I_4 \leq C e^{-\lambda_1 t} e^{(\lambda_1-\lambda)t} \left(\int_0^t (t-s)^{-2\beta} s^{2(\gamma-\nu-\omega)} ds\right)^{\frac{1}{2}} \left(\int_0^t E_\nu^2(s) ds\right)^{\frac{1}{2}}$$
$$\leq C e^{-\lambda t} \mathbf{B}^{\frac{1}{2}}(1-2\beta, 1+2(\gamma-\nu)) \left(t^{1-2\beta+2(\gamma-\nu-\omega)}\right)^{\frac{1}{2}} \left(\int_0^t E_\nu^2(s) ds\right)^{\frac{1}{2}}$$
$$\leq C \mathbf{B}_4^{\frac{1}{2}} e^{-\lambda t} \left(t^{1-2\beta+2(\gamma-\nu-\omega)}\right)^{\frac{1}{2}} \left(\int_0^t E_\nu^2(s) ds\right)^{\frac{1}{2}}.$$

From the above estimates and (16) and (17) we infer that

$$t^{\alpha-\gamma} e^{\lambda t} \left\|A^\alpha v(t)\right\|_p \leq C_{\alpha-\gamma,\lambda_1} e^{-(\lambda_1-\lambda)t} \left\|A^\gamma v_0\right\|_p$$
$$+M_1 e^{-\lambda t} t^{\frac{1}{2}-\delta+\gamma-2\mu} \left(\int_0^t E_\mu^4(s) ds\right)^{\frac{1}{2}} \left[or\ M_1 e^{-(\lambda_1-\lambda)t} t^{-\gamma-\delta} \left(\int_0^t E_\mu^4(s) ds\right)^{\frac{1}{2}} \right] \quad (21)$$
$$+M_2 t^{\frac{1}{2}+\hat{\gamma}-\hat{\mu}-\gamma} \left(\int_0^t \hat{E}_{\hat{\mu}}^2(s) ds\right)^{\frac{1}{2}},$$

according to the cases (i) or (ii), and

$$
\begin{aligned}
t^{\beta-\hat{\gamma}} e^{\lambda t} \left\| B^{\beta}\theta(t) \right\|_q \le \hat{C}_{\beta-\hat{\gamma},\lambda_1} e^{(\lambda-\lambda_1)t} \left\| B^{\hat{\gamma}}\theta_0 \right\|_q \\
+ M_3 e^{-(\lambda_1-\lambda)t} t^{-\hat{\gamma}-\hat{\delta}} \left(\int_0^t \hat{E}_{\hat{\nu}}^2(s) E_{\nu}^2(s) ds \right)^{\frac{1}{2}} \\
\left[or\ M_3 e^{-\lambda t} t^{\frac{1}{2}-\hat{\delta}+\gamma-\nu-\hat{\nu}} \left(\int_0^t \hat{E}_{\hat{\nu}}^2(s) E_{\nu}^2(s) ds \right)^{\frac{1}{2}} \right] \\
+ M_4 t^{\frac{1}{2}-\omega+\gamma-\nu-\hat{\gamma}} \left(\int_0^t E_{\nu}^2(s) ds \right)^{\frac{1}{2}},
\end{aligned} \tag{22}
$$

according to the cases (iii) or (iv). Here M_i, $i = 1, 2, 3, 4$ are positive constants. Therefore, if $\frac{1}{2} + \hat{\gamma} - \hat{\mu} - \gamma \le 0$, $\frac{1}{2} - \omega + \gamma - \nu - \hat{\gamma} \le 0$ (which is true for all $\omega \ge 0$ if $1 + \gamma - \hat{\gamma} > \nu \ge \frac{1}{2} + \gamma - \hat{\gamma}$. If $\nu < \frac{1}{2} + \gamma - \hat{\gamma}$ then it is true for $\omega \ge \frac{1}{2} + \gamma - \hat{\gamma} - \nu$ and with the first condition on ω we will have $\frac{1}{2} + \gamma - \hat{\gamma} - \nu \le \omega < \frac{1}{2} + \gamma - \nu$). By definitions (18), (19), the relations (21) and (22) imply that

$$
E_{\alpha}(t) \le C \left\{ \|A^{\gamma} v_0\|_p + \left(\int_0^t E_{\mu}^4(s) ds \right)^{\frac{1}{2}} + \left(\int_0^t \hat{E}_{\hat{\mu}}^2(s) ds \right)^{\frac{1}{2}} \right\}
$$

and

$$
\hat{E}_{\beta}(t) \le C \left\{ \left\| B^{\hat{\gamma}} \theta_0 \right\|_q + \left(\int_0^t \hat{E}_{\hat{\nu}}^2(s) E_{\nu}^2(s) ds \right)^{\frac{1}{2}} + \left(\int_0^t E_{\nu}^2(s) ds \right)^{\frac{1}{2}} \right\}
$$

for a positive constant C and for all $t \ge a > 0$. Using the algebraic inequality

$$
(a+b+c)^2 \le 3(a^2+b^2+c^2) \tag{23}
$$

we can write

$$
E_{\alpha}^2(t) \le C \left\{ \|A^{\gamma} v_0\|_p^2 + \int_0^t E_{\mu}^4(s) ds + \int_0^t \hat{E}_{\hat{\mu}}^2(s) ds \right\}
$$

and

$$
\hat{E}_{\beta}^2(t) \le C \left\{ \left\| B^{\hat{\gamma}} \theta_0 \right\|_q^2 + \int_0^t \hat{E}_{\hat{\nu}}^2(s) E_{\nu}^2(s) ds + \int_0^t E_{\nu}^2(s) ds \right\}
$$

for all $t \ge a > 0$. We set

$$
F(t) = \max \left\{ E_{\alpha}^2(t), E_{\beta}^2(t) E_{\mu}^2(t), E_{\nu}^2(t), E_{\hat{\mu}}^2(t), E_{\hat{\nu}}^2(t) \right\}.
$$

Then, by the embedding properties of $D(A^{\alpha})$ and $D(B^{\beta})$ we see that

$$
F(t) \le H \left\{ \|A^{\gamma} v_0\|_p^2 + \left\| B^{\hat{\gamma}} \theta_0 \right\|_p^2 + \int_0^t (F(s) + F(s)^2) ds \right\}
$$

for all $t \ge a > 0$ and some positive constant H which depends on the previous constants and parameters.

Applying Lemma 2 with $\omega(\sigma) = \sigma + \sigma^2$ we find

$$F(t) \le W^{-1}\left\{W\left[H\left(\|A^{\gamma}v_0\|_p^2 + \left\|B^{\hat{\gamma}}\theta_0\right\|_q^2\right)\right] + Ht\right\}.$$

Here $W(v) = \displaystyle\int_{v_0}^{v} \frac{d\sigma}{\omega(\sigma)} = \log\left(C_0 \frac{v}{v+1}\right)$ with $C_0 = \frac{v_0+1}{v_0}$ and $W^{-1}(z) = \frac{e^z}{C_0 - e^z}$. We obtain

$$F(t) \le \frac{v}{v+1} e^{Ht} \left\{1 - \frac{v}{v+1} e^{Ht}\right\}^{-1}$$

where $v = H\left(\|A^{\gamma}v_0\|_p^2 + \left\|B^{\hat{\gamma}}\theta_0\right\|_q^2\right)$. Thus $F(t)$ is bounded on any interval $[a, T]$ for all $a > 0$ provided that

$$HT < \log\left\{1 + \left(H\|A^{\gamma}v_0\|_p^2 + H\left\|B^{\hat{\gamma}}\theta_0\right\|_q^2\right)^{-1}\right\}.$$

Observe that T is small when $\|A^{\gamma}v_0\|_p^2 + \left\|B^{\hat{\gamma}}\theta_0\right\|_q^2$ is large and vice-versa it is large when the size of the initial data is small .

(b) Assume that $\alpha + \delta \ge 1/2$ and $\beta + \hat{\delta} \ge 1/2$. Let $y = \frac{1-(\alpha+\delta)}{\alpha+\delta}$ and $z = \frac{1-(\beta+\hat{\delta})}{\beta+\hat{\delta}}$. We use Hölder inequality instead of the Cauchy-Schwarz inequality everywhere we used this latter inequality in the proof of (a). Let us consider for instance the estimate of I_1, the first case (i). We have

$$\begin{aligned} I_1 &\le e^{-\lambda_1 t}\left(\int_0^t (t-s)^{-r(\alpha+\delta)} s^{2r(\gamma-\mu)} e^{r(\lambda_1 - 2\lambda)s} ds\right)^{\frac{1}{r}} \\ &\qquad \times \left(\int_0^t E_\mu^{2r^*}(s) ds\right)^{\frac{1}{r^*}} \\ &\le K_1^{\frac{1}{r}} e^{-\lambda_1 t} t^{-(\alpha+\delta)} \left(\int_0^t E_\mu^{2r^*}(s) ds\right)^{\frac{1}{r^*}}. \end{aligned}$$

The last inequality is justified since, if $r^* = \frac{2\xi+1}{\xi}$ then $r = \frac{2\xi+1}{\xi+1}$. Hence

$$1 - r(\alpha+\delta) = 1 - \frac{2\xi+1}{\xi+1}\frac{1}{y+1} \ge 1 - \frac{2y+1}{y+1}\frac{1}{y+1} = \frac{y^2}{(y+1)^2} > 0.$$

In addition, we need $\mu - \gamma < 1/2r$. In the estimate of I_3 we will need $1 - r(\beta+\hat{\delta}) > 0$. This also holds since

$$1 - r(\beta+\hat{\delta}) = 1 - \frac{2\xi+1}{\xi+1}\frac{1}{z+1} \ge 1 - \frac{2z+1}{z+1}\frac{1}{z+1} = \frac{z^2}{(z+1)^2} > 0.$$

The rest of the proof is similar to the proof of part (a). We will use the inequality

$$(a+b+c)^{r^*} \le 3^{r^*-1}(a^{r^*} + b^{r^*} + c^{r^*}),$$

instead of (23).

(c) The cases $\alpha + \delta \ge 1/2$, $\beta + \hat{\delta} < 1/2$ and $\alpha + \delta < 1/2$, $\beta + \hat{\delta} \ge 1/2$ can be treated in exactly the same manner. □

Our next theorem will show that if $\phi(t)$ satisfies a slightly stronger condition, that is $\phi(t)$ is supposed to be in a bit smaller class than the one considered in the previous theorem, namely

$$\|\nabla \phi(t)\|_\infty = O\left((t+1)^{-\omega}\right), \text{ as } t \to +\infty$$

then we get a polynomial decay rate of an arbitrary (positive) order for any value of $\omega \geq 0$.

Theorem 4. *Let the hypotheses of Theorem 1 hold. Suppose that*

$$\|\nabla \phi(t)\|_\infty = O\left((t+1)^{-\omega}\right), \text{ as } t \to +\infty$$

for some positive constant $\omega \geq 0$. Then the conclusion of Theorem 3 holds for any $\sigma > 0$ with

$$\|A^\alpha v(t)\|_p \leq C(t+1)^{-\sigma}\left(\|A^\gamma v_0\|_p^{r^*} + \left\|B^{\hat\gamma}\theta_0\right\|_q^{r^*}\right),$$
$$\left\|B^\beta \theta(t)\right\|_q \leq \hat{C}(t+1)^{-\sigma}\left(\|A^\gamma v_0\|_p^{r^*} + \left\|B^{\hat\gamma}\theta_0\right\|_q^{r^*}\right)$$

for all $t \geq a > 0$.

Proof. Let us define

$$E_\alpha(t) = (t+1)^\sigma \|A^\alpha v(t)\|_p,$$

and

$$\hat{E}_\beta(t) = (t+1)^\sigma \left\|B^\beta \theta(t)\right\|_q.$$

We will proceed as in the proof of Theorem 3.

Estimate of I_1 :

From the definition of I_1 (in proof of Theorem 3) we see that in case (a) we find

$$I_1 \leq \int_0^t (t-s)^{-(\alpha+\delta)}(s+1)^{-2\sigma} e^{-\lambda_1(t-s)} E_\mu^2(s) ds.$$

Using the Cauchy-Schwarz inequality and Lemma 5 we infer

$$I_1 \leq \left(\int_0^t (t-s)^{-2(\alpha+\delta)} e^{-2\lambda_1(t-s)}(s+1)^{-4\sigma} ds\right)^{\frac{1}{2}} \left(\int_0^t E_\mu^4(s) ds\right)^{\frac{1}{2}}$$
$$\leq L_1^{1/2}(2(\alpha+\delta), 2\lambda_1, 4\sigma, 1)(t+1)^{-2\sigma}\left(\int_0^t E_\mu^4(s) ds\right)^{\frac{1}{2}},$$

provided that $1-2(\alpha+\delta) > 0$ and $\sigma > 0$.

Estimate of I_2 :

In a similar manner we obtain

$$I_2 \leq \int_0^t (t-s)^{-\alpha}(s+1)^{-\sigma}e^{-\lambda_1(t-s)}\hat{E}_{\hat{\mu}}(s)ds$$
$$\leq L_2^{1/2}(2\alpha, 2\lambda_1, 2\sigma, 1)(t+1)^{-\sigma}\left(\int_0^t \hat{E}_{\hat{\mu}}^2(s)ds\right)^{\frac{1}{2}},$$

provided that $1-2\alpha > 0$ and $\sigma > 0$.

Estimate of I_3 :

Again following the steps in the estimate of I_1, we find

$$I_3 \leq \int_0^t (t-s)^{-(\beta+\hat{\delta})}(s+1)^{-2\sigma}e^{-\lambda_1(t-s)}E_\nu(s)\hat{E}_{\hat{\nu}}(s)ds$$
$$\leq L_3^{1/2}(2(\beta+\hat{\delta}), 2\lambda_1, 4\sigma, 1)(t+1)^{-2\sigma}\left(\int_0^t E_\nu^2(s)\hat{E}_{\hat{\nu}}^2(s)ds\right)^{\frac{1}{2}},$$

provided that $1-2(\beta+\hat{\delta}) > 0$ and $\sigma > 0$.

Estimate of I_4 :

Finally, we get as estimate for I_4

$$I_4 \leq \int_0^t (t-s)^{-\beta}(s+1)^{-\sigma}e^{-\lambda_1(t-s)}\left\|\nabla\phi(s)\right\|_\infty E_\nu(s)ds$$
$$\leq \hat{C}L_4^{1/2}(2\beta, 2\lambda_1, 2(\omega+\sigma), 1)(t+1)^{-\sigma-\omega}\left(\int_0^t E_\nu^2(s)ds\right)^{\frac{1}{2}},$$

provided that $1-2\beta > 0$ and $\omega+\sigma > 0$.

Note that all the above estimates are justified by the assumptions in the hypotheses. Multiplying the relations (16) and (17) by $(t+1)^\sigma$ and using the above new estimates for I_i, $i = 1, 2, 3, 4$, we obtain

$$(t+1)^\sigma \left\|A^\alpha v(t)\right\|_p \leq C_{\alpha-\gamma,\lambda_1}e^{-\lambda_1 t}(t+1)^\sigma t^{\gamma-\alpha}\left\|A^\gamma v_0\right\|_p$$
$$+L_1^{1/2}C_{\alpha+\delta,\lambda_1}C_1(t+1)^{-\sigma}\left(\int_0^t E_\mu^4(s)ds\right)^{\frac{1}{2}}$$
$$+L_2^{1/2}C_{\alpha,\lambda_1}C_3\left\|g\right\|_\infty\left(\int_0^t \hat{E}_\mu^2(s)ds\right)^{\frac{1}{2}}$$

and

$$(t+1)^\sigma \left\|B^\beta\theta(t)\right\|_q \leq \hat{C}_{\beta-\hat{\gamma},\lambda_1}e^{-\lambda_1 t}(t+1)^\sigma t^{\hat{\gamma}-\beta}\left\|B^{\hat{\gamma}}\theta_0\right\|_q$$
$$+\hat{C}_{\beta+\hat{\delta},\lambda_1}C_2(t+1)^{-\sigma}L_3^{1/2}\left(\int_0^t \hat{E}_{\hat{\nu}}^2(s)E_\nu^2(s)ds\right)^{\frac{1}{2}}$$
$$+\hat{C}_{\beta,\lambda_1}C_4(t+1)^{-\omega}L_4^{1/2}\left(\int_0^t E_\nu^2(s)ds\right)^{\frac{1}{2}}$$

for all $t \geq a > 0$. The rest of the proof is similar to that of Theorem 3. □

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