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WEIGHTED SOBOLEV SPACES FOR A DEGENERATED NONLINEAR ELLIPTIC EQUATION

M. AMARA [A], A. OBEID [B] & G. VALLET [A]

Abstract In this paper, we prove an existence result for a degenerated nonlinear elliptic equation posed in the upper half-space of $\mathbb{R}^2$. A logarithmic weighted Sobolev space is used as a framework to describe the behaviour of functions at infinity.

Keywords: *Weighted Sobolev spaces, variational methods, dual problem*

AMS subject classification: 35J70, 35Q35, 76H05

1. Introduction

In this work, we deal with the elliptic degenerated problem posed in $\mathbb{R}^2_+$ as follows:

$$(\mathcal{P})\begin{cases} \partial_x(g(\partial_x\varphi)) + \partial_y(\partial_y\varphi) = 0 \ \text{ in } \ \mathbb{R}^2_+, \\ \partial_y\varphi(x,0) = \sqrt{M_\infty}\, z'(x) \ \text{ on } \ ]-\frac{1}{2},\frac{1}{2}[, \\ \partial_y\varphi(x,0) = 0 \ \text{ on } \ \mathbb{R}\backslash\,]-\frac{1}{2},\frac{1}{2}[, \\ (\partial_x\varphi,\partial_y\varphi) \ \to \ (0,0) \ \text{ as } \ \sqrt{x^2+y^2} \ \to \ +\infty. \end{cases}$$

where, for any real number t, $g(t) = -\frac{\gamma+1}{2}[(u_{cr}-t)^+]^2 + \frac{\gamma+1}{2}u_{cr}^2$ (by t^+ we mean the positive part of t), γ and u_{cr} are positive constants, M_∞ is in $]0,1[$ and close to 1 and z belongs to $W_0^{1,\infty}(]-\frac{1}{2},\frac{1}{2}[)$ with $0 \le z \le 1$. This boundary value problem corresponds to the degenerated elliptic model for the Euler equations of a transonic steady state two-dimensional irrotational compressible flow around a thin profile. The considered flow is supposed to be uniform at infinity with speed $(1,0)$ and symmetric with respect to the axis (o,x). Thus the study is taken in the higher half-space. We recall that the Euler equations consist of the conservation law of mass and the Bernouilli law for the velocity.

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They can be written into the following second-order nonlinear elliptic-parabolic and hyperbolic equation for the velocity potential Φ defined on the exterior of the profile by:

$$\mathrm{div}(\rho(|\nabla\Phi|^2)\nabla\Phi) = 0$$

where ρ is the fluid density. This density is defined by:

$$\rho(|\nabla\Phi|^2) = \left[1 + \frac{\gamma-1}{2} M_\infty^2 (1 - |\nabla\Phi|^2)\right]^{\frac{1}{\gamma-1}}$$

where M_∞ is the Mach number at infinity and γ is the ratio of the specific heats ($\gamma = 1.4$ for the air). For some general mathematical modelizations, the reader can consult Pogu and al. [16].

The flow around the profile is considered as a perturbation of the one at infinity. After the decomposition $\nabla\Phi = (1,0) + \nabla\varphi$ where $\nabla\varphi$ is the perturbation of $\nabla\Phi$, we get the small disturbance model:

$$(\mathcal{P}_1)\begin{cases} \partial_x(f(\partial_x\varphi)) + \partial_y(\partial_y\varphi) = 0 \text{ in } \mathbb{R}^2_+, \\ \partial_y\varphi(x,0) = \sqrt{M_\infty}\, z'(x) \text{ on }]-\frac{1}{2}, \frac{1}{2}[, \\ \partial_y\varphi(x,0) = 0 \text{ on } \mathbb{R}\backslash]-\frac{1}{2}, \frac{1}{2}[, \\ (\partial_x\varphi, \partial_y\varphi) \to (0,0) \text{ as } \sqrt{x^2+y^2} \to +\infty. \end{cases}$$

with $f(t) = -\frac{\gamma+1}{2}(u_{cr} - t)^2 + \frac{\gamma+1}{2}u_{cr}^2$ and u_{cr} is the critical speed.

The systems $(\mathcal{P})$ and $(\mathcal{P}_1)$ are equivalent if $\partial_x\varphi \leq u_{cr}$ i.e. when the problem remains elliptic or parabolic.

Since forty years, many numerical and mathematical works have been done for the study of the exact equations of transonic flows of perfect compressible fluids. ¿From theoretical point of view, there are not complete results to ensure the existence and uniqueness of solutions. Currently, only methods based on conjectures are performed [8, 9, 12, 13, 15]. But from numerical point of view, many simulations are realized, the results obtained are in accordance with the experimentation (see for example [6, 10, 11]).

We know an existence result for the problem $(\mathcal{P})$ when the domain is reduced to the bounded rectangle $\Omega =]-R, R[\times]0, R_y[$ with $R, R_y > 1$ (see [1, 2]). In that case, the associated minimization problem provided a coercive functional with respect to the $W^{1,1}(\Omega) \cap L^2(-R, R; W^{1,2}(]0, R_y[))$-norm. This coerciveness is obtained under the condition of thin profile

$$R_y > \frac{2}{\gamma+1}\frac{\sqrt{M_\infty}}{u_{cr}^2} \tag{1}$$

and according to the Poincaré-Wirtinger's inequality. When the domain is unbounded, the arguments which can be used in the bounded case break down because of the lack of compactness. To overcome this difficulty, it is necessary to introduce a weighted Sobolev space as a variational setting of the problem $(\mathcal{P})$, extending that used in the bounded case and taking into account the compactnesses properties.
Our aim here is to prove the existence and the uniqueness of the solution to Problem $(\mathcal{P})$. For this purpose, we formulate it as a minimization of an associated total energy in an appropriate reflexive space. We show that the "energy" functional is coercive on the set of subsonic flows and that this associated solution is unique. But globally, the functional is not coercive. The difficulty of the problem is related to the lack of this property. Introducing perturbed problems of the total energy, existence of a unique couple $(g(u_*), v_*)$ for the dual problem with respect to the given perturbations, is proved. Next, by means of a minimizing sequence, existence of a function φ defining the solution of the problem is established. Notice that this function is linked to (u_*, v_*) by $\partial_x \varphi = u_* + \nu$ and $\partial_y \varphi = v_*$ where ν is a nonnegative bounded measure. As a consequence, the solution φ is not unique unless ν vanishes.
The paper is organized as follows. First, we present the notations and the functional spaces needed in the sequel. Therefore, we reduce the problem to the minimization of a suitable "energy" functional F. After the study of the functional settings, we give the properties of the functional F, in particular its coerciveness property. Using the concept of perturbed problems, (see for example Ekeland and al. [7]), we establish a duality approach and prove the existence of a weak solution. At last, we give some comments about the uniqueness.

2. Notations

Let Ω be an open set of $\mathbb{R}^2$. For any measurable positive function σ, we define the weighted p-norm $(1 \leq p < \infty)$ $\|\xi\|_{p,\Omega,\sigma} = \left(\int_\Omega \sigma(x,y)|\xi(x,y)|^p \, dx \, dy\right)^{\frac{1}{p}}$ and denote by $L^p(\Omega, \sigma)$ the space of all measurable functions ξ such that $\|\xi\|_{p,\Omega,\sigma}$ is finite. We consider the basic weights
$\rho(x,y) = \sqrt{x^2+y^2}$, $\omega(\rho) = \frac{1}{\sqrt{2+\rho^2}\,\log(2+\rho^2)}$, $\rho'(x) = \rho(x,0) = |x|$ and $\omega' = \omega(\rho')$.
Define the weighted spaces

$$\mathbb{H}(\Omega) = (L^2(\Omega) \cap L^3(\Omega)) \times L^2(\Omega), \tag{2}$$

$$V(\Omega, \omega) = \left\{\xi \in L^2(\Omega, \omega^2) \cap L^3(\Omega, \omega^3), \nabla\xi \in \mathbb{H}(\Omega)\right\}, \tag{3}$$

$$W^{1,2}(\Omega, \omega) = \left\{\xi \in L^2(\Omega, \omega^2), \nabla\xi \in (L^2(\Omega))^2\right\}. \tag{4}$$

Endowed respectively with their natural norms,

$$\begin{aligned}
\|(p,q)\|_{\mathbb{H}(\Omega)} &= \|p\|_{2,\Omega,1} + \|p\|_{3,\Omega,1} + \|q\|_{2,\Omega,1}, \\
\|\xi\|_{V(\Omega,\omega)} &= \|\xi\|_{2,\Omega,\omega^2} + \|\xi\|_{3,\Omega,\omega^3} + \|\nabla\xi\|_{\mathbb{H}(\Omega)}, \\
\|\xi\|_{W^{1,2}(\Omega,\omega)} &= \|\xi\|_{2,\Omega,\omega^2} + \|\nabla\xi\|_{2,\Omega,1},
\end{aligned}$$

$\mathbb{H}(\Omega)$, $V(\Omega,\omega)$ and $W^{1,2}(\Omega,\omega)$ are Banach spaces. Denote by $|\,.\,|_{V(\Omega,\omega)}$ and $|\,.\,|_{W^{1,2}(\Omega,\omega)}$ the semi-norms:

$$\begin{aligned}
|\xi|_{V(\Omega,\omega)} &= \|\nabla\xi\|_{\mathbb{H}(\Omega)}, \\
|\xi|_{W^{1,2}(\Omega,\omega)} &= \|\nabla\xi\|_{2,\Omega,1}.
\end{aligned}$$

We abbreviate $V(\Omega) = V(\Omega,1)$ and $W^{1,2}(\Omega) = W^{1,2}(\Omega,1)$. Denote by $V_0(\Omega,\omega)$ and $W_0^{1,2}(\Omega,\omega)$ the closure of $\mathcal{D}(\Omega)$ in $V(\Omega,\omega)$ and $W^{1,2}(\Omega,\omega)$ respectively. Let $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2; y > 0\}$ be the upper half-space of $\mathbb{R}^2$ and $\{(x,0); x \in \mathbb{R}\} = \mathcal{R}$ its boundary. It is known that $W^{\frac{1}{2},2}(\mathcal{R},\omega')$ is the trace space of $W^{1,2}(\mathbb{R}^2_+,\omega)$ (see Amrouche and al. [4]). It is defined by

$$W^{\frac{1}{2},2}(\mathcal{R},\omega') = \left\{\mu \in L^2(\mathcal{R},\omega'); \int_0^{+\infty} t^{-2}\,dt \int_{\mathbb{R}} |\mu(x+t,0) - \mu(x,0)|^2\,dx < \infty\right\}.$$

By adequate truncature and regularization procedures (see for example Amrouche and al. [3]), we prove that $\mathcal{D}(\overline{\mathbb{R}^2_+})$ is dense in $V(\mathbb{R}^2_+,\omega)$. It becomes from this density result and the inclusion of $V(\mathbb{R}^2_+,\omega)$ in $W^{1,2}(\mathbb{R}^2_+,\omega)$ that the mapping $\xi \longmapsto \gamma_0\xi$ defined on $\mathcal{D}(\overline{\mathbb{R}^2_+})$ can be extended to a linear and continuous mapping, still denoted by γ_0 from $V(\mathbb{R}^2_+,\omega)$ onto a subspace of $W^{\frac{1}{2},2}(\mathcal{R},\omega')$, denoted by $M(\mathcal{R},\omega')$. Equipped with the norm

$$\|\mu\|_{M(\mathcal{R},\omega')} := \inf_{\gamma_0\xi=\mu} \|\xi\|_{V(\mathbb{R}^2_+,\omega)},$$

$M(\mathcal{R},\omega')$ is a Banach space.
For any positive real number R, $B_R = B(0,R)$ denotes the open ball in $\mathbb{R}^2$ with center 0 and radius R and B'_R the exterior of $\overline{B}_R$.

3. Weak formulation of the boundary value problem

The *a priori* natural weighted space associated to $(\mathcal{P})$ is $V(\mathbb{R}^2_+,\omega)$ and we may see φ as a solution of a variational problem, associated with an integral convex functional:

$$\text{Minimize } F(\xi) \text{ such that } \xi \in V(\mathbb{R}^2_+,\omega) \text{ where}$$

• $\forall\xi \in V(\mathbb{R}^2_+,\omega)$, $F(\xi) = \displaystyle\int_{\mathbb{R}^2_+} \left[G(\partial_x\xi) + \frac{1}{2}(\partial_y\xi)^2\right] dx\,dy + \sqrt{M_\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} z'\xi\,dx.$

• $\forall t \in \mathbb{R}$, $G(t) = \frac{\gamma+1}{6}[(u_{cr} - t)^+]^3 - \frac{\gamma+1}{2}u_{cr}^2(u_{cr} - t) + \frac{\gamma+1}{3}u_{cr}^3$ and $G'(t) = g(t)$.

• F is Gâteaux-differentiable in $V(\mathbb{R}^2_+,\omega)$ with derivative:
$\forall\xi_1,\xi_2 \in V(\mathbb{R}^2_+,\omega)$,

$$< F'(\xi_1),\xi_2 >= \int_\Omega [g(\partial_x\xi_1)\partial_x\xi_2 + \partial_y\xi_1\partial_y\xi_2]\,dx\,dy + \sqrt{M_\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} z'\xi_2\,dx.$$

• As F is invariant with respect to the constants and zero is the highest degree of the polynomials contained in the reflexive space $V(\mathbb{R}^2_+,\omega)$, then the variational problem can be rewritten as:

$$\exists\,\varphi \in V(\mathbb{R}^2_+,\omega)/_{\mathbb{R}},\;\; F(\varphi) \le F(\xi)\;\; \forall\xi \in V(\mathbb{R}^2_+,\omega)/_{\mathbb{R}}. \tag{5}$$

4. Some properties of the functional settings

In this work, we are concerned with the existence and the uniqueness of the solution to $(\mathcal{P})$. To do this, it remains to establish the coerciveness of F on the quotient space $V(\mathbb{R}^2_+,\omega)/_{\mathbb{R}}$. Since F depends only on the gradient (this will be proved below), we have to prove a Poincaré type inequality on the space $V(\mathbb{R}^2_+,\omega)/_{\mathbb{R}}$. In order to, we proceed in three steps. The first step consists in proving this inequality on $V_0(B'_R,\omega)$, defined as the closure of $\mathcal{D}(B'_R)$ in $V(B'_R,\omega)$, where B'_R denotes the exterior of the closed ball in $\mathbb{R}^2$ with center 0 and radius R. Therefore, the second step consists in extending it on $V(\mathbb{R}^2,\omega)/\mathbb{R}$ by using an adequate partition of unity, that will enable to consider separately a bounded domain and the exterior of a closed ball. And the last step is to use a linear continuous extension operator from $V(\mathbb{R}^2_+,\omega)$ into $V(\mathbb{R}^2,\omega)$.
Before establishing the first step, some results are necessary and the first is given by:

Theorem 1. *(Amrouche and al. [3]) For any large enough real number R, there exists a positive constant C^1_R such that*

$$\forall\xi \in W^{1,2}_0(B'_R,\omega),\;\; \|\xi\|_{2,B'_R,\omega^2} \le C^1_R\|\nabla\xi\|_{2,B'_R,1}.$$

Lemma 1. *$V(B'_R,\omega)$ is compactly embedded into $L^3(B'_R,\omega^3)$.*

Proof. We verify that $V(B'_R,\omega)$ is continuously embedded in the weighted space W defined as follows:

$$W = \{\xi \in L^2(B'_R,(2+\rho^2)^{-\frac{1}{4}-1}\log^{-2}(2+\rho^2));\nabla\xi \in L^2(B'_R,(2+\rho^2)^{-\frac{1}{4}}\log^{-2}(2+\rho^2))\}.$$

Therefore W is compactly embedded into $L^3(B'_R,\omega^3)$. This compactness result is a particular case of that mentionned in Kufner and al. [14], 20.7. Example, p.290 which says:

Lemma 2. *(Kufner and Opic, pp. 289-290* [14]*) Let K be a compact set in $\mathbb{R}^2$ and let $\Omega = \mathbb{R}^2 \setminus K$. Set $a_* = \inf\{|X|; X \in \Omega\}$ and $a_* > 1$. Let $1 \le p \le q < \infty$, $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ and*

$$\omega(X) = |X|^\alpha \log^\gamma |X|, v_0(X) = |X|^{\beta-p}\log^\delta |X| \ \text{ and } \ v_1(X) = |X|^\beta \log^\delta |X|.$$

We consider $W^{1,p}(\Omega; v_0, v_1) = \{\xi \in L^p(\Omega, v_0), \nabla\xi \in L^p(\Omega, v_1)\}$. The embedding of $W^{1,p}(\Omega; v_0, v_1)$ in $L^q(\Omega,\omega)$ is compact if and only if

$$\frac{2}{q} - \frac{2}{p} + 1 > 0 \quad \text{and} \quad \frac{\alpha}{q} - \frac{\beta}{p} + \frac{2}{q} - \frac{2}{p} + 1 < 0.$$

We can now prove the Poincaré inequality in $V_0(B'_R,\omega)$.

Theorem 2. *For any large enough real number R, there exists a positive constant C_R^2 such that*

(6)

$$\forall \xi \in V_0(B'_R,\omega), \ \ \|\xi\|_{2,B'_R,\omega^2} + \|\xi\|_{3,B'_R,\omega^3} \le C_R^2(\|\nabla\xi\|_{2,B'_R,1} + \|\partial_x \xi\|_{3,B'_R,1}).$$

Proof. Since $\mathcal{D}(B'_R)$ is dense in $V_0(B'_R,\omega)$ then it suffices to prove this inequality on $\mathcal{D}(B'_R)$. To do this, we shall proceed by contradiction. If (6) is not true on $\mathcal{D}(B'_R)$ then for any $n \ge 1$, there exists ξ_n in $\mathcal{D}(B'_R)$ with:

$$\|\xi_n\|_{2,B'_R,\omega^2} + \|\xi_n\|_{3,B'_R,\omega^3} \ge n\left(\|\nabla\xi_n\|_{2,B'_R,1} + \|\partial_x\xi_n\|_{3,B'_R,1}\right).$$

Remark that $\xi_n \ne 0$, so that, using $\frac{\xi_n}{\|\xi_n\|_{2,B'_R,\omega^2} + \|\xi_n\|_{3,B'_R,\omega^3}}$ instead of ξ_n, we can suppose that $\|\xi_n\|_{2,B'_R,\omega^2} + \|\xi_n\|_{3,B'_R,\omega^3} = 1$ and $\{\xi_n\}_{n\ge1}$ is a bounded sequence in $V(B'_R,\omega)$. The compactness result of $V(B'_R,\omega)$ into $L^3(B'_R,\omega^3)$ in Lemma 1, combining with the reflexivity of $V(B'_R,\omega)$ imply the existence of a subsequence, still denoted by $\{\xi_n\}_{n\ge1}$, that converges strongly to an element ξ in $L^3(B'_R,\omega^3)$ and weakly in $V(B'_R,\omega)$. Therefore, it becomes from $\|\nabla\xi_n\|_{2,B'_R,1} + \|\partial_x\xi_n\|_{3,B'_R,1}$ tends to 0 that $\|\nabla\xi\|_{2,B'_R,1} + \|\partial_x\xi\|_{3,B'_R,1} = 0$. Thus $\xi = \text{cte}$ in B'_R. The continuity of the trace mapping from $V(B'_R,\omega)$ into $L^2(\partial B'_R)$ leads to $\xi = 0$ in B'_R. Now, we need a strong convergence of the sequence $\{\xi_n\}_{n\ge1}$ in $L^2(B'_R,\omega^2)$ to conclude by contradiction. For this, we use an adequate partition of unity:

$$\psi_1,\psi_2 \in \mathcal{C}^\infty(B'_R), \ \ 0 \le \psi_1,\psi_2 \le 1 \ \ \psi_1 + \psi_2 = 1 \ \text{ in } \ B'_R$$

with

$$\text{supp}\psi_1 \subset \overline{B}_{2R+2} \ \text{ and } \ \text{supp}\psi_2 \subset B'_{2R}.$$

Since, for fixed R, $V(B_{2R+2}\cap B'_R,\omega)$ coincides with $V(B_{2R+2}\cap B'_R)$, which is compactly embedded into $L^2(B_{2R+2}\cap B'_R)$ then

$$\begin{aligned}\xi_n &\to 0 \text{ strongly in } L^2(B_{2R+2}\cap B'_R),\\ \psi_1\xi_n &\to 0 \text{ strongly in } L^2(B_{2R+2}\cap B'_R).\end{aligned}$$

Using the fact that ψ_1 vanishes outside B_{2R+2}, it comes

$$\psi_1\xi_n \to 0 \text{ strongly in } L^2(B'_R)$$

and so

$$\psi_1\xi_n \to 0 \text{ strongly in } L^2(B'_R,\omega^2).$$

Applying Theorem 1 to $\psi_2\xi_n \in W_0^{1,2}(B'_{2R},\omega)$, we get

$$\begin{aligned}\|\psi_2\xi_n\|_{2,B'_{2R},\omega^2} &\leq C_R^1\|\nabla(\psi_2\xi_n)\|_{0,2,B'_{2R}}\\ &\leq C_R^1\left(\|\nabla(\psi_2\xi_n)\|_{0,2,B'_{2R}\cap B_{2R+2}}+\|\nabla(\psi_2\xi_n)\|_{0,2,B'_{2R+2}}\right).\end{aligned}$$

Notice that $\psi_2=0$ in $B_{2R}\cap B'_R$ and $\psi_2=1$ in B'_{2R+2}, then the last inequality leads to

$$\psi_2\xi_n \to 0 \text{ strongly in } L^2(B'_{2R},\omega^2) \text{ and in } L^2(B'_R,\omega^2).$$

This completes the proof.

Theorem 3. *The semi-norm* $|\,.\,|_{V(\mathbb{R}^2,\omega)}$ *is a norm on the space* $V(\mathbb{R}^2,\omega)/_{\mathbb{R}}$ *which is equivalent to the quotient norm.*

The proof is similar to that of Theorem 2. In fact, take an adequate partition of unity and consider separately the bounded domain B_R where the topology of $V(B'_R,\omega)$ coincides with that of $V(B'_R)$, and the exterior of $\overline{B}_R$ where the result of the Theorem 2 can be applied.

Corollary 1. *The semi-norm* $|\,.\,|_{V(\mathbb{R}^2_+,\omega)}$ *is a norm on the space* $V(\mathbb{R}^2_+,\omega)/_{\mathbb{R}}$ *which is equivalent to the quotient norm.*

This result can be proved via a linear continuous extension operator from $V(\mathbb{R}^2_+,\omega)$ in $V(\mathbb{R}^2,\omega)$ (by reflection).

At this stage, let us characterize the dual space of $M(\mathcal{R},\omega')$. Applying classical arguments, we can prove the following

Proposition 1. *(i) For any* ξ *in* $(M(\mathcal{R},\omega'))'$, *the dual space of* $M(\mathcal{R},\omega')$,

$$\|\zeta\|_{(M(\mathcal{R},\omega'))'}=\sup_{\{\xi\in V(\mathbb{R}^2_+,\omega),\xi\neq 0\}}\frac{<\zeta,\gamma_0\xi>_{\mathcal{R}}}{\|\xi\|_{V(\mathbb{R}^2_+,\omega)}}.$$

(ii) For any μ *in* $L^2(\mathcal{R},\omega')$, *we can identify* μ *as an element of* $(M(\mathcal{R},\omega'))'$ *and*

$$<\mu,\gamma_0\xi>_{(M(\mathcal{R},\omega'))',M(\mathcal{R},\omega')}=\int_{\mathcal{R}}\omega'\mu(\gamma_0\xi)\,dx$$

for any ξ *in* $V(\mathbb{R}^2_+, \omega)$.

Let us point out now some results on the divergence operator in the half-space. Consider the space

$$\mathbb{L}(\mathbb{R}^2_+, \omega^{-1}) = \Big\{ \mathbf{u} = (u_1, u_2) \in (L^{\frac{3}{2}}(\mathbb{R}^2_+) \cap L^2(\mathbb{R}^2_+)) \times L^2(\mathbb{R}^2_+); \\ \omega^{-1}\,\mathrm{div}\mathbf{u} \in L^{\frac{3}{2}}(\mathbb{R}^2_+) \cap L^2(\mathbb{R}^2_+) \Big\},$$

which has the Banach structure with the norm

$$\|\mathbf{u}\|_{\mathbb{L}(\mathbb{R}^2_+, \omega^{-1})} = \|u_1\|_{\frac{3}{2},\mathbb{R}^2_+,1} + \|u_1\|_{2,\mathbb{R}^2_+,1} + \|u_2\|_{2,\mathbb{R}^2_+,1} + \|\omega^{-1}\mathrm{div}\,\mathbf{u}\|_{\frac{3}{2},\mathbb{R}^2_+,1} + \|\omega^{-1}\mathrm{div}\,\mathbf{u}\|_{2,\mathbb{R}^2_+,1}.$$

By adapting the proof, based on the Hahn-Banach theorem and the density result of $\mathcal{D}(\mathbb{R}^2_+)$ in $V_0(\mathbb{R}^2_+, \omega)$, we get the following result:

Theorem 4. $(\mathcal{D}(\overline{\mathbb{R}^2_+}))^2$ *is dense in* $\mathbb{L}(\mathbb{R}^2_+, \omega^{-1})$.

The following lemma holds

Lemma 3. *Denoting by* $\mathbf{n}$ *the unit outer normal to* $\mathcal{R}$*, then the mapping* $\gamma_n : \mathbf{u} \longmapsto \mathbf{u}.\mathbf{n}$ *defined on* $(\mathcal{D}(\overline{\mathbb{R}^2_+}))^2$ *can be extended to a linear and continuous mapping, still denoted by* γ_n*, from* $\mathbb{L}\left(\mathbb{R}^2_+, \omega^{-1}\right)$ *in* $(M(\mathcal{R}, \omega'))'$*, the dual space of* $M(\mathcal{R}, \omega')$*. Moreover, we have the Green formula:*

$$\int_{\mathbb{R}^2_+} (\xi\, div\,\mathbf{u} + \mathbf{u}.\nabla\,\xi)\,d\,x\,d\,y = <\mathbf{u}.\mathbf{n},\ \gamma_0\xi>,\ \forall\,\xi \in V(\mathbb{R}^2_+, \omega),\ \forall\,\mathbf{u} \in \mathbb{L}\left(\mathbb{R}^2_+, \omega^{-1}\right),$$

where $< .,. >$ *is the duality* $(M(\mathcal{R}, \omega'))'$ *and* $M(\mathcal{R}, \omega')$.

5. Some properties of the functional F

In the sequel, we will denote, for any ξ in $V(\mathbb{R}^2_+, \omega)/_{\mathbb{R}}$,

$$b(\xi) = \sqrt{M_\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} z'(x)\xi(x)\,dx.$$

Lemma 4. *Let R_y be a real number which satisfies Hypothesis (1) and let $\Omega_0 =]-\frac{1}{2},\frac{1}{2}[\times]0,R_y[$. Then, for any ξ in $V(\mathbb{R}^2_+,\omega)/_{\mathbb{R}}$ we have*

$$\begin{aligned}
b(\xi) &= -\sqrt{M_\infty}\int_{\Omega_0} z'\left(1-\frac{y}{R_y}\right)\partial_y\xi\,d\,x\,d\,y - \frac{\sqrt{M_\infty}}{R_y}\int_{\Omega_0} z\,\partial_x\xi\,d\,x\,d\,y,\\
F(\xi) &\geq \frac{\gamma+1}{12}\|u_{cr}-(u_{cr}-\partial_x\xi)^+\|^3_{3,\mathbb{R}^2_+,1} + \frac{\gamma+1}{4}u_{cr}\|u_{cr}-(u_{cr}-\partial_x\xi)^+\|^2_{2,\mathbb{R}^2_+,1}\\
&+ \left(\frac{\gamma+1}{2}u_{cr}^2 - \frac{\sqrt{M_\infty}}{R_y}\right)\|(u_{cr}-\partial_x\xi)^-\|_{1,\mathbb{R}^2_+,1} + \frac{1}{4}\|\partial_y\xi\|^2_{2,\mathbb{R}^2_+,1}\\
&- \sqrt{M_\infty}\,u_{cr} - \frac{M_\infty c_1^2 R_y}{3} \quad \textit{with } c_1 = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |z'(x)|^2\,d\,x\right)^{\frac{1}{2}}.
\end{aligned}$$

Proof. After rewriting $b(\xi)$ as follows

$$b(\xi) = -\sqrt{M_\infty}\int_{\partial\Omega_0} z'\left(1-\frac{y}{R_y}\right)n_y\xi\,d\,\sigma$$

where $\mathbf{n} = (n_x, n_y)$ is the unit outer normal to $\partial\Omega_0$, we apply the Green formula in the bounded case and obtain the result.

Remark 1. i) In the context of subsonic flows, the problem is posed in the closed convex set $K = \{\xi \in V(\mathbb{R}^2_+,\omega)/_{\mathbb{R}};\, \partial_x\xi \leq u_{cr} \text{ a.e. in } \mathbb{R}^2_+\}$. More precisely, we have the existence of a unique minimizer φ of F in K satisfying $< F'(\varphi), \xi-\varphi >\geq 0$ for any $\xi \in K$ where $F'(\varphi)$ is the Gâteaux derivative of F at φ.
ii) If there exists a constant $\eta > 0$ such that $\partial_x\varphi \leq u_{cr} - \eta$ a.e. in $\mathbb{R}^2_+$ then the function φ satisfies

$$\text{(7)} \qquad \begin{cases} \partial_x(g(\partial_x\varphi)) + \partial_y(\partial_y\varphi) = 0 \ \text{ in } \ \mathbb{R}^2_+, \\[2ex] (g(\partial_x\varphi), \partial_y\varphi).\mathbf{n} = \sqrt{M_\infty}\,Z'(x) \ \text{ in } \ (M(\mathcal{R},\omega_1'))' \end{cases}$$

where $\mathbf{n}$ is the unit outer normal to $\mathcal{R}$ and Z is the extension of z by zero outside $]-\frac{1}{2},\frac{1}{2}[$.
iii) F is not coercive in $V(\mathbb{R}^2_+,\omega)/_{\mathbb{R}}$ as the following example shows: we consider for any $n \in \mathbb{N}^*$

$$\xi_n(x,y) = \begin{cases} -\frac{u_{cr}}{(1-x)(1+y)^2} & \text{if } \ x \leq -\frac{1}{n}, \\ \frac{n^2 u_{cr}}{n+1}\frac{x}{(1+y)^2} & \text{if } \ |x| \leq \frac{1}{n}, \\ \frac{u_{cr}}{(1+x)(1+y)^2} & \text{if } \ \ x \geq \frac{1}{n}, \end{cases}$$

then $F(\xi_n)$ remains bounded but $\|\xi_n\|_{V(\mathbb{R}^2_+,\omega)}$ tends to infinity.
It becomes that the problem (5) does not necessarily have a solution. Thus it is convenient to introduce the dual problem associated to (5).

Indeed, the dual problem is generally simpler than the initial problem and permits to obtain more information about the solutions of this one.

6. Perturbations and the associated dual problem

Let us set $\mathcal{L} = \inf_{\xi \in V(\mathbb{R}^2_+,\omega)} F(\xi)$, where F is the functional to be minimized. It can be rewritten as $F(\xi) = I(\nabla\xi) + b(\xi)$ with I is the functional defined of $\mathbb{H}(\mathbb{R}^2_+)$ into $\mathbb{R}$ by: $\forall\, \mathbf{q} = (q_1, q_2) \in \mathbb{H}(\mathbb{R}^2_+)$, $I(\mathbf{q}) = \int_{\mathbb{R}^2_+}(G(q_1) + \frac{1}{2}q_2^2)\,dx\,dy$. We introduce now the functional, denoted by ϕ, on $V(\mathbb{R}^2_+,\omega) \times \mathbb{H}(\mathbb{R}^2_+)$ into $\mathbb{R}$ such that $\phi(\xi,\mathbf{q}) = I(\nabla\xi + \mathbf{q}) + b(\xi)$. We see that $\phi(\xi,\mathbf{0}) = F(\xi)$.
We consider the minimization problem, for any $\mathbf{q} = (q_1, q_2)$ in $\mathbb{H}(\mathbb{R}^2_+)$,

$$J(\mathbf{q}) = \inf_{\xi\in V(\mathbb{R}^2_+,\omega)} \phi(\xi,\mathbf{q}). \tag{8}$$

The problems (8) are said the perturbed problems of (5). Let ϕ_* be the conjugate function of ϕ defined from $\mathbb{L}(\mathbb{R}^2_+,\omega^{-1}) \times \mathbb{H}'(\mathbb{R}^2_+)$ into $\mathbb{R}$; it is based on the duality between $V(\mathbb{R}^2_+,\omega) \times \mathbb{H}(\mathbb{R}^2_+)$ and $\mathbb{L}(\mathbb{R}^2_+,\omega^{-1}) \times \mathbb{H}'(\mathbb{R}^2_+)$. The function ϕ_* is defined, for every $\mathbf{p} \in \mathbb{L}(\mathbb{R}^2_+,\omega^{-1})$ and $\mathbf{u} \in \mathbb{H}'(\mathbb{R}^2_+)$, by

$$\phi_*(\mathbf{p},\mathbf{u}) = \sup_{\xi\in V(\mathbb{R}^2_+,\omega),\mathbf{q}\in\mathbb{H}(\mathbb{R}^2_+)} \left[< \mathbf{p},\xi >_{\mathbb{L}(\mathbb{R}^2_+,\omega^{-1}),V(\mathbb{R}^2_+,\omega)} + < \mathbf{u},\mathbf{q} >_{\mathbb{H}'(\mathbb{R}^2_+),\mathbb{H}(\mathbb{R}^2_+)} - \phi(\xi,\mathbf{q}) \right],$$

where $< \mathbf{p},\xi >_{\mathbb{L}(\mathbb{R}^2_+,\omega^{-1}),V(\mathbb{R}^2_+,\omega)} = \int_{\mathbb{R}^2_+}(\xi\,\mathrm{div}\,\mathbf{u} + \mathbf{u}.\nabla\,\xi)\,dx\,dy$.
The dual problem consists in finding $\mathbf{U}$ in $\mathbb{H}'(\mathbb{R}^2_+)$ such that

$$-\phi_*(\mathbf{0},\mathbf{U}) = \sup_{\mathbf{u}\in\mathbb{H}'(\mathbb{R}^2_+)} [-\phi_*(\mathbf{0},\mathbf{u})]. \tag{9}$$

Theorem 5. *There exists* (U_1, U_2) *in* $\mathbb{H}'(\mathbb{R}^2_+)$ *such that*
i) $\forall\,\xi \in V(\mathbb{R}^2_+,\omega)$, $\displaystyle\int_{\mathbb{R}^2_+} U_1\partial_x\xi\,dx\,dy + \int_{\mathbb{R}^2_+} U_2\partial_y\xi\,dx\,dy + b(\xi) = 0,$
ii) $-U_2 + \sqrt{M_\infty}\,Z'(x) = 0$ *in* $(M(\mathcal{R},\omega'))'$,
iii) $U_1 \le g(u_{cr})$ *a.e. in* $\mathbb{R}^2_+$,
iv) $\displaystyle\mathcal{L} = \int_{\mathbb{R}^2_+}\left(G(g^{-1}(U_1)) - g^{-1}(U_1)U_1 + \frac{1}{2}U_2^2\right)\,dx\,dy,$

where Z denotes the extension of z by zero outside $]-\frac{1}{2},\frac{1}{2}[$.
Proof. Since F is convex, $\inf_{\xi\in V(\mathbb{R}^2_+,\omega)} F(\xi)$ is finite and I is continuous on $\mathbb{H}(\mathbb{R}^2_+)$, the conditions of theorem 4.1 of (Ekeland and al. [7], p.58) are satisfied. Hence problem (5) is stable i.e.

$$\inf_{\xi\in V(\mathbb{R}^2_+,\omega)} F(\xi) = \sup_{\mathbf{u}\in\mathbb{H}'(\mathbb{R}^2_+)} [-\phi_*(\mathbf{0},\mathbf{u})],$$

and (9) has at least one solution $\mathbf{U}$ in $\mathbb{H}'(\mathbb{R}^2_+)$.
Therefore,

$$\mathcal{L} = -\phi_*(\mathbf{0},\mathbf{U}) \;=\; \inf_{\xi \in V(\mathbb{R}^2_+,\omega),\mathbf{q}\in \mathbb{H}(\mathbb{R}^2_+)} \left[\phi(\xi,\mathbf{q}) - <\mathbf{U},\mathbf{q}>_{\mathbb{H}'(\mathbb{R}^2_+),\mathbb{H}(\mathbb{R}^2_+)}\right]$$

and get

$$\mathcal{L} \;\leq\; \inf_{\mathbf{q}\in \mathbb{H}(\mathbb{R}^2_+)} \left[J(\mathbf{q}) - <\mathbf{U},\mathbf{q}>_{\mathbb{H}'(\mathbb{R}^2_+),\mathbb{H}(\mathbb{R}^2_+)}\right].$$

In particular, for any ξ in $V(\mathbb{R}^2_+,\omega)$, we have

$$\mathcal{L} \;\leq\; J(\nabla \xi) - <\mathbf{U},\nabla\xi>_{\mathbb{H}'(\mathbb{R}^2_+),\mathbb{H}(\mathbb{R}^2_+)}\;.$$

Moreover,

$$\begin{aligned} & J(\nabla\xi) \\ = \quad & \inf_{\zeta\in V(\mathbb{R}^2_+,\omega)} \left[\int_{\mathbb{R}^2_+} G(\partial_x(\xi+\zeta))\,dxdy + \frac{1}{2}\int_\Omega (\partial_y(\xi+\zeta))^2\,dxdy + b(\zeta+\xi)\right] - b(\xi) \\ = \quad & \mathcal{L} - b(\xi), \end{aligned}$$

and, since $V(\mathbb{R}^2_+,\omega)$ is a linear space,

$$<\mathbf{U},\nabla\xi>_{\mathbb{H}'(\mathbb{R}^2_+),\mathbb{H}(\mathbb{R}^2_+)} + b(\xi) = 0. \tag{10}$$

Whence, $\mathrm{div}\mathbf{U} = 0$ a.e in $\mathbb{R}^2_+$.
2. Since

$$\phi_*(\mathbf{0},\mathbf{U}) \;=\; \sup_{\xi\in V(\mathbb{R}^2_+,\omega),\mathbf{q}\in\mathbb{H}(\mathbb{R}^2_+)} \left[<\mathbf{U},\mathbf{q}>_{\mathbb{H}'(\mathbb{R}^2_+),\mathbb{H}(\mathbb{R}^2_+)} - I(\nabla\xi+\mathbf{q}) - b(\xi)\right],$$

we can write

$$\begin{aligned} \phi_*(\mathbf{0},\mathbf{U}) &= \sup_{\xi\in V(\mathbb{R}^2_+,\omega),\mathbf{p}\in\mathbb{H}(\mathbb{R}^2_+)} \left[<\mathbf{U},\mathbf{p}-\nabla\xi>_{\mathbb{H}'(\mathbb{R}^2_+),\mathbb{H}(\mathbb{R}^2_+)} - I(\mathbf{p}) - b(\xi)\right] \\ &= \sup_{\xi\in V(\mathbb{R}^2_+,\omega),\mathbf{p}\in\mathbb{H}(\mathbb{R}^2_+)} \left[<\mathbf{U},\mathbf{p}>_{\mathbb{H}'(\mathbb{R}^2_+),\mathbb{H}(\mathbb{R}^2_+)} - <\mathbf{U},\nabla\xi>_{\mathbb{H}'(\mathbb{R}^2_+),\mathbb{H}(\mathbb{R}^2_+)} - I(\mathbf{p}) - b(\xi)\right]. \end{aligned}$$

Thus, if we set for $\mathbf{p} = (p_1,p_2)$ and $\mathbf{U} = (U_1,U_2)$, Equality (10) leads to

$$\begin{aligned} \phi_*(\mathbf{0},\mathbf{U}) &= \sup_{\mathbf{p}\in\mathbb{H}(\mathbb{R}^2_+)} \left[<\mathbf{U},\mathbf{p}>_{\mathbb{H}'(\mathbb{R}^2_+),\mathbb{H}(\mathbb{R}^2_+)} - I(\mathbf{p})\right] \\ &= \sup_{\substack{p_1\in L^3(\mathbb{R}^2_+)\cap L^2(\mathbb{R}^2_+),\\ p_2\in L^2(\mathbb{R}^2_+)}} \left[\int_{\mathbb{R}^2_+} (U_1p_1+U_2p_2)\,dxdy - \int_{\mathbb{R}^2_+} G(p_1)\,dxdy - \frac{1}{2}\int_{\mathbb{R}^2_+} p_2^2\,dxdy\right] \\ &= \sup_{p_1\in L^3(\mathbb{R}^2_+)\cap L^2(\mathbb{R}^2_+)} \left[\int_{\mathbb{R}^2_+} (U_1p_1 - G(p_1))\,dxdy\right] + \sup_{p_2\in L^2(\mathbb{R}^2_+)} \left[\int_{\mathbb{R}^2_+} \left(U_2p_2 - \frac{1}{2}p_2^2\right)\,dxdy\right]. \end{aligned}$$

Whence, by using (Ekeland and al. [7], proposition IV.1.2), we get $U_1 \leq \frac{\gamma+1}{2}u_{cr}^2 = g(u_{cr})$ and

$$\begin{aligned}
\phi_*(\mathbf{0},\mathbf{U}) &= \int_{\mathbb{R}^2_+} U_1\left(u_{cr} - \sqrt{u_{cr}^2 - \frac{2}{\gamma+1}U_1}\right)\,dxdy \\
&- \int_{\mathbb{R}^2_+}\left[G\left(u_{cr} - \sqrt{u_{cr}^2 - \frac{2}{\gamma+1}U_1}\right) - \frac{1}{2}U_2^2\right]\,dxdy \\
&= \int_{\mathbb{R}^2_+}\left(U_1 g^{-1}(U_1) - G(g^{-1}(U_1)) + \frac{1}{2}U_2^2\right)\,dxdy.
\end{aligned}$$

3. Thanks to the Green formula, Equality (10) leads to point ii).

Remark 2. i) If we define $u_* = g^{-1}(U_1)$ and $v_* = U_2$, we get $u_* \leq u_{cr}$ and

$$\mathcal{L} = \int_{\mathbb{R}^2_+}\left(G(u_*) - u_* g(u_*) - \frac{1}{2}v_*^2\right)\,dxdy.$$

ii) Problem (5) may have or not a solution. But if it has a solution φ such that $\partial_x\varphi \leq u_{cr}$ a.e. in $\mathbb{R}^2_+$, then $\nabla\varphi = (u_*, v_*)$.

Proposition 2. *Assume that Hypothesis (1) is verified. Then, every minimizing sequence $\{\xi_n\}_{n\geq 1}$ in $V(\mathbb{R}^2_+,\omega)/_{\mathbb{R}}$ for the functional F satisfies:*

$$\begin{cases}
\{u_{cr} - (u_{cr} - \partial_x\xi_n)^+\}_{n=1}^{\infty} \text{ converges strongly to } u_* \\
\text{in } L^3(\mathbb{R}^2_+) \text{ and in } L^2(\mathbb{R}^2_+), \\
\{\partial_y\xi_n\}_{n=1}^{\infty} \text{ converges strongly to } v_* \text{ in } L^2(\mathbb{R}^2_+), \\
\{(u_{cr} - \partial_x\xi_n)^-\}_{n=1}^{\infty} \text{ is bounded in } L^1(\mathbb{R}^2_+), \\
\lim\limits_{n\to+\infty}\int_{\mathbb{R}^2_+}(u_{cr} - u_*)^2(u_{cr} - \partial_x\xi_n)^-\,d\,x\,d\,y = 0.
\end{cases}$$

Proof. Any minimizing sequence $\{\xi_n\}_{n\geq 1}$ for (5) in $V(\mathbb{R}^2_+,\omega)/_{\mathbb{R}}$ satisfies

$$\begin{aligned}
F(\xi_n) - \mathcal{L} &= \int_{\mathbb{R}^2_+}\left(G(\partial_x\xi_n) - G(u_*) - (\partial_x\xi_n - u_*)g(u_*)\right)dxdy \\
&+ \frac{1}{2}\int_{\mathbb{R}^2_+}(\partial_y\xi_n - v_*)^2\,dxdy \\
&\geq \frac{\gamma+1}{6}\int_{\mathbb{R}^2_+}|u_{cr} - (u_{cr} - \partial_x\xi_n)^+ - u_*|^3\,d\,x\,d\,y \\
&+ \frac{\gamma+1}{2}\int_{\mathbb{R}^2_+}(u_{cr} - u_*)^2(u_{cr} - \partial_x\xi_n)^-\,d\,x\,d\,y + \frac{1}{2}\int_{\mathbb{R}^2_+}(\partial_y\xi_n - v_*)^2\,d\,x\,d\,y.
\end{aligned}$$

Moreover, since for any $s, t \in \mathbb{R},\ s, t \leq u_{cr}$,

$$G(s)-G(t)-(s-t)g(t) = \frac{\gamma+1}{2}u_{cr}(s-t)^2-\frac{\gamma+1}{6}(s-t)^3-\frac{\gamma+1}{2}t(s-t)^2,$$

then

$$\begin{aligned}&\frac{\gamma+1}{2}u_{cr}\int_{\mathbb{R}^2_+}|u_{cr}-(u_{cr}-\partial_x\xi_n)^+-u_*|^2\,d\,x\,d\,y\\ \leq\ & \frac{2}{n}+\frac{\gamma+1}{2}\left(\int_{\mathbb{R}^2_+}|u_*|^3\,d\,x\,d\,y\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^2_+}|u_{cr}-(u_{cr}-\partial_x\xi_n)^+-u_*|^3\,d\,x\,d\,y\right)^{\frac{2}{3}}.\end{aligned}$$

Then $\{u_{cr}-(u_{cr}-\partial_x\xi_n)^+\}_{n\geq 1}$ converges to u_* in $L^2(\mathbb{R}^2_+)\cap L^3(\mathbb{R}^2_+)$ and $\{\partial_y\xi_n\}_{n\geq 1}$ converges to v_* in $L^2(\mathbb{R}^2_+)$. Therefore, according to Lemma 4, $\{(u_{cr}-\partial_x\xi_n)^-\}_{n\geq 1}$ is bounded in $L^1(\mathbb{R}^2_+)$. This ends the proof.

Remark 3. Let $\{\xi_n\}_{n\geq 1}$ be as above. It results form

$$\begin{aligned}\int_{\mathbb{R}^2_+}\omega^2|\xi_n|\,d\,x\,d\,y &\leq \sqrt{\frac{\pi}{\log 2}}\|\xi_n\|_{2,\mathbb{R}^2_+,\omega^2},\\ \int_{\mathbb{R}^2_+}\omega^2|u_{cr}-(u_{cr}-\partial_x\xi_n)^+|\,d\,x\,d\,y &\leq \frac{1}{\log^3 2}\sqrt{\frac{\pi}{2}}\|u_{cr}-(u_{cr}-\partial_x\xi_n)^+\|_{2,\mathbb{R}^2_+,1},\\ \int_{\mathbb{R}^2_+}\omega^2|(u_{cr}-\partial_x\xi_n)^-|\,d\,x\,d\,y &\leq \frac{1}{2\log^2 2}\|(u_{cr}-\partial_x\xi_n)^-\|_{1,\mathbb{R}^2_+,1},\\ \int_{\mathbb{R}^2_+}\omega^2|\partial_y\xi_n|\,d\,x\,d\,y &\leq \frac{1}{\log^3 2}\sqrt{\frac{\pi}{2}}\|\partial_y\xi_n\|_{2,\mathbb{R}^2_+,1},\end{aligned}$$

that $\{\omega^2\xi_n\}_{n\geq 1}$ belongs to $L^1(\mathbb{R}^2_+)$, $\{\omega^2\nabla\xi_n\}$ is bounded in $(L^1(\mathbb{R}^2_+))^2$ and no conclusion on the boundeness of $\{\omega^2\xi_n\}_{n\geq 1}$.
As a consequence of the Gagliardo-Nirenberg-Sobolev inequality, there are two positive constants C and C' (C' depends on C) such that

$$\|\omega^2\xi_n\|_{2,\mathbb{R}^2_+,1} \leq C'\|\omega^2\xi_n\|_{1,\mathbb{R}^2_+,1}+C\|\omega^2\nabla\xi_n\|_{1,\mathbb{R}^2_+,1}.$$

In order to derive a sufficient condition for the existence of solutions to Problem (5), a first suitable way is to ensure the boundeness of the sequence $\{\omega^2\xi_n\}_{n\geq 1}$ in $L^1(\mathbb{R}^2_+)$. Such a condition can be obtained, in the bounded case, by establishing a Poincaré-Wirtinger's inequality on a non reflexive $W^{1,1}$ type space (see [1, 2]).

Theorem 6. *If $\{\xi_n\}_{n\geq 1}$ is bounded in $L^1(\mathbb{R}^2_+,\omega^2)$ then there exists a function φ in $L^2(\mathbb{R}^2_+,\omega^4)$ and a bounded nonnegative measure ν on $\mathbb{R}^2_+$ such that $\partial_x\varphi = u_*+\nu$, $\partial_y\varphi = v_*$ and $\nu\{(x,y)\in\mathbb{R}^2_+; u_*(x,y)<u_{cr}\} = 0$.*

Proof. According to the previous remark, there exists a subsequence of $\{\xi_n\}_{n\geq 1}$ that converges weakly to an element φ in $L^2(\mathbb{R}^2_+, \omega^4)$. Furthemore, we have $v_* = \partial_y \varphi$ a.e. in $\mathbb{R}^2_+$. Now, for any nonnegative ψ in $\mathcal{D}(\mathbb{R}^2_+)$, we have:

$< \partial_x \varphi - u_*, \psi >= \lim\limits_{n\to\infty} \int_{\mathbb{R}^2_+} (u_{cr} - \partial_x \xi_n)^- \psi \, d\,x\, d\,y$. So, $\partial_x \varphi - u_* \geq 0$ in the sense of distributions and $\partial_x \varphi = u_* + \nu$, where ν is a nonnegative measure defined for any nonnegative ψ in $\mathcal{D}(\mathbb{R}^2_+)$ by: $\nu(\psi) = \lim\limits_{n\to\infty} \int_{\mathbb{R}^2_+} (u_{cr} - \partial_x \xi_n)^- \psi \, d\,x\, d\,y$.

We set $A = \{(x,y) \in \mathbb{R}^2_+ ; u_*(x,y) < u_{cr}\}$. Then $A = \cup_{m\geq 1} A_m$, where $A_m = \{(x,y) \in \mathbb{R}^2_+ ; u_*(x,y) \leq u_{cr} - \frac{1}{m}\}$. Since $\{A_m\}_{m\geq 1}$ is a nondecreasing sequence of $\nu-$measurable sets then $\nu(A) = \lim\limits_{m\to\infty} \nu(A_m)$ and

$$\begin{aligned}
\frac{1}{m^2}\nu(A_m) &= \frac{1}{m^2} \lim_{n\to\infty} \int_{A_m} (u_{cr} - \partial_x \xi_n)^- \, d\,x\, d\,y \\
&\leq \lim_{n\to\infty} \int_{A_m} (u_{cr} - u_*)^2 (u_{cr} - \partial_x \xi_n)^- \, d\,x\, d\,y \\
&\leq \lim_{n\to\infty} \int_{\mathbb{R}^2_+} (u_{cr} - u_*)^2 (u_{cr} - \partial_x \xi_n)^- \, d\,x\, d\,y = 0.
\end{aligned}$$

Remark 4. If the condition of Theorem 6 holds true, then

• Using the properties of each minimizing sequence $\{\xi_n\}_{n\geq 1}$ of the problem (5), we prove the uniqueness of the couple (u_*, v_*) in $\mathbb{H}(\mathbb{R}^2_+)$.

• If $\partial_x \varphi < u_{cr}$ a.e.in $\mathbb{R}^2_+$ then $u_* = \partial_x \varphi$ a.e. in $\mathbb{R}^2_+$, $\nu = 0$ and φ is the unique solution of (5).

• If $u_* < u_{cr}$ a.e. in $\mathbb{R}^2_+$ then $\partial_x \varphi = u_*$ a.e. in $\mathbb{R}^2_+$, $\nu = 0$ and φ is the unique solution of (5).

• If φ and ψ are two solutions of (5), then there exists f in $L^2\left(\mathbb{R}, \int_0^{+\infty} \omega^4 \, dy\right)$ such that

$$\varphi(x,y) = \psi(x,y) + f(x) \quad \text{for almost every} \quad (x,y) \in \mathbb{R}^2_+.$$

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