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Multiple solutions with changing sign energy to a nonlinear elliptic equation

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Abstract

In this paper, the existence of multiple solutions to a nonlinear elliptic equation with a parameter λ is studied. Initially, the existence of two nonnegative solutions is showed for $0 < \lambda < \widehat{\lambda}$. The first solution has a negative energy while the energy of the second one is positive for $0 < \lambda < \lambda_0$ and negative for $\lambda_0 < \lambda < \widehat{\lambda}$. The values λ_0 and $\widehat{\lambda}$ are given under variational form and we show that every corresponding critical point is solution of the nonlinear elliptic problem (with a suitable multiplicative term). Finally, the existence of two classes of infinitely many solutions is showed via the Lusternik-Schnirelman theory.

Keywords: Ekeland's principle, p-Laplacian operator, Palais-Smale condition, Lusternik-Schnirelman theory

1 Introduction

In this paper, we deal with the existence of multiple solutions to the mixed boundary value problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{r-2}u \ \text{ in } \Omega, \\ \\ \varepsilon |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + a(x)|u|^{p-2}u = 0, \ \text{ on } \partial\Omega, \end{cases} \tag{1}$$

with respect to the real parameter λ. Here, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$, Δ_p is the p-Laplacian, $\frac{\partial}{\partial \nu}$ is the outer normal derivative and $\varepsilon \in \{0, 1\}$. Throughout this paper, the function a is assumed to be in $L^\infty(\partial\Omega)$, $a(s) \geq a_0 > 0$ for every s in $\partial\Omega$, and

$$\lambda > 0, \ \ 1 < q < p < r < p^*, \text{ where } \ p^* = \begin{cases} \frac{Np}{N-p} & \text{if } \ p < N, \\ +\infty & \text{if } \ p \geq N. \end{cases}$$

When the parameter $\varepsilon = 0$ (resp. $\varepsilon = 1$), we deal with a Dirichlet (resp. mixed) boundary value problem which is posed in the framework of the Sobolev space $W = W_0^{1,p}(\Omega)$ (resp.

$W = W^{1,p}(\Omega)$). To take into account these two situations, the space W will be endowed with the norm

$$||u|| = \left(\int_\Omega |\nabla u|^p dx + \varepsilon \int_{\partial\Omega} a(s)\, |u|^p \ ds \right)^{1/p},$$

which gives to W the structure of Banach space.

For solutions of (1) we understand critical points of the associated Euler-Lagrange (energy) functional $E_\lambda \in C^1(W)$, given by

$$E_\lambda(u) = \frac{1}{p}P(u) - \frac{\lambda}{q}Q(u) - \frac{1}{r}R(u),$$

where

$$P(u) = ||u||^p, \ \ Q(u) = \int_\Omega |u|^q dx \ \text{ and } \ R(u) = \int_\Omega |u|^r dx.$$

Such kind of problems with combined concave and convex nonlinearities were studied recently by several authors [1, 2, 3, 4, 10, 11, 20], we can refer the reader to the valuable survey [5]. Our main results here can be summarized as follows:

• First, we find two characteristic values λ_0 and $\widehat{\lambda}$ ($\lambda_0 < \widehat{\lambda}$) under *variational form, i.e.*

$$\lambda_0 = C_0(p,q,r) \inf_{u \in W\setminus\{0\}} \mathcal{F}(u) \ \text{ and } \ \widehat{\lambda} = \widehat{C}(p,q,r) \inf_{u \in W\setminus\{0\}} \mathcal{F}(u), \tag{2}$$

such that two branches of nonnegative solutions to (1) exist for $\lambda \in]0, \widehat{\lambda}[$ (the functional $\mathcal{F}$ will be given below). Moreover, the energy of the first nonnegative solution is negative for $\lambda \in]0, \widehat{\lambda}[$ while the energy of the second nonnegative solution changes sign at λ_0, *i.e.* it is positive for $\lambda \in]0, \lambda_0[$ and negative for $\lambda \in]\lambda_0, \widehat{\lambda}[$. Notice that these two nonnegative solutions are found simultaneously and that our approach does not use the mountain-pass lemma.

• On the other hand, we show that every solution of (2) is a solution of the problem (1) (with a suitable multiplicative term). This second point lets expect that the *first nonlinear eigenvalue* Λ of (1), *i.e.*

$$\Lambda := \sup\{\lambda > 0 : \ (1) \text{ has a nonnegative solution}\}$$

may satisfy a variational problem similar to (2) (see [3] for $p = 2$, $\varepsilon = 0$ and [11] for $p > 1$, $\varepsilon = 0$). Let us precise that $\widehat{\lambda}$ coincides with Λ when $q \longrightarrow p$ and that $\widehat{\lambda}$ constitutes a good minoration of Λ in the general case $1 < q < p$.

In the sequel, $||\ ||_q$ and $||\ ||_r$ will denote the norms on $L^q(\Omega)$ and $L^r(\Omega)$ respectively. As in [15, 26], we introduce the modified Euler-Lagrange functional $\widetilde{E}_\lambda$ defined on $\mathbb{R} \times W$ by

$$\widetilde{E}_\lambda(t,u) := E_\lambda(tu).$$

If u is an arbitrary element of W, $\partial_t \widetilde{E}_\lambda(.,u)$ (resp. $\partial_{tt}\widetilde{E}_\lambda(.,u)$) are the first (resp. second) derivative of the real valued function: $t \mapsto \widetilde{E}_\lambda(t,u)$.

2 Preliminary results

Since the functional $\widetilde{E}_\lambda$ is even in t and that we are interested by the nontrivial solutions of (1), we limit our study for $t > 0$ and $u \in W \setminus \{0\}$.

Lemma 1. *For every $u \in W \setminus \{0\}$, there is a unique $\lambda(u) > 0$ such that the real valued function $t \mapsto \partial_t \widetilde{E}_\lambda(t,u)$ has exactly two positive zeros* (resp. *one positive zero*) *if* $0 < \lambda < \lambda(u)$ (resp. $\lambda = \lambda(u)$). *This function has no zero for $\lambda > \lambda(u)$.*

Proof. Let u be an arbitrary element of $W \setminus \{0\}$ and let us write

$$\partial_t \widetilde{E}_\lambda(t,u) = t^{q-1} \widetilde{F}_\lambda(t,u), \text{ where } \widetilde{F}_\lambda(t,u) = t^{p-q} P(u) - \lambda Q(u) - t^{r-q} R(u).$$

Then

$$\partial_{tt} \widetilde{E}_\lambda(t,u) = (q-1) t^{q-2} \widetilde{F}_\lambda(t,u) + t^{q-1} \partial_t \widetilde{F}_\lambda(t,u),$$

holds true, with

$$\partial_t \widetilde{F}_\lambda(t,u) = t^{p-q-1} \left\{ (p-q) P(u) - (r-q) t^{r-p} R(u) \right\}.$$

It is clear that the real valued function $t \longmapsto \widetilde{F}_\lambda(t,u)$ is increasing on $]0, t(u)[$, decreasing on $]t(u), +\infty[$ and attains its unique maximum for $t = t(u)$, where

$$t(u) = \left(\frac{p-q}{r-q} \frac{P(u)}{R(u)} \right)^{\frac{1}{r-p}}. \tag{3}$$

Thus, the function $t \longmapsto \widetilde{F}_\lambda(t,u)$ has two positive zeros (resp. one positive zero) if $\widetilde{F}_\lambda(t(u),u) > 0$ (resp. if $\widetilde{F}_\lambda(t(u),u) = 0$) and has no zero if $\widetilde{F}_\lambda(t(u),u) < 0$. On the other hand, a direct computation gives

$$\widetilde{F}_\lambda(t(u),u) = \frac{r-p}{p-q} \left(\frac{p-q}{r-q} \frac{P(u)}{R(u)} \right)^{\frac{r-q}{r-p}} R(u) - \lambda Q(u).$$

Similarly, $\widetilde{F}_\lambda(t(u),u) > 0$ (resp. $\widetilde{F}_\lambda(t(u),u) < 0$) if $\lambda < \lambda(u)$ (resp. $\lambda > \lambda(u)$) and $\widetilde{F}_{\lambda(u)}(t(u),u) = 0$, where

$$\lambda(u) = \widehat{C} \frac{P^{\frac{r-q}{r-p}}(u)}{Q(u)\, R^{\frac{p-q}{r-p}}(u)}, \tag{4}$$

with

$$\widehat{C} = \frac{r-p}{p-q} \left(\frac{p-q}{r-q} \right)^{\frac{r-q}{r-p}}.$$

Hence, if $\lambda \in]0, \lambda(u)[$, the real valued function $t \longmapsto \partial_t \widetilde{E}_\lambda(t,u)$ has two positive zeros, denoted by $\underline{t}(u,\lambda)$ and $\overline{t}(u,\lambda)$, verifying $0 < \underline{t}(u,\lambda) < t(u) < \overline{t}(u,\lambda)$.
Since, $\widetilde{F}_\lambda(\underline{t}(u,\lambda),u) = \widetilde{F}_\lambda(\overline{t}(u,\lambda),u) = 0$, $\partial_t \widetilde{F}_\lambda(t,u) > 0$ for $t < t(u)$ and $\partial_t \widetilde{F}_\lambda(t,u) < 0$ for $t > t(u)$, it follows that

$$\partial_{tt} \widetilde{E}_\lambda(\underline{t}(u,\lambda),u) > 0 \text{ and } \partial_{tt} \widetilde{E}_\lambda(\overline{t}(u,\lambda),u) < 0.$$

This means that the real valued function $t \longmapsto \widetilde{E}_\lambda(t,u)$, $t>0$, achieves its unique local minimum at $t = \underline{t}(u,\lambda)$ and its global maximum at $t = \overline{t}(u,\lambda)$. $\square$

Let us precise that for every $u \in W \setminus \{0\}$ and $\lambda \in]0, \lambda(u)[$, $\underline{t}(u,\lambda)u$ and $\overline{t}(u,\lambda)u$ belong to the Nehari manifold [19, 26] defined by

$$\mathcal{N} := \{u \in W \setminus \{0\} \,:\, E'_\lambda(u)u = 0\}.$$

Now, we introduce

$$\widehat{\lambda} := \inf_{u \in W\setminus\{0\}} \lambda(u). \tag{5}$$

If $c_q(\Omega)$ (resp. $c_r(\Omega)$) denotes the best Sobolev constant of the embedding $W \subset L^q(\Omega)$ (resp. $W \subset L^r(\Omega)$), then

$$\widehat{\lambda} \geq \widehat{C}[c_q(\Omega)]^{q/p}[c_r(\Omega)]^{\frac{r(p-q)}{p(r-p)}} > 0.$$

Since $\partial_t \widetilde{E}_\lambda(\underline{t}(u,\lambda),u) = 0$ (resp. $\partial_t \widetilde{E}_\lambda(\overline{t}(u,\lambda),u) = 0$) for every $u \in W \setminus \{0\}$, it follows that the functional $u \longmapsto \widetilde{E}_\lambda(\underline{t}(u,\lambda),u)$ (resp. $u \longmapsto \widetilde{E}_\lambda(\overline{t}(u,\lambda),u)$) is coercive on $W\setminus\{0\}$. Thus, for every $\lambda \in]0, \widehat{\lambda}[$, we define

$$\underline{\alpha}(\lambda) = \inf_{u \in W\setminus\{0\}} \widetilde{E}_\lambda(\underline{t}(u,\lambda),u), \tag{6}$$

$$\overline{\alpha}(\lambda) = \inf_{u \in W\setminus\{0\}} \widetilde{E}_\lambda(\overline{t}(u,\lambda),u). \tag{7}$$

Lemma 2. *Let $(u_n) \subset W \setminus \{0\}$ be a minimizing sequence of* (6) (resp. *of* (7)) *and $\underline{U}_n := \underline{t}(u_n,\lambda)u_n$* (resp. $\overline{U}_n := \overline{t}(u_n,\lambda)u_n$). *Then*

$$\text{(i)} \quad \limsup_{n\to+\infty} ||\underline{U}_n|| < +\infty \quad (\text{resp.} \quad \limsup_{n\to+\infty} ||\overline{U}_n|| < +\infty),$$

$$\text{(ii)} \quad \liminf_{n\to+\infty} ||\underline{U}_n|| > 0 \quad (\text{resp.} \quad \liminf_{n\to+\infty} ||\overline{U}_n|| > 0).$$

Proof.
(i) Let $(u_n) \subset W \setminus \{0\}$ be a minimizing sequence of (6). Since $\partial_t \widetilde{E}_\lambda(\underline{t}(u_n,\lambda),u_n) = 0$, it follows that

$$||\underline{U}_n||^p = \lambda ||\underline{U}_n||_q^q + ||\underline{U}_n||_r^r. \tag{8}$$

Similarly, since $\partial_{tt} \widetilde{E}_\lambda(\underline{t}(u_n,\lambda),u_n) > 0$, it follows that

$$(p-1)||\underline{U}_n||^p - \lambda(q-1)||\underline{U}_n||_q^q - (r-1)||\underline{U}_n||_r^r > 0. \tag{9}$$

Combining (8) and (9), we get $E_\lambda(\underline{U}_n) < 0$, for every n.

Suppose that there is a subsequence of $(\underline{U}_n)$, still denoted by $(\underline{U}_n)$ such that $\lim\limits_{n\to+\infty} ||\underline{U}_n|| = +\infty$. It is well known that there is some constant C such that $||\underline{U}_n||_q \leq C||\underline{U}_n||_r$ for every n, then $\lim\limits_{n\to+\infty} ||\underline{U}_n||_r = +\infty$. Using the fact that $0<q<r$ we get $||\underline{U}_n||_q^q = \mathrm{o}_n\left(||\underline{U}_n||_r^r\right)$, and consequently

$$||\underline{U}_n||^p = ||\underline{U}_n||_r^r(1 + \mathrm{o}_n(1)).$$

Thus,

$$E_\lambda(\underline{U}_n) = ||\underline{U}_n||_r^r \left(\frac{1}{p} - \frac{1}{r} + \mathrm{o}_n(1)\right),$$

which implies that $E_\lambda(\underline{U}_n)$ tends to $+\infty$ as n goes to $+\infty$ and this is impossible. Hence, we conclude that $\limsup\limits_{n\to+\infty} ||\underline{U}_n|| < +\infty$.

The same arguments with a minimizing sequence (u_n) of (7) show that $\limsup\limits_{n\to+\infty} ||\overline{U}_n|| < +\infty$.

(ii) Let $(u_n) \subset W \setminus \{0\}$ be a minimizing sequence of (6) and suppose that there is a subsequence of $(\underline{U}_n)$, still denoted by $(\underline{U}_n)$ such that $\lim\limits_{n\to+\infty} ||\underline{U}_n|| = 0$. It follows that $\lim\limits_{n\to+\infty} E_\lambda(\underline{U}_n) = 0$ *i.e.* $\underline{\alpha}(\lambda) = 0$, which is impossible since $\widetilde{E}_\lambda(\underline{t}(u_n, \lambda), u_n) < 0$ for every n.

Let $(u_n) \subset W \setminus \{0\}$ be a minimizing sequence of (7). Since $\partial_t \widetilde{E}_\lambda(\overline{t}(u_n, \lambda), u_n) = 0$ and $\partial_{tt} \widetilde{E}_\lambda(\overline{t}(u_n, \lambda), u_n) < 0$ it follows that

$$\begin{cases} ||\overline{U}_n||^p - \lambda ||\overline{U}_n||_q^q - ||\overline{U}_n||_r^r = 0, \\ (p-1)||\overline{U}_n||^p - \lambda(q-1)||\overline{U}_n||_q^q - (r-1)||\overline{U}_n||_r^r < 0. \end{cases}$$

Combining the two last inequalities we obtain, for every n

$$(p-q)||\overline{U}_n||^p < (r-q)||\overline{U}_n||_r^r \le C'||\overline{U}_n||^r,$$

via the continuous embedding $W \subset L^r(\Omega)$. Then $(p-q) \le C'||\overline{U}_n||^{r-p}$. Now, suppose that there is a subsequence of $(\overline{U}_n)$, still denoted by $(\overline{U}_n)$ such that $\lim\limits_{n\to+\infty} ||\overline{U}_n|| = 0$. This implies that $p - q \le 0$, which is impossible. $\square$

Remark 1. *Since, for every real number $\gamma > 0$, we have*

$$\widetilde{E}_\lambda\left(\gamma t, \frac{u}{\gamma}\right) = \widetilde{E}_\lambda(t, u),$$

$$\partial_t \widetilde{E}_\lambda\left(\gamma t, \frac{u}{\gamma}\right) = \frac{1}{\gamma}\partial_t \widetilde{E}_\lambda(t, u),$$

$$\partial_{tt} \widetilde{E}_\lambda\left(\gamma t, \frac{u}{\gamma}\right) = \frac{1}{\gamma^2}\partial_{tt} \widetilde{E}_\lambda(t, u),$$

it follows that

$$\underline{t}(u, \lambda) = \frac{1}{\gamma}\underline{t}\left(\frac{u}{\gamma}, \lambda\right), \tag{10}$$

$$\overline{t}(u, \lambda) = \frac{1}{\gamma}\overline{t}\left(\frac{u}{\gamma}, \lambda\right). \tag{11}$$

Hence, we conclude that

$$\underline{\alpha}(\lambda) = \inf_{u\in\mathbb{S}} \widetilde{E}_\lambda(\underline{t}(u, \lambda), u), \tag{12}$$

$$\overline{\alpha}(\lambda) = \inf_{u\in\mathbb{S}} \widetilde{E}_\lambda(\overline{t}(u, \lambda), u), \tag{13}$$

where $\mathbb{S}$ is the unit sphere of W.

Lemma 3. *Let* $(u_n) \subset \mathbb{S}$ *be a minimizing sequence of* (12) (resp. *of* (13)). *Then,* $(\underline{U}_n) := (\underline{t}(u_n,\lambda)u_n)$ (resp. $(\overline{U}_n) := (\overline{t}(u_n,\lambda)u_n)$) *are Palais-Smale sequences for the functional* E_λ.

Proof. We will show this lemma only for the sequence $(\underline{U}_n)$, the proof for $(\overline{U}_n)$ can be done in the same way.
First, according to the previous lemma, it is clear that $(\underline{U}_n)$ is bounded in W. On the other hand, notice that for every $u \in W \setminus \{0\}$ and $\lambda \in]0, \widehat{\lambda}[$, we have $\partial_t \widetilde{E}_\lambda(\underline{t}(u,\lambda), u) = 0$ and $\partial_{tt}\widetilde{E}_\lambda(\underline{t}(u,\lambda), u) \neq 0$. The implicit function theorem implies that $\underline{t}(u,\lambda)$ is C^1 with respect to u since $\widetilde{E}$ is. Let us introduce the C^1 functional $\underline{\mathcal{E}}_\lambda$ defined on $\mathbb{S}$ by

$$\underline{\mathcal{E}}_\lambda(u) = \widetilde{E}_\lambda(\underline{t}(u,\lambda), u) \equiv E_\lambda(\underline{t}(u,\lambda)u).$$

Then

$$\underline{\alpha}(\lambda) = \inf_{u\in\mathbb{S}} \underline{\mathcal{E}}_\lambda(u) \text{ and } \lim_{n\to+\infty} \underline{\mathcal{E}}_\lambda(u_n) = \underline{\alpha}(\lambda).$$

Using the Ekeland variational principle on the complete manifold $(\mathbb{S}, ||\ ||)$ to the functional $\underline{\mathcal{E}}_\lambda$, we conclude that

$$|\underline{\mathcal{E}}'_\lambda(u_n)(\varphi_n)| \leq \frac{1}{n}||\varphi_n||, \text{ for every } \varphi_n \in \mathrm{T}_{u_n}\mathbb{S},$$

where $\mathrm{T}_{u_n}\mathbb{S}$ is the tangent space to $\mathbb{S}$ at the point u_n. Moreoever, for every $\varphi_n \in \mathrm{T}_{u_n}\mathbb{S}$, one has

$$\begin{aligned}\underline{\mathcal{E}}'_\lambda(u_n)(\varphi_n) &= \partial_t\widetilde{E}_\lambda(\underline{t}(u_n,\lambda),u_n)\underline{t}'(u_n,\lambda)(\varphi_n) + \partial_u\widetilde{E}_\lambda(\underline{t}(u_n,\lambda),u_n)(\varphi_n),\\ &= \partial_u\widetilde{E}_\lambda(\underline{t}(u_n,\lambda),u_n)(\varphi_n),\end{aligned}$$

since $\partial_t\widetilde{E}_\lambda(\underline{t}(u_n,\lambda),u_n) \equiv 0$, where $\underline{t}'(u_n,\lambda)$ denotes the derivative of $\underline{t}(.,\lambda)$ with respect to its first variable at the point (u_n,λ).
Furthermore, let

$$\begin{aligned}\pi : \ W\setminus\{0\} &\longrightarrow \mathbb{R}\times\mathbb{S}\\ u &\longmapsto \left(||u||, \tfrac{u}{||u||}\right) := (\pi_1(u), \pi_2(u)).\end{aligned}$$

Applying Hölder's inequality, we get for every $(u,\varphi) \in (W\setminus\{0\}) \times W$:

$$\begin{cases} |\pi_1'(u)(\varphi)| \leq ||\varphi||,\\ ||\pi_2'(u)(\varphi)|| \leq 2\frac{||\varphi||}{||u||}.\end{cases}$$

>From Lemma 2, there is a positive constant C such that

$$\underline{t}(u_n,\lambda) \geq C, \ \forall\, n \in \mathbb{N}.$$

Then for every $\varphi \in W$, there are $\varphi_n^1 \in \mathbb{R}$ and $\varphi_n^2 \in \mathrm{T}_{u_n}\mathbb{S}$ such that $|\varphi_n^1| \leq ||\varphi||$, $||\varphi_n^2|| \leq \frac{2}{C}||\varphi||$ and

$$\begin{aligned}E'_\lambda(\underline{t}(u_n,\lambda)u_n)(\varphi) &= \partial_t\widetilde{E}_\lambda(\underline{t}(u_n,\lambda),u_n)(\varphi_n^1) + \partial_u\widetilde{E}_\lambda(\underline{t}(u_n,\lambda),u_n)(\varphi_n^2),\\ &= \partial_u\widetilde{E}_\lambda(\underline{t}(u_n,\lambda),u_n)(\varphi_n^2),\\ &= \underline{\mathcal{E}}'_\lambda(u_n)(\varphi_n^2).\end{aligned}$$

Therefore,

$$\begin{aligned} E'_\lambda(\underline{t}(u_n,\lambda)u_n)(\varphi) &\leq \frac{1}{n}||\varphi_n^2|| \\ &\leq \frac{2}{nC}||\varphi||. \end{aligned}$$

We easily conclude that

$$\lim_{n\to\infty} ||E'_\lambda(\underline{U}_n)||_* = 0.$$

□

Remark 2. *Until now, the minimizing sequences we consider are not nonnegative. Notice that for every* $u \in W\backslash\{0\}$ *and* $0 < \lambda < \widehat{\lambda}$*, one has* $\widetilde{E}_\lambda(t,|u|) = \widetilde{E}_\lambda(t,u)$*,* $\underline{t}(|u|,\lambda) = \underline{t}(u,\lambda)$ *and* $\overline{t}(|u|,\lambda) = \overline{t}(u,\lambda)$*. Thus, every minimizing sequence* $(u_n) \subset \mathbb{S}$ *of* (12) *or* (13) *can be considered as a sequence of nonnegative functions.*

Hereafter, we will assume the sequences $\underline{U}_n$ and $\overline{U}_n$, defined in Lemma 3, to be nonnegative.

3 Existence results

Theorem 1. *Let* $1 < q < p < r < p^*$ *and* $\lambda \in]0,\widehat{\lambda}[$*. Then the problem* (1) *has at least two nonnegative solutions.*

Proof. We will use the notations of the previous lemmas. According to Lemma 3, we know that $E_\lambda(\underline{U}_n)$ converges to $\underline{\alpha}(\lambda)$, $||E'_\lambda(\underline{U}_n)||_*$ converges to 0 as n tends to $+\infty$ and that $(\underline{U}_n)$ is bounded in W. Passing if necessary to a subsequence, we have

$$\begin{aligned} \underline{U}_n &\rightharpoonup \underline{U} \quad \text{in} \quad W, \\ \underline{U}_n &\to \underline{U} \quad \text{in} \quad L^r(\Omega), \quad \left(\text{also in} \quad L^q(\Omega)\right), \\ \underline{U}_n &\to \underline{U} \quad \text{a.e} \quad \Omega. \end{aligned}$$

Let $v_n = \underline{U}_n - \underline{U}$, then using a lemma due to Brézis-Lieb [12], we get

$$\begin{aligned} ||v_n|| &= ||\underline{U}_n|| - ||\underline{U}|| + \mathrm{o}_n(1), \\ ||v_n||_q &= ||\underline{U}_n||_q - ||\underline{U}||_q + \mathrm{o}_n(1), \\ ||v_n||_r &= ||\underline{U}_n||_r - ||\underline{U}||_r + \mathrm{o}_n(1). \end{aligned}$$

It follows that

$$\begin{aligned} E_\lambda(v_n) = E_\lambda(\underline{U}_n) - E_\lambda(\underline{U}) + \mathrm{o}_n(1), \\ E'_\lambda(v_n) = E'_\lambda(\underline{U}_n) - E'_\lambda(\underline{U}) + \mathrm{o}_n(1), \end{aligned}$$

and consequently $E'_\lambda(v_n)v_n \to 0$ as $n \to +\infty$, which implies that

$$||\underline{v}_n||^p = \lambda||\underline{v}_n||_q^q + ||\underline{v}_n||_r^r + \mathrm{o}_n(1).$$

Therefore, $||v_n||^p \to 0$ as $n \to +\infty$. The same argument can be used for $\overline{U}_n$, which ends the proof. □

Remark 3. *We showed in the previous theorem that for any minimizing sequence* (u_n) *of* (12)*, there is a subsequence still denoted by* (u_n) *such that*

$$\underline{U}_n := \underline{t}(u_n, \lambda)u_n \longrightarrow \underline{U} \ \text{ in } \ W.$$

Moreover, according to Lemma 2 and Theorem 1, there is $\underline{t} \in]0, +\infty[$ *such that*

$$\begin{cases} \underline{t}(u_n, \lambda) & \longrightarrow & \underline{t} \ \text{ in } \ \mathbb{R}, \\ u_n & \longrightarrow & \underline{u} = \underline{U}/\underline{t} \ \text{ in } \ W, \end{cases}$$

with $\underline{u} = \underline{U}/\underline{t} \in \mathbb{S}$, *(i.e.* $\underline{t} = ||\underline{U}||$*) and* $\underline{t} = \underline{t}(\underline{u}, \lambda)$. *In the same way, for any minimizing sequence* (w_n) *of* (13)*, passing if necessary to a subsequence, there is* $\overline{t} \in]0, +\infty[$ *such that*

$$\begin{cases} \overline{t}(w_n, \lambda) & \longrightarrow & \overline{t} \ \textit{ in } \ \mathbb{R}, \\ w_n & \longrightarrow & \overline{u} = \overline{U}/\overline{t} \ \textit{ in } \ W, \end{cases}$$

with $\overline{u} = \overline{U}/\overline{t} \in \mathbb{S}$, *(i.e.* $\overline{t} = ||\overline{U}||$*) and* $\overline{t} = \overline{t}(\overline{u}, \lambda)$.
It is interesting to notice that $\underline{U} \neq \overline{U}$. *Indeed, since* $\partial_{tt}\widetilde{E}_\lambda(\underline{t}(\underline{u}, \lambda), \underline{u}) > 0$ *and* $\partial_{tt}\widetilde{E}_\lambda(\overline{t}(\overline{u}, \lambda), \overline{u}) < 0$, *it follows that* $\partial_{tt}\widetilde{E}_\lambda(\underline{t}, \underline{U}/\underline{t}) > 0$ *and* $\partial_{tt}\widetilde{E}_\lambda(\overline{t}, \overline{U}/\overline{t}) < 0$. *However,* $\underline{U}$ *and* $\overline{U}$ *verify* $\partial_t\widetilde{E}_\lambda(\underline{t}, \underline{U}/\underline{t}) = \partial_t\widetilde{E}_\lambda(\overline{t}, \overline{U}/\overline{t}) = 0$.

In the sequel the solutions $\underline{U}$ and $\overline{U}$ of (1), for $\lambda \in]0, \widehat{\lambda}[$, will be denoted by $\underline{U}_\lambda$ and $\overline{U}_\lambda$. Also, $\underline{t}_\lambda$, $\overline{t}_\lambda$, $\underline{u}_\lambda$ and $\overline{u}_\lambda$ will stand for $\underline{t}(\underline{u}, \lambda)$, $\overline{t}(\overline{u}, \lambda)$, $\underline{u}$ and $\overline{u}$ respectively.

Theorem 2. *Let* $1 < q < p < r < p^*$. *Then*

$$\text{(i)} \quad E_\lambda(\underline{U}_\lambda) < 0 \ \textit{ for } \ \lambda \in]0, \widehat{\lambda}[,$$

$$\text{(ii)} \quad \begin{cases} E_\lambda(\overline{U}_\lambda) & > 0 & \textit{for } \lambda \in]0, \lambda_0[, \\ E_\lambda(\overline{U}_\lambda) & < 0 & \textit{for } \lambda \in]\lambda_0, \widehat{\lambda}[, \end{cases}$$

where

$$\lambda_0 := \frac{q}{r}\left(\frac{r}{p}\right)^{\frac{r-q}{r-p}}\widehat{\lambda}.$$

Proof.
(i) Let us recall that $\partial_t\widetilde{E}_\lambda(\underline{t}, \underline{U}/\underline{t}) = 0$ and $\partial_{tt}\widetilde{E}_\lambda(\underline{t}, \underline{U}/\underline{t}) > 0$, (see Remark 3). Then

$$\begin{cases} P(\underline{U}_\lambda) - \lambda Q(\underline{U}_\lambda) - R(\underline{U}_\lambda) = 0, \\ (p-1)P(\underline{U}_\lambda) - \lambda(q-1)Q(\underline{U}_\lambda) - (r-1)R(\underline{U}_\lambda) > 0. \end{cases}$$

Using the fact that $1 < q < p < r$, we get

$$\frac{1}{p}P(\underline{U}_\lambda) - \frac{\lambda}{q}Q(\underline{U}_\lambda) - \frac{1}{r}R(\underline{U}_\lambda) < 0,$$

and consequently $E_\lambda(\underline{U}_\lambda) < 0$.
(ii) Let u be an arbitrary element of $W \setminus \{0\}$ and let us write

$$\widetilde{E}_\lambda(t, u) = t^q\widetilde{G}_\lambda(t, u), \ \text{ where } \widetilde{G}_\lambda(t, u) = t^{p-q}\frac{P(u)}{p} - \lambda\frac{Q(u)}{q} - t^{r-q}\frac{R(u)}{r}.$$

It follows that

$$\partial_t \widetilde{E}_\lambda(t,u) = qt^{q-1}\widetilde{G}_\lambda(t,u) + t^q \partial_t \widetilde{G}_\lambda(t,u),$$

with

$$\partial_t \widetilde{G}_\lambda(t,u) = t^{p-q-1}\left\{\frac{p-q}{p}P(u) - \frac{r-q}{r}t^{r-p}R(u)\right\}.$$

It is clear that the real valued function $t \longmapsto \widetilde{G}_\lambda(t,u)$ is increasing on $]0,t_0(u)[$, decreasing on $]t_0(u),+\infty[$ and attains its unique maximum for $t = t_0(u)$, where

$$t_0(u) = \left(\frac{r}{p}\right)^{\frac{1}{r-p}} t(u), \tag{14}$$

and $t(u)$ is defined in (3). On the other hand, a direct computation gives

$$\widetilde{G}_\lambda(t_0(u),u) = \left(\frac{p-q}{r-q}\frac{P(u)}{R(u)}\right)^{\frac{r-q}{r-p}} R(u) - \lambda Q(u).$$

Similarly, $\widetilde{G}_\lambda(t_0(u),u) > 0$ (resp. $\widetilde{G}_\lambda(t_0(u),u) < 0$) if $\lambda < \lambda_0(u)$ (resp. $\lambda > \lambda_0(u)$) and $\widetilde{G}_{\lambda_0(u)}(t_0(u),u) = 0$, where

$$\lambda_0(u) = \frac{q}{r}\left(\frac{r}{p}\right)^{\frac{r-q}{r-p}} \lambda(u), \tag{15}$$

with $\lambda(u)$ given by (4). Thus, we get

$$\begin{cases} \widetilde{E}_\lambda(t_0(u),u) > 0 & \text{if} \quad \lambda < \lambda_0(u), \\ \widetilde{E}_\lambda(t_0(u),u) = 0 & \text{if} \quad \lambda = \lambda_0(u), \\ \widetilde{E}_\lambda(t_0(u),u) < 0 & \text{if} \quad \lambda > \lambda_0(u). \end{cases} \tag{16}$$

First, since the function

$$\begin{array}{ccc}]0,1[& \longrightarrow & \mathbb{R} \\ t & \longmapsto & \frac{\ln t}{1-t} \end{array}$$

is increasing, then for every real numbers x, y such that $0 < x < y < 1$, one has

$$\ln\left[\frac{1}{x}\right] > \frac{1-x}{1-y}\ln\left[\frac{1}{y}\right] = \ln\left[\left(\frac{1}{y}\right)^{\frac{1-x}{1-y}}\right],$$

and consequently

$$0 < x\left(\frac{1}{y}\right)^{\frac{1-x}{1-y}} < 1.$$

In the particular case $x = q/r$ and $y = p/r$ we get

$$0 < \frac{q}{r}\left(\frac{r}{p}\right)^{\frac{r-q}{r-p}} < 1,$$

and therfore $0 < \lambda_0(u) < \widehat{\lambda}(u)$.

Moreover, for every $u \in W \setminus \{0\}$, one has $\widetilde{G}_{\lambda_0(u)}(t,u) < 0$ for $t \in]0,+\infty[\setminus\{t_0(u)\}$ and $\widetilde{G}_{\lambda_0(u)}(t_0(u),u) = 0$. Hence, the real valued function $t \longmapsto \widetilde{E}_{\lambda_0(u)}(t,u)$, $(t > 0)$, attains its unique maximum at $t = t_0(u)$ and we obtain the following interesting identity

$$\bar{t}(u, \lambda_0(u)) = t_0(u). \tag{17}$$

On the other hand, it is clear that the functional $\lambda_0(u)$ is weakly lower semi-continuous on $W \setminus \{0\}$. Thus, the value

$$\lambda_0 := \inf_{u \in W\setminus\{0\}} \lambda_0(u) \tag{18}$$

is achieved on $W \setminus \{0\}$. Since $\lambda_0(u)$ is homogeneous in u, we can assume that there is some $u^* \in \mathbb{S}$ such that $\lambda_0 = \lambda_0(u^*)$.
Now, let λ is such that $0 < \lambda < \lambda_0$. Then, for every $u \in W \setminus \{0\}$ one has $\lambda < \lambda_0(u)$ and consequently $\widetilde{E}_\lambda(t_0(u),u) > 0$ holds from (16). But, $t \longmapsto \widetilde{E}_\lambda(t,u)$, $(t > 0)$ attains its unique maximum for $t = \bar{t}(u,\lambda)$, hence $\widetilde{E}_\lambda(\bar{t}(u,\lambda),u) > 0$, for every $u \in W \setminus \{0\}$. In particular, we have $\widetilde{E}_\lambda(\bar{t}(\overline{u}_\lambda,\lambda),\overline{u}_\lambda) > 0$, *i.e.* $E_\lambda(\overline{U}_\lambda) > 0$.

If $\lambda = \lambda_0$, then

$$\begin{aligned}
E_{\lambda_0}(\overline{U}_{\lambda_0}) &= \widetilde{E}_{\lambda_0}(\bar{t}(\overline{u}_{\lambda_0},\lambda_0),\overline{u}_{\lambda_0}),\\
&= \inf_{u\in\mathbb{S}} \widetilde{E}_{\lambda_0}(\bar{t}(u,\lambda_0),u),\\
&\le \widetilde{E}_{\lambda_0}(\bar{t}(u^*,\lambda_0(u^*)),u^*),\\
&= \widetilde{E}_{\lambda_0(u^*)}(t_0(u^*),u^*),\\
&= 0,
\end{aligned}$$

which implies that $E_{\lambda_0}(\overline{U}_{\lambda_0}) \le 0$. In addition, it is known from (16) that $\widetilde{E}_{\lambda_0}(t_0(u),u) \ge 0$ and $\widetilde{E}_{\lambda_0}(\underline{t}(u,\lambda_0),u) < 0$ for every $u \in W \setminus \{0\}$. Then

$$t_0(u) > \underline{t}(u,\lambda_0), \quad \forall u \in W \setminus \{0\}.$$

It follows that

$$\widetilde{E}_{\lambda_0}(\bar{t}(\overline{u}_{\lambda_0},\lambda_0),\overline{u}_{\lambda_0}) \ge \widetilde{E}_{\lambda_0}(t_0(\overline{u}_{\lambda_0}),\overline{u}_{\lambda_0}) \ge 0.$$

Hence,

$$E_{\lambda_0}(\overline{U}_{\lambda_0}) = \widetilde{E}_{\lambda_0}(\bar{t}(\overline{u}_{\lambda_0},\lambda_0),\overline{u}_{\lambda_0}) = 0.$$

Finally, assume that $\lambda_0 < \lambda < \widehat{\lambda}$. Since, for every $(t,u) \in]0,+\infty[\times (W \setminus \{0\})$, the real valued function $\lambda \longmapsto \widetilde{E}_\lambda(t,u)$ is decreasing, it follows that

$$\widetilde{E}_\lambda(t,u) < \widetilde{E}_{\lambda_0}(t,u), \quad \text{for every } \ t > 0 \ \text{ and } \ u \in W \setminus \{0\}. \tag{19}$$

In addition, we have

$$\begin{aligned}
\widetilde{E}_\lambda(\bar{t}(\overline{u}_\lambda,\lambda),\overline{u}_\lambda) &= \inf_{u\in\mathbb{S}} \widetilde{E}_\lambda(\bar{t}(u,\lambda),u),\\
&\le \widetilde{E}_\lambda(\bar{t}(u^*,\lambda),u^*),\\
&< \widetilde{E}_{\lambda_0}(\bar{t}(u^*,\lambda),u^*),
\end{aligned}$$

where the last inequality follows from (19). Moreover, the real valued function $t \longmapsto \widetilde{E}_{\lambda_0}(t,u^*)$, $(t > 0)$, achieves its unique maximum at $t = t_0(u^*)$. Thus, $\widetilde{E}_{\lambda_0}(\bar{t}(u^*,\lambda),u^*) \le$

$\widetilde{E}_{\lambda_0}(t_0(u^*), u^*) = \widetilde{E}_{\lambda_0(u^*)}(t_0(u^*), u^*) = 0$. Hence $\widetilde{E}_\lambda(\bar{t}(\overline{u}_\lambda, \lambda), \overline{u}_\lambda) < 0$, which ends the proof. □

The following reslut shows that the variational character of (5) leads λ_0 to share a specifi c property to eignevalues.

Theorem 3. *If u is a solution of (18) then $t_0(u)u$ is a solution of (1) for $\lambda = \lambda_0$.*

Proof.
Let u be a solution of (18), then $\lambda_0 = \lambda_0(u)$ and for every $\varphi \in W$, we have

$$\begin{aligned} E'_{\lambda_0}(t_0(u)u)(\varphi) &= \frac{1}{p}P'(t_0(u)u)(\varphi) - \frac{\lambda_0}{q}Q'(t_0(u)u)(\varphi) - \frac{1}{r}R'(t_0(u)u)(\varphi), \\ &= \frac{P(u)[t_0(u)]^{p-1}}{p}\left(\frac{P'(u)(\varphi)}{P(u)} - \frac{r-p}{r-q}\frac{Q'(u)(\varphi)}{Q(u)} - \frac{p-q}{r-q}\frac{R'(u)(\varphi)}{R(u)}\right), \\ &= K\left(\frac{r-q}{r-p}\frac{P'(u)(\varphi)}{P(u)} - \frac{Q'(u)(\varphi)}{Q(u)} - \frac{p-q}{r-p}\frac{R'(u)(\varphi)}{R(u)}\right), \end{aligned}$$

where

$$K := \frac{r-p}{r-q}\frac{P(u)}{p}[t_0(u)]^{p-1}.$$

Furthermore, we know that $\lambda'_0(u)(\varphi) = 0$, for every $\varphi \in W$, and

$$\lambda'_0(u)(\varphi) = \lambda_0\left(\frac{r-q}{r-p}\frac{P'(u)(\varphi)}{P(u)} - \frac{Q'(u)(\varphi)}{Q(u)} - \frac{p-q}{r-p}\frac{R'(u)(\varphi)}{R(u)}\right).$$

Therefore, we obtain that

$$E'_{\lambda_0}(t_0(u)u)(\varphi) = \frac{K}{\lambda_0}\lambda'_0(u)(\varphi) = 0,$$

for every $\varphi \in W$, which implies that $t_0(u)u$ is a solution of (1) for $\lambda = \lambda_0$. □

Remark 4. *It is very interesting to notice that in the case of homogeneous Dirichlet boundary condition ($\varepsilon = 0$), we have*

$$\lim_{q\to p}\widehat{\lambda} = \inf_{u\in W_0^{1,p}\setminus\{0\}}\frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u|^p\,dx}.$$

Hence, in the case where $p = q$, $\widehat{\lambda}$ is the first eigenvalue of the problem (1), i.e. the problem (1) has positive solutions for $\lambda \in]0, \widehat{\lambda}]$ and has no positive solution for $\lambda > \widehat{\lambda}$.

In the following section, the existence of infi nitely many solutions is studied via the Lusternik-Schnirelman theory [7, 22, 23, 25, 24].

4 Existence of infinitely many solutions

We show here the existence of two disjoint and infinite sets of solutions to (1). The first set contains only solutions with negative energy while the second one contains solutions with changing sign energy.

For the reader's convenience, we recall some background facts used here. Let us define

$$\mathcal{A} = \{A \subset \mathbb{S} : \quad A \text{ closed}, \ A = -A\}$$

to be the class of closed and symmetric subsets of the complete smooth submanifold $(\mathbb{S}, ||\ ||)$ [7, 22, 23, 25, 24]. For every $A \in \mathcal{A}$, $A \neq \emptyset$, let

$$\gamma(A) = \inf \left\{k \in \mathbb{N};\ \ \exists\, \varphi \in C^0(A, \mathbb{R}^k \setminus \{0\}),\ \forall\, u \in A,\ \varphi(-u) = -\varphi(u)\right\}$$

be the Krasnoselskii genus [22]. When there does not exist a finite such integer, set $\gamma(A) = +\infty$. Finally, set $\gamma(\emptyset) = 0$. For each positive integer k, let us define

$$\Gamma_k = \{A \in \mathcal{A};\ A \text{ compact},\ \gamma(A) \geq k\}\,,$$

$$\underline{c}_k = \inf_{A \in \Gamma_k} \ \max_{u \in A} \ \underline{\mathcal{E}}_\lambda(u) \ \text{ and } \ \overline{c}_k = \inf_{A \in \Gamma_k} \ \max_{u \in A} \ \overline{\mathcal{E}}_\lambda(u),$$

where

$$\underline{\mathcal{E}}_\lambda(u) = E_\lambda(\underline{t}(u, \lambda)u)$$

and

$$\overline{\mathcal{E}}_\lambda(u) = E_\lambda(\overline{t}(u, \lambda)u).$$

It is well known that $(\underline{c}_k)$ (resp. $(\overline{c}_k)$) is a nondecreasing sequence of critical values of $\underline{\mathcal{E}}_\lambda$ (resp. $\overline{\mathcal{E}}_\lambda$) [24]. Recall that if the sequence $(\underline{c}_k)$ (resp. $(\overline{c}_k)$) is increasing, then $\underline{\mathcal{E}}_\lambda$ (resp. $\overline{\mathcal{E}}_\lambda$) has infinitely many critical points $(\underline{u}_{\lambda,k})$ (resp. $(\overline{u}_{\lambda,k})$) corresponding to the sequence of distinct levels $(\underline{c}_k)$ (resp. $(\overline{c}_k)$). And if there are two positive integers j and p such that $\underline{c}_j = \underline{c}_{j+1} = \cdots = \underline{c}_{j+p}$ (resp. $\overline{c}_j = \overline{c}_{j+1} = \cdots = \overline{c}_{j+p}$), then the set of critical points for $\underline{\mathcal{E}}_\lambda$ (resp. $\overline{\mathcal{E}}_\lambda$) corresponding to the level $\underline{c}_j$ (resp. $\overline{c}_j$) is infinite. In what follows, for every $k \in \mathbb{N}^*$, we set

$$\underline{U}_{\lambda,k} = \underline{t}(\underline{u}_{\lambda,k}, \lambda)\underline{u}_{\lambda,k}$$

and

$$\overline{U}_{\lambda,k} = \overline{t}(\overline{u}_{\lambda,k}, \lambda)\overline{u}_{\lambda,k}.$$

It is clear that $\underline{U}_{\lambda,k}$ and $\overline{U}_{\lambda,k}$ are solutions of (1), for every $k \in \mathbb{N}^*$.

Theorem 4. *Let $1 < q < p < r < p^*$ and $0 < \lambda < \widehat{\lambda}$. Then, there are two disjoint and infinite sets of solutions to (1) $\left\{\underline{U}_{\lambda,k};\ \ k \in \mathbb{N}^*\right\}$ and $\left\{\overline{U}_{\lambda,k};\ \ k \in \mathbb{N}^*\right\}$. Moreover,*

$$\textbf{(i)} \quad E_\lambda(\underline{U}_{\lambda,k}) < 0 \ \textit{ and } \ \lim_{k \to +\infty} E_\lambda(\underline{U}_{\lambda,k}) = 0,$$

$$\textbf{(ii)} \quad \lim_{k \to +\infty} E_\lambda(\overline{U}_{\lambda,k}) = +\infty,$$

hold true.

Proof. i) Since $\partial_{tt}\widetilde{E}_\lambda(\underline{t}(\underline{u}_{\lambda,k},\lambda),\underline{u}_{\lambda,k}) > 0$ and $\partial_{tt}\widetilde{E}_\lambda(\overline{t}(\overline{u}_{\lambda,k},\lambda),\overline{u}_{\lambda,k}) < 0$, it follows that the sets $\left\{\underline{U}_{\lambda,k};\ \ k\in\mathbb{N}^*\right\}$ and $\left\{\overline{U}_{\lambda,k};\ \ k\in\mathbb{N}^*\right\}$ are disjoint. On the other hand, we know that $E_\lambda(\underline{U}_{\lambda,k}) < 0$ for every k. Let us show that $(\underline{c}_k) := \left(\underline{\mathcal{E}}_\lambda(\underline{u}_{\lambda,k})\right) := \left(E_\lambda(\underline{U}_{\lambda,k})\right)$ converges to zero as k goes to infinity. Suppose that

$$\lim_{k\to+\infty}\underline{c}_k = \underline{c} < 0.$$

Consider the set

$$K_{\underline{c}} = \left\{u\in\mathbb{S};\ \ \underline{\mathcal{E}}_\lambda(u) = \underline{c}\ \text{ and }\ \underline{\mathcal{E}}'_\lambda(u) = 0\right\}.$$

Since $\underline{\mathcal{E}}$ satisfies that Palais-Smale condition on $\mathbb{S}$ and $\underline{c} < 0$ then $K_{\underline{c}}$ is not empty, compact and symmetric wich implies that $\gamma(K_{\underline{c}}) < +\infty$. Let N be a closed neighborhood of $K_{\underline{c}}$ in $\mathbb{S}$ such that $\gamma(N) = \gamma(K_{\underline{c}})$. Applying the deformation Lemma [23, 25, 24, 26], there are an odd homeomorphism Φ from $\mathbb{S}$ to $\mathbb{S}$ and $\varepsilon > 0$ such that

$$\Phi\left(A_{\underline{c}+\varepsilon}\setminus\overset{\circ}{N}\right)\subset A_{\underline{c}-\varepsilon}.$$

Thus,

$$\begin{aligned}\gamma\left(A_{\underline{c}+\varepsilon}\right) &\le \gamma\left(A_{\underline{c}+\varepsilon}\setminus\overset{\circ}{N}\right)+\gamma(N)\\ &\le \gamma\left(\Phi\left(A_{\underline{c}+\varepsilon}\setminus\overset{\circ}{N}\right)\right)+\gamma(N)\\ &\le \gamma\left(A_{\underline{c}-\varepsilon}\right)+\gamma(N).\end{aligned}$$

Furthermore, there is a positive integer j such that $\underline{c}-\varepsilon < c_j \le \underline{c}$, then $\gamma\left(A_{\underline{c}-\varepsilon}\right) < j$ and consequently

$$\gamma\left(A_{\underline{c}+\varepsilon}\right) < j + \gamma(K_{\underline{c}}) < +\infty.$$

This contradicts the fact that $\gamma\left(A_{\underline{c}+\varepsilon}\right) = +\infty$, which ends the proof of **(i)**.
(ii) This last point can be shown in the same way. ☐

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