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On the stationary solutions of generalized reaction diffusion equations with p & q-Laplacian

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On the stationary solutions of generalized reaction diffusion equations with p & q- Laplacian

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Abstract

In this paper, we analyse a family of stationary nonlinear equations with p&q- Laplacian $-\Delta_p u - \Delta_q u = \lambda c(x,u)$ which have a wide spectrum of applications in many areas of science. We introduce a new type of variational principles corresponding to this family of equations. Using this formalism, we exhibit intervals for the scalar parameter λ where there exist positive solutions of the considered problems. Furthermore, we prove, in another interval, the nonexistence of nontrivial solutions. These results are different from those of existence and nonexistence for stationary equations with single Laplacian.

AMS Subject Classifications: 35J70, 35J65, 47H17

1 Introduction

This paper is devoted to the study of the existence and non-existence results for solutions to the following stationary nonlinear equation

$$-\operatorname{div}\left(\left[|\nabla u|^{p-2}+|\nabla u|^{q-2}\right]\nabla u\right)+q(x)|u|^{p-2}u+w(x)|u|^{q-2}u=\lambda f(x)|u|^{\gamma-2}u \quad (1)$$

considered in a bounded domain $\Omega\subset\mathbb{R}^n$, $n\geq 1$, under Dirichlet boundary condition

$$u=0 \quad \text{on} \quad \partial\Omega. \qquad (2)$$

We study problem (1)-(2) with respect to the real parameter λ. Hereafter, We suppose that the boundary $\partial\Omega$ is sufficiently smooth, and that the coefficients p, q, γ satisfy

$$1 < p < \gamma < q \quad \text{and}\ \gamma < p^*, \ \text{where} \ \ p^* = \begin{cases} \frac{pn}{n-p} & \text{if} \ \ p < n, \\ +\infty & \text{if} \ \ p \geq n. \end{cases} \tag{3}$$

We also assume that

$$f,\ q,\ w\ \in L^r(\Omega), \quad \text{with} \begin{cases} r > \frac{p^*}{p^*-\gamma} & \text{if} \ \ p < n, \\ r > 1 & \text{if} \ \ p \geq n. \end{cases} \tag{4}$$

$$\text{and } q(x) \geq 0, w(x) \geq 0 \text{ a. e. on } \Omega.$$

The boundary value problem (1)-(2) arises in the study of stationary solutions of reaction diffusion systems. A general reaction diffusion system has the form

$$u_t = \operatorname{div}[D(u)\nabla u] + c(x,u). \tag{5}$$

This equation has a wide range of applications in physical and related sciences, e.g. in biophysics [11], plasma physics [16], and chemical reaction design [2]. In such applications, the function u in (5) describes a concentration, the first term corresponds to the diffusion with a (generally non-constant) diffusion coefficient $D(u)$, whereas the second one is the reaction and relates to source and loss processes. Boundary conditions are taken as zero flux, that is, the domain's boundary is assumed impermeable to chemical species. Typically, in chemical and biological applications, the reaction term $c(x,u)$ in (5) has a polynomial form with respect to the concentration u.

Thus, in our case, we discuss stationary solutions of (5), with a diffusion coefficient having a power law dependency $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$ and reaction terms $c(x,u) = -q(x)|u|^{p-2}u - w(x)|u|^{q-2}u + \lambda f(x)|u|^{\gamma-2}u$.

Another important example where equation (1) arises is the study of soliton-like solutions of the nonlinear Schrödinger equation

$$i\psi_t = -\Delta\psi + q(x)\psi - \lambda f(x)|\psi|^{\gamma-2}\psi - \Delta_q\psi + W'(x,\psi), \tag{6}$$

where $\Delta_q\psi = \operatorname{div}(|\nabla\psi|^{q-2}\nabla\psi)$ is a q-Laplacian. This class of equations was proposed by C.H. Derrick [7] as a model for elementary particles (see [5, 3, 4] for details). If we consider standing waves of (6) in a bounded domain Ω

$$\psi(t,x) = u(x)e^{-i\omega t}$$

then, in order to find $\lambda \in \mathbb{R}$ and u one has the following eigenvalue problem

$$\begin{cases} -\Delta u + (V(x) - \omega)u - \Delta_q u + W'(x,u) = \lambda f(x)|u|^{\gamma-2}u, \quad x \in \Omega, \\ \\ u = 0, \quad \text{on} \ \ \partial\Omega. \end{cases} \tag{7}$$

which belongs to the class of problems (1)-(2) for $p = 2 < \gamma < 2^*$, $(V(x) - \omega) = q(x)$, and $W(x, u) = w(x)|u|^q$.

Equation (1), which contains only one of the Laplacian, i.e., the p-Laplacian $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ or the q-Laplacian $\Delta_q u = \mathrm{div}(|\nabla u|^{q-2}\nabla u)$ has been widely considered (see e.g. [15] and the references therein). However, there does not seem to have been much work when both p-Laplacian and q-Laplacian occur in (1). As far as we know, only the papers [5, 3, 4] deal with such a problem. Note that, in general, problem (1)-(2) is non-coercive, non-homogeneous, the corresponding Euler functional is unbounded both from above and below, and the nonlinear term $f(x)|u|^{\gamma-2}u$ is indefinite, i.e., $f(x)$ may change sign in Ω. Another obstacle here is that only little is known about the regularity of the solution of (1) (see [8]), which makes it difficult to apply for example powerful tools such as the super-sub solutions or bifurcation methods. Nonhomogeneity and indefinitness of the nonlinearity makes it difficult to apply the constrained minimization method and the Mountain-Pass Lemma for arbitrary λ.

In our approach, the main idea to overcome these difficulties lies on the nonlinear spectral analysis recently developed in [12] by one of the authors of the present paper. Based on this idea, we introduce a well-defined variational principle which, in general, corresponds to problem (1)-(2) on a discrete subset of the values λ. Firstly, this variational principle enables us to prove the existence of non-negative solutions to (1)-(2) for some specific characteristic values $\lambda = \lambda_1^+, \lambda_1^-$. Then we prove the existence of non-negative solutions in the whole intervals $\lambda > \lambda_1^+, \lambda < \lambda_1^-$. This variational principle enables us also to find a priori bifurcation points $-\infty \leq \Lambda_1^- < 0 < \Lambda_1^+ \leq +\infty$ and to prove the non-existence result of any non trivial solutions for (1)-(2) as $\lambda \in]\Lambda_1^-, \Lambda_1^+[$.

It should be noted that, in case of a single Laplacian in (1), the behaviour with respect to λ is distinguished. For instance, when equation (1) contains only the p-Laplacian and $w(x) \equiv 0$, $q(x) \geq 0$, $f(x) > 0$, it is well known (see for example [15]) that for all $\lambda > 0$ (1)-(2) possesses a positive solution. Nevertheless, under the same conditions, but with the addition of the q-Laplacian term in (1), one has (see below) a gap $\lambda \in (0, \Lambda_1^+)$ of non-existence of solution for (1)-(2).

The paper is organised as follows. In section 2, we briefly present our results. In section 3, we introduce a 0-homogeneous variational problem which corresponds to (1)-(2). In section 4, we show that there exists an interval for λ in which problem (1)-(2) only admits the trivial solution. In section 5, we prove the existence of a non-trivial non-negative solution for two characteristic values of λ. In section 6, we apply the Mountain-Pass Lemma in order to achieve the existence of a whole branch of non-negative solutions, starting from these two characteristic values. Finally in section 7, we prove that problem (14)-(2) has a unique weak solution in $W_0^{1,q}(\Omega)$.

2 Statements of the Results

Introduce the following functionals

$$H_p(v) = \int_\Omega (|\nabla v|^p + q(x)|v|^p)dx, \quad H_q(v) = \int_\Omega (|\nabla v|^q + w(x)|v|^q)dx,$$

$$\text{and} \quad F_\gamma(v) = \int_\Omega f(x)|v|^\gamma dx, \quad v \in W, \tag{8}$$

where $W = W_0^{1,q}(\Omega)$ is the usual Sobolev space endowed with the norm $||u||_W = (\int_\Omega |\nabla u|^q dx)^{1/q}$. Recall that there is a continuous embedding $W^{1,q}(\Omega) \subset L^{q^*}(\Omega)$ if $q^* < +\infty$ and observe also that if $p < q < n$ then $p^* < q^*$. Hence under assumption (3) the functionals in (8) are well-defined on the Sobolev space W and belong to the class $C^1(W)$.

The problem (1)-(2) has a variational form with the following Euler functionnal on W

$$I_\lambda(u) = \frac{1}{p}H_p(u) + \frac{1}{q}H_q(u) - \frac{\lambda}{\gamma}F_\gamma(u). \tag{9}$$

We will deal with a *non-trivial weak solution* u_0 of the problem (1)-(2), i.e., with a function $u_0 \in W$ that is not identically zero on Ω and satisfies

$$\frac{\partial}{\partial u} I_\lambda(u_0)(\psi) \;=\; I_\lambda'(u_0)(\psi) \;=0 \tag{10}$$

for every $\psi \in C^\infty(\Omega)$. Thus the existence of weak solutions of problem (1)-(2) is equivalent to the existence of critical points for the Euler functional I_λ defined above.

First, based on the nonlinear spectral analysis [12], we introduce the following two characteristic points

$$\Lambda_1^+ = \inf\{ \frac{C_{p,q,\gamma} H_p(v)^{\frac{q-\gamma}{q-p}} H_q(v)^{\frac{\gamma-p}{q-p}}}{F_\gamma(v)} \;:\; F_\gamma(v) > 0, \; v \in W \setminus \{0\}\}, \tag{11}$$

and

$$\Lambda_1^- = \sup\{ \frac{C_{p,q,\gamma} H_p(v)^{\frac{q-\gamma}{q-p}} H_q(v)^{\frac{\gamma-p}{q-p}}}{F_\gamma(v)} \;:\; F_\gamma(v) < 0, \; v \in W \setminus \{0\}\} \tag{12}$$

where in case meas$\{x \in \Omega : \; f(x) > 0\} = 0$ or meas$\{x \in \Omega : \; f(x) < 0\} = 0$ we put $\Lambda_1^+ = +\infty$ or $\Lambda_1^- = -\infty$, respectively. Here we denote

$$C_{p,q,\gamma} = \frac{q-p}{(\gamma-p)^{\frac{\gamma-p}{q-p}}(q-\gamma)^{\frac{q-\gamma}{q-p}}}\,.$$

It can be shown that $\Lambda_1^- \le -K < 0$ and $\Lambda_1^+ \ge K > 0$ (see below) where K is a constant depending on $p, q, \gamma, r, \Omega,$ and $\|f\|_{0,r}$.

Our main result on the non existence of solutions of (1)-(2) is the following

Theorem 2.1 *Under the assumption (3) and (4), the problem (1)-(2) has no non-trivial solution when* $\lambda \in]\Lambda_1^-, \Lambda_1^+[$.

Actually, in (11), (12) we deal with general variational principles (see below Theorem 3.1) so that in particular every solutions of the variational problems (11) (resp. (12)) correspond to weak solutions of (1)-(2) with a specific characteristic value of λ which can be expressed as following

$$\lambda_1^+ = \frac{\gamma}{p^\alpha q^{1-\alpha}} \Lambda_1^+, \quad (\text{resp. } \lambda_1^- = \frac{\gamma}{p^\alpha q^{1-\alpha}} \Lambda_1^-) \tag{13}$$

Here and from now on we denote $\alpha = \frac{q-\gamma}{q-p}$. Remark that under the assumption (3) we have $0 < \alpha < 1$. Notice also that the strong inequalities $\Lambda_1^+ < \lambda_1^+$ and $\lambda_1^- < \Lambda_1^-$ hold (see below).

We now state our main result on the existence of non-negative solutions

Theorem 2.2 *Let suppose that (3) and (4) hold.*
(1) *Assume that* $meas(\{x \in \Omega : \ f(x) > 0\}) \neq 0$. *Then for every* $\lambda \geq \lambda_1^+$ *there exists a non-negative weak solution* $u_\lambda \in W \setminus \{0\}$ *for the problem (1)-(2). Moreover,* $I_{\lambda_1^+}(u_{\lambda_1^+}) = 0$ *and* $I_\lambda(u_\lambda) > 0$ *for every* $\lambda > \lambda_1^+$.
(2) *Assume that* $meas(\{x \in \Omega : \ f(x) < 0\}) \neq 0$. *Then for every* $\lambda \leq \lambda_1^-$ *there exists a non-negative weak solution* $u_\lambda \in W \setminus \{0\}$ *for the problem (1)-(2). Moreover,* $I_{\lambda_1^-}(u_{\lambda_1^-}) = 0$ *and* $I_\lambda(u_\lambda) > 0$ *for every* $\lambda < \lambda_1^-$.

To prove this theorem we use the following idea. First, using the variational principles (11), (12), we prove the existence of non-trivial solution of (1)-(2) for the separate given values $\lambda = \lambda_1^+, \lambda_1^-$. These results enable us to apply the Mountain-Pass Lemma and then, to prove existence results for $\lambda > \lambda_1^+$ and $\lambda < \lambda_1^-$, respectively.

Our last result deals with the existence and the uniqueness of the solution for the following reaction-diffusion equation

$$-\operatorname{div}\left(\left[|\nabla u|^{p-2} + |\nabla u|^{q-2}\right] \nabla u\right) + q(x)|u|^{p-2}u + w(x)|u|^{q-2}u = g(x), \ x \in \Omega \tag{14}$$

with Dirichlet boundary condition (2). We will prove the following

Theorem 2.3 *Let assume that (3) and (4) are satisfied, and that* $g \in W^{-1,q'}(\Omega)$. *Then the problem (14)-(2) has a unique weak solution.*

Throughout the article, we assume that (3) and (4) hold, we set $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$, and we denote by $\langle .,. \rangle_{q',q}$ (resp. $\langle .,. \rangle_{p',p}$) the duality product between the spaces $W^{-1,q'}(\Omega)$ and $W_0^{1,q}(\Omega)$ (resp. between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.) We also denote by $\|.\|_{0,\gamma}$ the norm in $L^\gamma(\Omega)$, and by $\|.\|_{0,\infty}$ the norm in $L^\infty(\Omega)$.

3 Homogeneous variational principles

In this section, following the ideas of the spectral analysis [12], we introduce a 0-homogeneous variational problem which corresponds to (1)-(2).

As in the fibering scheme [12, 14] we associate the original functional I_λ with a new fibering functional $\tilde{I}_\lambda$ on $]0,+\infty[\times W\setminus\{0\}$ defined by

$$\begin{array}{rcl} \tilde{I}_\lambda :]0,+\infty[\times W\setminus\{0\} & \longrightarrow & \mathbb{R}, \\ (t\,,\,v) & \mapsto & I_\lambda(tv) = \dfrac{t^p}{p}\,H_p(v) + \dfrac{t^q}{q}H_q(v) - \lambda\dfrac{t^\gamma}{\gamma}F_\gamma(v). \end{array} \tag{15}$$

Then $\tilde{I}_\lambda$ is a C^1 - functional and we are able to extract the C^1-submanifold

$$\Sigma_\lambda = \{(t,v)\ \in]0,+\infty[\times W\setminus\{0\}\ :\ \frac{\partial}{\partial t}\tilde{I}_\lambda(t,v) = 0,\ \frac{\partial^2}{\partial t^2}\tilde{I}_\lambda(t,v)\neq 0\}.$$

We recall that, according to the base lemma of the fibering scheme [12, 14], if $(t_\lambda, v_\lambda)\in\Sigma_\lambda$ is a critical point of $\tilde{I}_\lambda$ on $]0,+\infty[\times W\setminus\{0\}$ then $u_\lambda = t_\lambda v_\lambda$ is a critical point of the functional I_λ.

In accordance with the conception of the spectral analysis [12], we first consider the boundary (in the strong topology of W) of the set Σ_λ, i.e., the subset $d_\lambda = \partial\Sigma_\lambda \subset \{(t,v)\in[0,+\infty)\times W\}$ of the solutions of the system

$$\left\{\begin{array}{rcl} \dfrac{\partial}{\partial t}\tilde{I}_\lambda(t,v) & = & 0, \\ \dfrac{\partial^2}{\partial t^2}\tilde{I}_\lambda(t,v) & = & 0. \end{array}\right. \tag{16}$$

Then to find possible (a priori) bifurcation points it is natural to extract extreme points λ_i^* of the set

$$\{\lambda : d_\lambda \neq \phi\}\subset\mathbb{R}. \tag{17}$$

Observe that in our case the system (16) on $d_\lambda = \partial\Sigma_\lambda$ is equivalent to the system

$$\left\{\begin{array}{rcl} H_p(v) + t^{q-p}H_q(v) - \lambda t^{\gamma-p}F_\gamma(v) & = & 0 \\ (p-1)H_p(v) + (q-1)t^{q-p}H_q(v) - \lambda(\gamma-1)t^{\gamma-p}F_\gamma(v) & = & 0. \end{array}\right. \tag{18}$$

It is easy to see that this system is solvable, i.e., for each v in $W\setminus\{0\}$, we are able to define the real numbers $t(v)>0$ and $\lambda(v)$ which are the unique solution of (18). Moreover, very simple calculations lead to

$$\left\{\begin{array}{l} \lambda(v)\ =\ C_{p,q,\gamma}\dfrac{H_p(v)^\alpha H_q(v)^{1-\alpha}}{F_\gamma(v)}, \\ t(v)\ =\ \left\{\dfrac{H_p(v)}{H_q(v)}\,\dfrac{(\gamma-p)}{(q-\gamma)}\right\}^{\frac{1}{q-p}}, \\ C_{p,q,\gamma} = \dfrac{q-p}{(\gamma-p)^{1-\alpha}(q-\gamma)^\alpha}. \end{array}\right. \tag{19}$$

Thus the set (17) can be exactly described and we are able to introduce the following characteristic points:

$$\Lambda_1^+ = \inf\{\lambda(v) : F_\gamma(v) > 0,\ v \in W \setminus \{0\}\}, \tag{20}$$
$$\Lambda_1^- = \sup\{\lambda(v) :\ F_\gamma(v) < 0,\ v \in W \setminus \{0\}\}. \tag{21}$$

Here we deal with variational principles, i.e., variational problems (20), (21) corresponding to the problem (1)-(2) in the following sense

Theorem 3.1 *Let assume that (3) and (4) hold. Suppose $v_0 \in W \setminus \{0\}$ is a critical point of $\lambda(v)$ on W such that $F_\gamma(v_0) \neq 0$. Let*

$$s_0 =: s(v_0) \ = \ \left(\frac{q}{p}\right)^{\frac{1}{q-p}} t(v_0), \quad \lambda_0 =: \lambda_0(v_0) = \frac{\gamma}{p^\alpha q^{1-\alpha}} \lambda(v_0). \tag{22}$$

Then $u_{\lambda_0} = s_0 v_0 \in W \setminus \{0\}$ is a non-trivial critical point of I_{λ_0} on W. Furthermore, it holds $I_{\lambda_0}(u_{\lambda_0}) = 0$.

Proof Let $v_0 \in W\setminus\{0\}$ be a critical point of λ. Then we have $\langle \lambda'(v_0), \varphi\rangle_{q',q} = 0, \quad \forall \varphi \in W$. By direct calculations, we obtain:

$$\begin{aligned} &\alpha\ p\ C_{p,q,\gamma}\Big(\frac{H_p(v_0)}{H_q(v_0)}\Big)^{\alpha-1} \Big(\int_\Omega |\nabla v_0|^{p-2} (\nabla v_0, \nabla\varphi)\, dx \ + \ \int_\Omega q|v_0|^{p-2} v_0\, \varphi\ dx \Big) \\ &+ (1-\alpha)\ q\ C_{p,q,\gamma}\ \Big(\frac{H_p(v_0)}{H_q(v_0)}\Big)^{\alpha} \Big(\int_\Omega |\nabla v_0|^{q-2} (\nabla v_0, \nabla\varphi)\, dx + \int_\Omega w|v_0|^{q-2} v_0 \varphi\ dx \Big) \\ &\qquad\qquad - \gamma\ \lambda(v_0) \int_\Omega f|v_0|^{\gamma-2}\, v_0 \varphi\, dx \ = \ 0. \end{aligned}$$

for every $\varphi \in W$. Multiplying this equality by $\dfrac{1}{p^\alpha q^{1-\alpha}}$ and setting $u_{\lambda_0} = s_0 v_0$, we obtain after simplifications the weak equality (10). Using (15) and (19),we derive straightforwardly that $I_{\lambda_0}(u_{\lambda_0}) = 0$.

Let us prove the following

Proposition 3.1 *Let assume that (3) and (4) hold. Then there exist two constants $C_1 = C_1(p, \gamma, r, \Omega) > 0$ and $C_2 = C_2(q, \gamma, r, \Omega) > 0$ such that*

$$\begin{aligned} &(i)\ \ 0 < C_{p,q,\gamma}\ \frac{C_1^\alpha\ \ C_2^{1-\alpha}}{\|f\|_{0,r}} \le \Lambda_1^+ \ < \ \lambda_1^+, \\ &(ii)\ \ \lambda_1^- < \Lambda_1^- \le \ - \ C_{p,q,\gamma}\ \frac{C_1^\alpha\ \ C_2^{1-\alpha}}{\|f\|_{0,r}} < 0. \end{aligned}$$

Proof Let us prove the first estimation. Let $v \in W \setminus \{0\}$ and $F_\gamma(v) > 0$. By the following continuous embeddings

$$W_0^{1,p}(\Omega) \ \hookrightarrow \ L^s(\Omega) \ \hookrightarrow \ L^\gamma(\Omega) \quad \forall\, \gamma < s < p^*,$$

it can easily be shown that we have

$$H_p(v) \ \geq \ C_1(p,\gamma,r,\Omega)\Big(\frac{F_\gamma(v)}{\|f\|_{0,r}}\Big)^{\frac{p}{\gamma}} \text{ and } H_q(v) \ \geq \ C_2(\gamma,q,r,\Omega)\Big(\frac{F_\gamma(v)}{\|f\|_{0,r}}\Big)^{\frac{q}{\gamma}}, \quad (23)$$

where $0 < C_1(p,\gamma,r,\Omega)$ and $0 < C_2(\gamma,q,r,\Omega)$ are some Sobolev's constants occuring in Hölder's inequalities and Poincaré's inequalities. Hence, since $\alpha\frac{p}{\gamma} + (1-\alpha)\frac{q}{\gamma} - 1 \ = 0$, we get by (19) and (23) that

$$\lambda(v) \ \geq \ C_{p,q,\gamma} \ \frac{C_1(p,\gamma,r,\Omega)^\alpha \ \ C_2(\gamma,q,r,\Omega)^{1-\alpha}}{\|f\|_{0,r}}.$$

It remains to prove that $\lambda_1^+ \ > \ \Lambda_1^+$, which also can be written as following (see (13))

$$\frac{\gamma}{p^\alpha q^{1-\alpha}} > 1 \quad \Longleftrightarrow \quad \big(\frac{\gamma}{p}\big)^{q-p} \ > \ \big(\frac{q}{p}\big)^{\gamma-p}.$$

Setting $\mu = \dfrac{q-p}{p} > 0, \quad \eta = \dfrac{\gamma-p}{p} > 0$, the last inequality yields to $(1+\eta)^{\frac{1}{\eta}} > (1+\mu)^{\frac{1}{\mu}}$. Observing now that the function $x \mapsto (1+x)^{\frac{1}{x}}$ is decreasing on $]0,+\infty[$ and since $\mu > \eta$ we derive the proof.
The proof of the second estimation is similar.

4 Non-existence result

In this section we prove Theorem 2.1, i.e. the non existence of non trivial solutions for every $\Lambda_1^- < \lambda < \Lambda_1^+$.
Let $\lambda \in [0,\Lambda_1^+[$. Assume the converse that there exists a solution $u \in W \setminus 0$ of the problem (1)-(2) Let $v \ = u/\|u\|$ and $t = \|u\|$. Then by assumption it should be

$$\frac{\partial}{\partial t} I_\lambda(tv) \ \ = \ \langle I_\lambda^{'}(tv), v\rangle_{q',q} \ = 0. \quad (24)$$

It is easy to see that in case $F_\gamma(v) < 0$ and $\lambda > 0$ this equality is impossible. Suppose that $F_\gamma(v) > 0$. We set

$$\frac{\partial}{\partial t}\tilde{I}_\lambda(t,v) \ = \ t^{\gamma-1}\, R_\lambda(t,v), \quad (25)$$

with $R_\lambda(t,v) \ = \ t^{p-\gamma} \ H_p(v) \ + \ t^{q-\gamma} \ H_q(v) \ - \ \lambda\, F_\gamma(v)$. By (16), (18), (19), we infer $R_{\lambda(v)}(t(v),v) \ = \ 0$. It is easy to see that $R_\lambda(t,v) \ \geq \ R_\lambda(t(v),v)$ for every $t > 0$. Moreover, according to (11), (19) and since $\lambda < \Lambda_1^+ \leq \lambda(v)$ it follows that $R_\lambda(t,v) \ > \ R_{\lambda(v)}(t,v)$ for every $t > 0$. Hence we get

$$R_\lambda(t,v) \ \geq \ R_\lambda(t(v),v) \ \ > \ \ R_{\lambda(v)}(t(v),v) \ \ = 0$$

and from (25) we conclude that for every $\lambda < \Lambda_1^+$ and every $v \in W$ such that $F_\gamma(v) > 0$

$$\frac{\partial}{\partial t}\tilde{I}_\lambda(t,v) \neq 0 \quad \forall t > 0.$$

Thus we get a contradiction with (24) and therefore the functional I_λ has no critical point in $W \setminus \{0\}$ as $\lambda \in]0, \Lambda_1^+[$. The proof of the theorem for $\lambda \in]\Lambda_1^-, 0]$ is similar.

5 Existence of solutions for $\lambda = \lambda_1^+$ and $\lambda = \lambda_1^-$

In this section we prove the existence of a non-negative, non-trivial solution of (1)-(2) for the separate given values $\lambda = \lambda_1^+$ and $\lambda = \lambda_1^-$. To this aim, we first prove the following

Lemma 5.1 *Assume that (3) and (4) hold.*
(i) If meas $(\{x \in \Omega,\, f(x) > 0\}) \neq 0$, *then there exists* $v_1^+ \in W \setminus \{0\}$, *non-negative, such that*

$$\lambda(v_1^+) = \Lambda_1^+ = \inf\{\lambda(v)\,;\, F_\gamma(v) > 0,\ v \in W \setminus \{0\}\}.$$

(ii) If meas $(\{x \in \Omega,\, f(x) < 0\}) \neq 0$, *then there exists* $v_1^- \in W \setminus \{0\}$, *non-negative, such that*

$$\lambda(v_1^-) = \Lambda_1^- = \sup\{\lambda(v)\,;\, F_\gamma(v) < 0,\ v \in W \setminus \{0\}\}.$$

Proof We start with the proof of i). Remark that the application $\lambda : v \mapsto \lambda(v)$ is a 0-homogeneous function, i. e.

$$\lambda(tv) = \lambda(v) \quad \forall t > 0,\ \forall v \in W \text{ such that } F_\gamma(v) \neq 0.$$

Moreover, for any $v \in W \setminus \{0\}$, we introduce the strictly positive real value

$$\hat{t} =: \hat{t}(v) = \frac{1}{\left(H_p(v)^\alpha\, H_q(v)^{(1-\alpha)}\right)^{\frac{1}{\gamma}}}$$

such that

$$\begin{aligned}(H_p(\hat{t}v))^\alpha\, (H_q(\hat{t}v))^{(1-\alpha)} &= \hat{t}^\gamma\, H_p(v)^\alpha\, H_q(v)^{(1-\alpha)} \\ &= 1.\end{aligned}$$

As a consequence, if we set

$$M = \{v \in W \setminus \{0\}\,;\, H_p(v)^\alpha\, H_q(v)^{1-\alpha} = 1\}, \tag{26}$$

the definition of the characteristic point Λ_1^+ (see 20) reduces to

$$\Lambda_1^+ = \inf\{\lambda(v) :\ F_\gamma(v) > 0,\ v \in M\}.$$

Let us now prove that Λ_1^+ is achieved. Let $(v_n)_{n\in\mathbb{N}}$ be a minimizing sequence of $\lambda(v)$, i.e. satisfying

$$\lambda(v_n) \underset{n\to+\infty}{\longrightarrow} \Lambda_1^+, \quad \text{with } v_n \in M \text{ and } F_\gamma(v_n) > 0. \tag{27}$$

Since $v_n \in M$, we deduce that

$$\|v_n\|_{W_0^{1,p}}^{\alpha p} \, \|v_n\|_W^{(1-\alpha)q} \leq 1. \tag{28}$$

So the continuous embedding $W \hookrightarrow W_0^{1,p}(\Omega)$ implies that

$$\|v_n\|_{W_0^{1,p}}^{(1-\alpha)q} \leq C \, \|v_n\|_W^{(1-\alpha)q}, \quad C > 0.$$

This yields

$$\|v_n\|_{W_0^{1,p}}^{\gamma} \leq C,$$

and we deduce that v_n is bounded in $W_0^{1,p}(\Omega)$.
Consequently, using the compact embedding $W_0^{1,p}(\Omega) \subset\subset L^\gamma(\Omega)$, we infer the existence of $v_1 \in W_0^{1,p}(\Omega)$ and a subsequence (v_{n_j}) converging to v_1 weakly in $W_0^{1,p}$, and strongly in L^γ. Moreover, since F_γ is a continuous function from L^γ into $\mathbb{R}$, we have

$$F_\gamma(v_{n_j}) \underset{n_j\to+\infty}{\longrightarrow} F_\gamma(v_1). \tag{29}$$

Thus it results that $F_\gamma(v_1) \geq 0$. Let us show that $F_\gamma(v_1) \neq 0$. Assume the converse that $F_\gamma(v_1) = 0$. Then (8), (19) yield

$$\lambda(v_{n_j}) = C_{p,q,\gamma} \frac{H_p(v_{n_j})^\alpha \, H_q(v_{n_j})^{1-\alpha}}{F_\gamma(v_{n_j})} = C_{p,q,\gamma} \frac{1}{F_\gamma(v_{n_j})} \underset{n_j\to+\infty}{\longrightarrow} +\infty,$$

that is in contradiction with $\lambda_1(v_{n_j}) \underset{n_j\to+\infty}{\longrightarrow} \Lambda_1^+$. So we conclude that $F_\gamma(v_1) \neq 0$, and therefore $v_1 \neq 0$.
Since (v_{n_j}) converges to $v_1 \neq 0$ strongly in $L^\gamma(\Omega)$, we derive that for n_j large enough $\|v_{n_j}\|_{0,\gamma} \geq c > 0$, c being a constant. Therefore, the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$ leads to

$$0 < c' < \|v_{n_j}\|_{W_0^{1,p}},$$

and then, using (28), we finally get

$$\|v_{n_j}\|_W \leq c'' < +\infty.$$

We deduce from the last inequality that $v_1 \in W$, and that at least for a subsequence that we still denote v_{n_j},

$$v_{n_j} \underset{n_j\to+\infty}{\rightharpoonup} v_1, \quad \text{weakly in } W. \tag{30}$$

Since the functions H_q and H_p are weakly lower semi-continuous in W and $W_0^{1,p}(\Omega)$ respectively. Then,

$$\begin{cases} H_p(v_1) & \leq & \liminf_{n_j \to +\infty} H_p(v_{n_j}) & = & \overline{H_p}, \\ H_q(v_1) & \leq & \liminf_{n_j \to +\infty} H_q(v_{n_j}) & = & \overline{H_q}. \end{cases} \tag{31}$$

Hence by (19), we deduce

$$\liminf_{n_j \to +\infty} \lambda(v_{n_j}) = \liminf_{n_j \to +\infty} \left[C_{p,q,\gamma} \frac{H_p(v_{n_j})^\alpha \, H_q(v_{n_j})^{1-\alpha}}{F_\gamma(v_{n_j})} \right],$$

and consequently, with (27), (29), (31) and since $F_\gamma(v_1) \neq 0$

$$\begin{aligned} \Lambda_1^+ & \geq C_{p,q,\gamma} \liminf_{n_j \to +\infty} \left(H_p(v_{n_j})^\alpha\right) \liminf_{n_j \to +\infty} \left(H_q(v_{n_j})^{1-\alpha}\right) \liminf_{n_j \to +\infty} \left(\frac{1}{F_\gamma(v_{n_j})}\right) \\ & \geq C_{p,q,\gamma} \left(\liminf_{n_j \to +\infty} H_p(v_{n_j})\right)^\alpha \left(\liminf_{n_j \to +\infty} H_q(v_{n_j})\right)^{1-\alpha} \frac{1}{F_\gamma(v_1)} \\ & \geq C_{p,q,\gamma} \frac{\overline{H_p}^\alpha \, \overline{H_q}^{1-\alpha}}{F_\gamma(v_1)} \geq C_{p,q,\gamma} \frac{(H_p(v_1))^\alpha \, (H_q(v_1))^{1-\alpha}}{F_\gamma(v_1)} = \lambda(v_1) \end{aligned}$$

Thus, we get $\lambda(v_1) = \Lambda_1^+$. Furthermore, the function $\lambda : W \setminus \{0\} \to \mathbb{R}$ being even, we have $\lambda(v_1) = \lambda(|v_1|) = \Lambda_1^+$, and we infer the existence of at least one non-negative solution $v_1^+ = |v_1|$ of the minimizing problem.

Notice that the characteristic point Λ_1^- can also be viewed as:

$$-\Lambda_1^- = \inf\{-\lambda(v); \; v \in W, \; F_\gamma(v) < 0, \}.$$

Thus, our previous arguments are still suitable for proving that Λ_1^- is achieved.
A consequence of Lemma 5.1 is the following

Lemma 5.2 *Assume that (3) and (4) hold, and that meas* $(\{x \in \Omega; \; f(x) > 0\}) \neq 0$ *(resp. meas* $(\{x \in \Omega; \; f(x) < 0\}) \neq 0$.*)*
Then, for $\lambda = \lambda_1^+$ *(resp.* $\lambda = \lambda_1^-$*), there exists* $u_{\lambda_1^+} \neq 0$ *(resp.* $u_{\lambda_1^-} \neq 0$*), non-negative weak solution of problem (1)-(2), satisfying* $I_{\lambda_1^+}(u_{\lambda_1^+}) = 0$ *(resp.* $I_{\lambda_1^-}(u_{\lambda_1^-}) = 0$*).*

Proof ¿From Lemma 5.1, we infer the existence of v_1^+ and v_1^-, non-negative critical points of $\lambda(v)$ on W. Then, it results from Theorem 3.1 that $u_{\lambda_1^+} = (\frac{q}{p})^{\frac{1}{q-p}} t(v_1^+) \, v_1^+$ is a non negative weak solution of problem (1)-(2) associated with λ_1^+ $(= \dfrac{\gamma}{p^\alpha q^{1-\alpha}} \lambda(v_1^+))$. In the same way, $u_{\lambda_1^-} = (\frac{q}{p})^{\frac{1}{q-p}} t(v_1^-) \, v_1^-$ is a non negative weak solution of problem (1)-(2) associated with λ_1^- $(= \dfrac{\gamma}{p^\alpha q^{1-\alpha}} \lambda(v_1^-))$. Moreover we have $I_{\lambda_1^+}(u_{\lambda_1^+}) = 0$ and $I_{\lambda_1^-}(u_{\lambda_1^-}) = 0$.

6 Existence of solutions for $\lambda > \lambda_1^+$, $\lambda < \lambda_1^-$

This section is devoted to the proof of Theorem 2.1, i.e. we prove the existence of non-negative, non-trivial solutions of (1)-(2) for $\lambda > \lambda_1^+$ and $\lambda < \lambda_1^-$. We first prove three Lemmas which will be useful afterwards for the application of the Mountain-Pass Lemma [1].

We say that $(u_n) \subset W$ is a Palais-Smale (P.-S.)-sequence if it satifies the following condition

$$\begin{cases} |I_\lambda(u_n)| \ \leq \ D \quad \forall n \geq 0, \\ \|I'_\lambda(u_n)\|_{W^{-1,q'}} \ \underset{n\to+\infty}{\longrightarrow} \ 0, \end{cases} \tag{32}$$

with some constant $D > 0$.

Lemma 6.1 *We assume that (3) and (4) are satisfied, and $\lambda \in \mathbb{R}$. Then any (P.-S.)-sequence $(u_n) \subset W$ is bounded in W.*

Proof Let $(u_n) \subset W$ be a (P.-S.)-sequence, i.e. it satisfies conditions (32). Then by (9) we have

$$\begin{aligned} \langle I'_\lambda(u_n), \varphi\rangle_{q',q} \ &= \ \int_\Omega |\nabla u_n|^{p-2}(\nabla u_n, \nabla\varphi)\, dx + \int_\Omega q|u_n|^{p-2}u_n\varphi\, dx + \\ &+ \int_\Omega |\nabla u_n|^{q-2}(\nabla u_n, \nabla\varphi)\, dx + \int_\Omega w|u_n|^{q-2}u_n\varphi\, dx - \lambda\int_\Omega f|u_n|^{\gamma-2}u_n\varphi\, dx. \end{aligned}$$

for every $\varphi \in W$. Thus

$$\langle I'_\lambda(u_n), u_n\rangle_{q',q} - \gamma\, I_\lambda(u_n) \ = \ (1 - \frac{\gamma}{p})\ H_p(u_n) \ + \ (1 - \frac{\gamma}{q})\ H_q(u_n).$$

Using (32), it follows that

$$\begin{aligned} (1 - \frac{\gamma}{p})\ H_p(u_n) \ + \ (1 - \frac{\gamma}{q})\ H_q(u_n) \ &\leq \ \|I'_\lambda(u_n)\|_{W^{-1,q'}}\ \|u_n\|_W \ + \ \gamma\, D, \\ &\leq \ o(1)\ \|u_n\|_W \ + \ \gamma\, D. \end{aligned}$$

Furthermore, since $1 < p < \gamma < p^*$ and $\gamma < q$, then $1 - \frac{\gamma}{p} < 0$, and $1 - \frac{\gamma}{q} > 0$. So, there exists a constant $c > 0$ such that $(1 - \frac{\gamma}{p})\, H_p(u_n) \ \geq \ (-c)\, \|u_n\|_W^p$. Finally, we get

$$(1 - \tfrac{\gamma}{q})\ \|u_n\|_W^q \ - \ c\ \|u_n\|_W^p \ - \ o(1)\ \|u_n\|_W \ \leq \ \gamma\, D, \quad \text{with } 1 - \tfrac{\gamma}{q} > 0.$$

Consequently we deduce that the sequence (u_n) is necessarily bounded in W.

Lemma 6.2 *Under the assumptions (3) and (4), if $(u_n) \subset W$ is a Palais-Smale sequence of I_λ, $\lambda \in \mathbb{R}$, then (u_n) has a strong convergent subsequence.*

Proof The case $q \geq n$ is simple. We only detail the proof in the case $q < n$. If $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence, then $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n) \underset{n\to+\infty}{\longrightarrow} 0$, and we infer from Lemma 6.1 that u_n is bounded in W. Thus, there exists $u \in W$, and we can extract a subsequence (u_{n_k}) satisfying:

$$u_{n_k} \rightharpoonup u \ \text{(weakly) in } W_0^{1,q}(\Omega), \tag{33}$$

$$u_{n_k} \longrightarrow u \ \text{(strongly) in } L^\eta(\Omega) \quad \text{with } \eta < q^*. \tag{34}$$

Furthermore, since u_n is bounded in W and $I'_\lambda(u_{n_k}) \underset{n_k\to+\infty}{\longrightarrow} 0$ we have:

$$\left|\langle I'_\lambda(u_{n_k}),\ u_{n_k} - u\ \rangle_{q',q}\right| \ \leq\ \|I'_\lambda(u_{n_k})\|_{W^{-1,q'}} \left(\|u_{n_k}\|_W \ +\ \|u\|_W\right) \underset{n_k\to+\infty}{\longrightarrow} 0$$

Therefore

$$\langle I'_\lambda(u_{n_k}), u_{n_k} - u\rangle_{q',q} \underset{n_k\to+\infty}{\longrightarrow} 0. \tag{35}$$

Moreover we have

$$\int_\Omega w\ |u_{n_k}|^{q-1}\ |u_{n_k} - u|\ dx\ \leq\ \|w\|_{0,r}\ \|u_{n_k}\|_{0,q*}^{q-1}\ \|u_{n_k} - u\|_{0,\eta}.$$

By assumption $r > \dfrac{p^*}{p^* - \gamma} > \dfrac{q^*}{q^* - q}$. This implies that

$$\eta = \frac{rq^*}{(r-1)q^* - r(q-1)} <\ q^*.$$

Since u_{n_k} is bounded in W, it follows from (34) that

$$\int_\Omega w|u_{n_k}|^{q-2}u_{n_k}(u_{n_k} - u)\,dx \underset{n_k\to+\infty}{\longrightarrow} 0.$$

Proceeding as above, we also obtain

$$\int_\Omega f|u_{n_k}|^{\gamma-2}u_{n_k}(u_{n_k}-u)\,dx \underset{n_k\to+\infty}{\longrightarrow} 0 \quad\text{and}\quad \int_\Omega g|u_{n_k}|^{p-2}u_{n_k}(u_{n_k}-u)\,dx \underset{n_k\to+\infty}{\longrightarrow} 0.$$

This, combined with (35) lead to

$$\int_\Omega |\nabla u_{n_k}|^{p-2}\nabla u_{n_k}\nabla(u_{n_k} - u)\,dx + \int_\Omega |\nabla u_{n_k}|^{q-2}\nabla u_{n_k}\nabla(u_{n_k} - u)\,dx \underset{n_k\to+\infty}{\longrightarrow} 0$$

which also can be written

$$\langle -\Delta_p u_{n_k}, u_{n_k} - u\rangle_{p',p} + \langle -\Delta_q u_{n_k}, u_{n_k} - u\rangle_{q',q} \underset{n_k\to+\infty}{\longrightarrow} 0. \tag{36}$$

We record that $-\Delta_q$ (resp. $-\Delta_p$) can be seen as a duality mapping between $W_0^{1,q}(\Omega)$ and $W^{-1,q'}$. (resp. between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$,) corresponding to the normalization function $\varphi(t) = t^{q-1}$ (resp. $\varphi(t) = t^{p-1}$)(cf. [13], [9]). As a consequence, we have

$$\begin{aligned}\langle -\Delta_q u_{n_k} + \Delta_q u,\ u_{n_k} - u\rangle_{q',q} & \\ \geq \left(\|u_{n_k}\|_W^{q-1} - \|u\|_W^{q-1}\right)\left(\|u_{n_k}\|_W - \|u\|_W\right) & \\ \geq 0 \ \ \text{(because } \varphi(t) = t^{q-1} \text{ is an increasing function)}, &\end{aligned} \tag{37}$$

and

$$\begin{aligned}\langle -\Delta_p u_{n_k} + \Delta_p u,\ u_{n_k} - u\rangle_{p',p} & \\ \geq \left(\|u_{n_k}\|_{W_0^{1,p}}^{p-1} - \|u\|_{W_0^{1,p}}^{p-1}\right)\left(\|u_{n_k}\|_{W_0^{1,p}} - \|u\|_{W_0^{1,p}}\right) & \qquad \geq 0.\end{aligned} \tag{38}$$

Furthermore, since $\Delta_q u \in W^{-1,q'}(\Omega)$ and $\Delta_p u \in W^{-1,p'}(\Omega)$ then (33) and (36) yield the following

$$\langle -\Delta_p u_{n_k} + \Delta_p u, u_{n_k} - u\rangle_{p',p} + \langle -\Delta_q u_{n_k} + \Delta_q u, u_{n_k} - u\rangle_{q',q} \underset{n_k \to +\infty}{\longrightarrow} 0. \tag{39}$$

Using (39), (37), and (38), we obtain

$$0 \leq \left(\|u_{n_k}\|_W^{q-1} - \|u\|_W^{q-1}\right)\left(\|u_{n_k}\|_W - \|u\|_W\right) \underset{n_k \to +\infty}{\longrightarrow} 0,$$

and we conclude

$$\|u_{n_k}\|_W \underset{n_k \to +\infty}{\longrightarrow} \|u\|_W \tag{40}$$

According to (33) and (40), we finally achieve the strong convergence of (u_{n_k}) in the space $W = W_0^{1,q}(\Omega)$.

Lemma 6.3 *Assume that (3) and (4) hold, and that $\lambda \neq 0$. Then there exist constants $\rho > 0$, $\alpha > 0$ such that $I_\lambda(u) \geq \alpha$ for each $u \in W$ satisfying $\|u\|_W = \rho$.*

Proof We consider the case $\lambda > 0$. The case $\lambda < 0$ is very similar and will be omitted. According to the following embeddings

$$W \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega), \quad \forall\, s < p^*,$$

there exist two constants $C_1, C_2 > 0$ such that

$$\begin{array}{rrcl} & \|u\|_{W_0^{1,p}} & \leq & C_1\ \|u\|_W, \\ \text{and} & \int_\Omega f|u|^\gamma\, dx & \leq & C_2\ \|f\|_{0,r}\ \|u\|_{W_0^{1,p}}^\gamma. \end{array} \tag{41}$$

If we assume that $u \in W$ satisfy $\|u\|_W = \rho,\ \rho > 0$, then, using (9) we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{\rho^q}{q} + \frac{1}{p}\|u\|_{W_0^{1,p}}^p - \frac{\lambda C_2}{\gamma}\|f\|_{0,r}\ \|u\|_{W_0^{1,p}}^\gamma, \\ &\geq \frac{\rho^q}{q} + \frac{1}{p}\|u\|_{W_0^{1,p}}^p \left(1 - \frac{\lambda C_2 p}{\gamma}\ \|f\|_{0,r}\ \|u\|_{W_0^{1,p}}^{\gamma-p}\right). \end{aligned}$$

Let us set
$$\rho^* = \frac{1}{C_1}\Big(\frac{\gamma}{|\lambda| C_{2p}\ \|f\|_{0,r}}\Big)^{\frac{1}{\gamma-p}}.$$
Then, for every $0 < \rho \leq \rho^*$, it follows from (41) that
$$1 - \frac{\lambda C_{2p}}{\gamma}\ \|f\|_{0,r}\ \|u\|^{\gamma-p}_{W_0^{1,p}}\ \geq 0.$$
Consequently, we get
$$I_\lambda(u)\ \geq\ \frac{\rho^q}{q},$$
and the proof is complete, provided $\alpha = \dfrac{\rho^q}{q}$.

Theorem 6.1 *Let assume that the hypotheses (3) and (4) are satisfied.*

(i) Under the assumptions meas $(\{x \in \Omega;\, f(x) > 0\}) \neq 0$ and for all $\lambda > \lambda_1^+$, there exists $u_\lambda \in W \setminus \{0\}$, non-negative weak solution of problem (1)-(2), such that $I_\lambda(u_\lambda) > 0$.

(ii) Under the assumptions meas $(\{x \in \Omega;\, f(x) < 0\}) \neq 0$, and for all $\lambda < \lambda_1^-$, there exists $u_\lambda \in W \setminus \{0\}$, non-negative weak solution of problem (1)-(2), such that $I_\lambda(u_\lambda) > 0$.

Proof We only prove (i), the case (ii) being very similar. The proof relies on the application of the Mountain Pass Lemma. Indeed, according to Lemma 6.2, we know that any Palais-Smale sequence for I_λ has a strong convergent subsequence in W. Moreover, for every $\lambda > \lambda_1^+$, the functional I_λ satisfies :

(i) $I_\lambda(0) = 0$,

(ii) Let take ρ satisfying $0 < \rho < \min\{\ \rho^*,\ \|u_{\lambda_1^+}\|_W\ \}$, with ρ^* as in Lemma 6.3 and $u_{\lambda_1^+} \in W$ defined in Lemma 5.2. Then there exists $\alpha > 0$ such that
$$\forall u \in W \text{ s. t. } \|u\|_W = \rho, \text{then } I_\lambda(u) \geq \alpha \qquad \text{(cf. Lemma 6.3)}.$$

(iii) Furthermore, we have $\|u_{\lambda_1^+}\|_W > \rho$, and (cf Lemma 5.2)
$$\forall \lambda > \lambda_1^+ \quad I_\lambda(u_{\lambda_1^+}) < I_{\lambda_1^+}(u_{\lambda_1^+}) = 0.$$

Hence we can apply the Mountain Pass Lemma [1] and therefore we conclude that for every $\lambda > \lambda_1^+$ the functional I_λ possesses a (non trivial) critical point, i.e.
$$\exists\ u_\lambda\ \in W \setminus \{0\} \text{ s.t. } I_\lambda(u_\lambda) \geq \alpha > 0 \text{ and } I_\lambda'(u_\lambda) = 0.$$
Moreover, u_λ for all $\lambda > \lambda_1^+$ is non-negative. Indeed it follows from the arguments
$$I_\lambda(u) = I_\lambda(|u|) \quad \forall\, u \in W \qquad \text{and } u_{\lambda_1^+} \text{ is non-negative.}$$

(cf. [6]). Thus for every $\lambda > \lambda_1^+$, the existence of non trivial non-negative (weak) solutions for problem (1)-(2) is ensured. And the proof of Theorem 6.1 is complete. This also ends the proof of our main statement, since Theorem 2.2 is a direct consequence of the Lemma 5.2 and Theorem 6.1.

7 Proof of Theorem 2.3

In this section, we prove the existence and the uniqueness of the solution of problem (14)-(2).

The existence of the solution follows from standard arguments. Indeed, let us consider

$$J(u) = \frac{1}{p}\, H_p(u) \;+\; \frac{1}{q}\, H_q(u) \;-\; \int_\Omega gu\, dx,$$

with H_p, H_q defined in (8).
It is easy to check that the functional $J: W \to \mathbb{R}$ is weakly lower semi-continuous and of class C^1 on W. Let us show briefly that it is also coercive. The Young's inequality implies

$$\Big|\int_\Omega gu\, dx\Big| \;\le\; \|g\|_{W^{-1,q'}}\;\|u\|_W \;\le\; \frac{1}{2q}\|u\|_W^q \;+\; C\;\|g\|_{W^{-1,q'}}^{q'},$$

with $C > 0$. Then it follows that

$$J(u) \;\ge\; \frac{1}{2q}\,\|u\|_W^q \;-C\;\|g\|_{W^{-1,q'}}^{q'}.$$

which tends to $+\infty$ as $\|u\|_W \to +\infty$. Thus J is a coercive and the existence of the solution of (14)-(2) can be obtained by the minimizing problem (see Theorem I.1.2 in [15])

$$J(u) \;=\; \inf_{v\in W}\; J(v). \tag{42}$$

Let us prove now the uniqueness of the solution. Suppose on the contrary that there exist two distinct solutions u_1, u_2 in W for problem (14)-(2). It yields

$$\begin{aligned}&\langle -\Delta_p u_1 + \Delta_p u_2,\, u_1 - u_2\rangle_{p',p} \;+\; \langle q\,|u_1|^{p-2}u_1 - q\,|u_2|^{p-2}u_2, u_1 - u_2\rangle_{p',p} +\\ &\langle -\Delta_q u_1 + \Delta_q u_2, u_1 - u_2\rangle_{q',q} \;+ \langle w\,|u_1|^{q-2}u_1 - w\,|u_2|^{q-2}u_2,\, u_1 - u_2\rangle_{q',q} \;= 0.\end{aligned}$$

The function $x \mapsto |x|^{r-2}\,x$ being increasing when $r > 1$, then, the last equality reduces to

$$\langle -\Delta_p u_1 + \Delta_p u_2,\, u_1 - u_2\rangle_{p',p} \;+\; \langle -\Delta_q u_1 + \Delta_q u_2, u_1 - u_2\rangle_{q',q} \;\le 0. \tag{43}$$

A consequence of the strict convexity of the spaces W and $W_0^{1,p}(\Omega)$ is that the duality mappings $-\Delta_q$ and $-\Delta_p$ are strictly monotone (see [13], [9]), i.e., since $u_1 \neq u_2$

$$\begin{aligned}&\langle -\Delta_q u_1 + \Delta_q u_2, u_1 - u_2\rangle_{q',q} \;> 0,\\ &\langle -\Delta_p u_1 + \Delta_p u_2,\, u_1 - u_2\rangle_{p',p} > 0.\end{aligned} \tag{44}$$

Thus, combining (44) with (43), we get a contradiction.

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