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Université de La Rochelle
Avenue Michel Crépeau
17042 La Rochelle Cedex 1
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Jean-Christophe Breton et Youri Davydov

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Local limit theorem for supremum of an empirical processes for i.i.d. random variables

Jean-Christophe Breton* and Youri Davydov†

Abstract

In this paper, we establish the convergence in total variation norm of the law of the supremum of an empirical process constructed from a sequence of i.i.d. random variables to the law of the supremum of a (generalized) Brownian bridge.

Introduction

Let $(\xi_i)_{i\in\mathbb{N}^*}$ be a sequence of independent identically distributed random variables with common distribution function F assumed to be smooth. Consider the empirical process given by:

$$\zeta_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \big[\mathbf{1}_{(-\infty,t]}(\xi_i) - F(t)\big]. \tag{0.1}$$

The well known invariance principle [B, Th. 14.3] states the following weak convergence of $(\zeta_n)_n$ in $\mathbb{D}(\mathbb{R})$ (the space of right-continuous functions with left-hand limits):

$$\zeta_n \Longrightarrow W_F^0,\ n \to \infty, \tag{0.2}$$

where W_F^0 is a generalized Brownian bridge, that is a continuous Gaussian process with independent increments and with covariance function given by

$$F(t) \wedge F(s) - F(t)F(s). \tag{0.3}$$

In the uniform case ($F(t) = t,\ t \in [0,1]$), we note W for the standard Brownian motion on $[0,1]$, and the related standard Brownian bridge can be realized taking $W^0(t) = W(t) - tW(1)$, $t \in [0,1]$.

*Laboratoire de Mathématiques et Applications, Université de La Rochelle

†Laboratoire de Mathématiques Paul Painlevé, Université des Sciences et Technologies de Lille

The goal of this work is to study the behaviour of the sup of empirical processes and their asymptotics in law. Of course, (0.2) yields immediately when $n \to \infty$, the weak convergence

$$\sup_{t\in\mathbb{R}} \zeta_n(t) \Longrightarrow \sup_{t\in\mathbb{R}} W_F^0(t).$$

In this paper, our aim is to strengthen this convergence to the convergence in variation. In the sequel, this later convergence will be denoted $\xrightarrow{var}$ and the relative norm will be denoted simply $\|\cdot\|$. Since the laws are absolutely continuous, such convergence are equivalent to convergence in $L^1(\mathbb{R})$ of their densities, which yields a local limit theorem for this law.

In the case of linear piecewise processes, such stronger convergences in variation have been yet derived in [DLS, Sect. 20] and [BD] for a broad class of functionals for which the key is the existence and non-degeneracy of directional derivatives. For instance, in this class, there are smooth functionals and supremum or integral type functionals. In the case of empirical process, the problem is much more difficult since even with the simplest functionals of estimation, $f_{t_0}(x) = x(t_0)$, the law of estimated empirical process $f_{t_0}(\zeta_n)$ is atomic and can thus not converge in variation to the Gaussian law of $W_F^0(t_0)$.

For this reason, we deal with specific functionals of interest and we begin in this paper with supremum-type functionals. The main result is the following:

Theorem 1 *Let $(\xi_i)_{i>0}$ be a sequence of i.i.d. random variables with a continuous distribution F, and ζ_n^F be the related empirical process given in* (0.1)*. Then*

$$\mathcal{L}\big(\sup_{t\in\mathbb{R}} \zeta_n^F(t)\big) \xrightarrow{var} \mathcal{L}\big(\sup_{t\in\mathbb{R}} W_F^0(t)\big), \quad n \to +\infty$$

where $\mathcal{L}(X)$ stands for the law of a random variable X.

Remark: It is enough to prove Theorem 1 for a Gaussian i.i.d. sequence since when $(\xi_i)_{i\in\mathbb{N}^*}$ is a sequence of i.i.d. random variables with continuous distribution F, the random variable $F(\xi_i)$ are i.i.d. uniformly distributed and we have easily

$$\zeta_n^F(t) = \zeta_n^U(F(t))$$

where ζ_n^F and ζ_n^U stand for the empirical processes related to the F and the uniform distribution; but in the same way, $\zeta_n^\Phi(t) = \zeta_n^U(\Phi(t))$ where Φ is the standard normal distribution. Since F and Φ are surjective from $\mathbb{R}$ to $[0,1]$ we have

$$\sup_{t\in\mathbb{R}} \zeta_n^F(t) = \sup_{t\in\mathbb{R}} \zeta_n^\Phi(t). \tag{0.4}$$

Moreover, W_F^0 and $W_\Phi^0(\Phi^{-1}\circ F)$ have the same finite dimensional law since both have the same covariance function, namely (0.3). Finally,

$$\mathcal{L}\big(\sup_{t\in\mathbb{R}} W_F^0(t)\big) \quad = \quad \mathcal{L}\big(\sup_{t\in\mathbb{R}} W_\Phi^0\big).$$

So that Theorem 1 holds for any i.i.d. sequence $(\xi_i)_{i\in\mathbb{N}^*}$ with a continuous distribution F whenever it holds for i.i.d. Gaussian sequences.

The sequel is thus devoted to the (long) proof of Theorem 1 for an i.i.d standard Gaussian sequence $(\xi_i)_{i>0}$. In order to derive such convergence in variation, we shall apply the so-called superstructure method relying on the behaviour of the law of empirical processes in admissible directions for the asymptotic law (of Brownian bridge). For a complete account on this method, we refer to [DLS]. In particular, we shall make a fondamental use of the following result:

Theorem A ([DLS, th. 18.4]) *Consider a sequence of probability measures $\left\{P_n, n \in \overline{\mathbb{N}}\right\}$ defined on the Borel σ-algebra $\mathcal{B}_{\mathcal{X}}$ of a complete separable metric space $(\mathcal{X}, d)$. Suppose that $P_n \Rightarrow P_\infty$. Furthermore suppose that, for P_∞-almost all x, there exists an open ball V centered at x, a number $\epsilon > 0$ and also a family $\left(G_{n,c},\ n \in \overline{\mathbb{N}},\ c \in (0, \epsilon]\right)$ of measurable transformations of $\mathcal{X}$ such that the following five conditions are fulfilled:*

(i) for each $c \in (0, \epsilon)$ and each $\delta > 0$,

$$\lim_{n\to\infty} P_n \left\{x \mid |G_{n,c}\, x - G_{\infty,c}\, x| \geq \delta\right\} = 0;$$

(ii) for each $c \in (0, \epsilon)$, the mapping $G_{\infty,c}$ is P_∞-almost everywhere continuous; moreover suppose that $\rho(S, c) = \sup_{z\in S} d\left(z, G_{\infty,c} z\right) \to 0$ when $c \to\ 0$, for each open ball S;

(iii) $\lim_{c\to 0} \overline{\lim}_{n\to\infty} \|P_n G_{n,c}^{-1} - P_n\| \ =\ 0$;

(iv) for each $\delta \in (0, \epsilon)$,

$$\int_V \|\lambda_{[0,\delta]} \varphi_{n,z}^{-1} - \lambda_{[0,\delta]} \varphi_{\infty,z}^{-1}\|\ P_n(dz) \ \longrightarrow\ 0 \quad \text{as } n \to \infty,$$

where $\varphi_{n,z}(c) = f(G_{n,c} z)$ with $n \in \overline{\mathbb{N}}$ and $c \in (0, \epsilon]$;

(v) for each $\delta \in (0, \epsilon)$, the mapping $z \longmapsto \lambda_{[0,\delta]} \varphi_{\infty,z}^{-1}$ of V into $\mathcal{Z}(\mathbb{R})$, the Banach space of signed measures on $\mathbb{R}$ normed with the total variation, is P_∞-almost everywhere continuous.

Then

$$P_n f^{-1} \xrightarrow{var} P_\infty f^{-1}, \quad n \to +\infty.$$

The rest of the paper is organized as follows: as empirical processes lie in the Skorokhod space $\mathbb{D}(\mathbb{R})$ (that is the space of right-continuous function with left-hand limit), we first define suitable transformations $G_{n,c}$ and our purpose is to show the conditions of Theorem A are fulfilled. We start describing a setting where Theorem A applies and each condition is derived in this setting. The proof of conditions (i) and (iv) in Sections 1, 4 are intricate and lenthly.

Notation

First, we note in the sequel Φ for the standard Gaussian distribution function and $p(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ for its density.

Let remember some notation for the Skorokhod space. The space of sample path of the empirical processes (0.1) is the Skorokhod space $\mathbb{D}(\mathbb{R})$. But since $\zeta_n(t) \to 0$ as $t \to \pm\infty$, we shall rather consider as sample space the subspace $\mathbb{D}_0(\mathbb{R}) = \big\{x \in \mathbb{D}(\mathbb{R}) \mid \lim_{t\to\pm\infty} x(t) = 0\big\}$.

We equipp this space with a complete separable topology brought from the usual Skorokhod space $\mathbb{D}([0,1])$ as follows: consider a bijection from $\mathbb{R}$ to $]0,1[$; in our setting the simplest choice (we will do!) is to consider the bijection

$$\Phi(x) = \int_{-\infty}^{x} p(t)dt,$$

whose inverse is denoted Ψ. Define then the following Skorokhod metrics on $\mathbb{D}_0(\mathbb{R})$:

$$\begin{array}{rcl} d_{0,\mathbb{R}}(x,y) & = & d_0(x\circ\Psi, y\circ\Psi), \\ d_{\mathbb{R}}(x,y) & = & d(x\circ\Psi, y\circ\Psi), \end{array} \qquad \text{for } x,y \in \mathbb{D}_0(\mathbb{R}), \tag{0.5}$$

where d_0 and d are the basic Skorokhod metrics of $\mathbb{D}([0,1])$ given by (see[B]):

$$\begin{array}{rcl} d_0(x,y) & = & \inf_{\lambda\in\Lambda}\left(\sup_{t\in[0,1]} |x(\lambda(t)) - y(t)| + \sup_{t\in[0,1]} |\lambda(t) - t|\right), \\ d(x,y) & = & \inf_{\lambda\in\Lambda}\left(\sup_{t\in[0,1]} |x(\lambda(t)) - y(t)| + \sup_{\substack{s<t,\\ s,t\in[0,1]}} \log\left|\frac{\lambda t - \lambda s}{t-s}\right|\right), \end{array}$$

$x,y \in \mathbb{D}([0,1])$ and $\Lambda = \{\lambda : [0,1] \to [0,1]$ nondecreasing continuous bijection$\}$ and where by convention, we take

$$x\circ\Psi(0) = \lim_{t\to 0} x\circ\Psi(t) = \lim_{s\to-\infty} x(s) = 0,\ x\circ\Psi(1) = \lim_{t\to 1} x\circ\Psi(t) = \lim_{s\to\infty} x(s) = 0.$$

Since d is a complete metric for $\mathbb{D}([0,1])$, it is an easy matter now to see that $(\mathbb{D}_0(\mathbb{R}), d_{\mathbb{R}})$ is a complete separable metric space. But, since d_0 and d define the same convergence on $\mathbb{D}([0,1])$, we will work henceforth with $d_{0,\mathbb{R}}$. We refer to [B] for the study of the Skorokhod space $\mathbb{D}([0,1])$.

The first step consists in the definition of the transformations $G_{n,c}$ and $G_{\infty,c}$. To this way, we consider the following transformations acting as translations on the underlying Gaussian variables:

$$G_{n,c}\zeta_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \big[\mathbf{1}_{(-\infty,t]}(\xi_i + c/\sqrt{n}) - \Phi(t)\big]. \tag{0.6}$$

More precisely, since the support of P_n is $\mathcal{B}(\mathbb{R})$-measurable, take $G_{n,c}x$ to be given by (0.6) if $x = \zeta_n(\omega) \in$ Supp(P_n) and 0 if $x \notin$ Supp(P_n).

The asymptotic transformation is chosen to be a mere translation:

$$G_{\infty,c}x(t) = x(t) - c\Phi'(t). \tag{0.7}$$

Applying the Skorokhod's representation theorem [B, Th. 6.7], we can suppose we are on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the weak convergence (0.2) turns in the almost sure one $\zeta_n \longrightarrow W_\Phi^0$ in $\mathbb{D}_0(\mathbb{R})$, $n \to \infty$. But since W_Φ^0 is a continuous process, the convergence turns out to holds also for the uniform metric (see [B, p. 124]).

The only point to take care about, when we apply Skorokhod's theorem, is the fact, we have to consider an array of variables $(\tilde{\xi}_{1,1}, \dots, \tilde{\xi}_{n,n})$ rather than a mere sequence $(\xi_i)_{i\in\mathbb{N}}$, the n-th empirical process being constructed from the n-th line of the random array $(\tilde{\xi}_{1,1}, \dots, \tilde{\xi}_{n,n})$. In our study, this is not a cumbersome point since we shall work anyway with the triangular array of order statistics.

Henceforth, we forget the array $(\tilde{\xi}_{1,1}, \dots, \tilde{\xi}_{n,n})$ and note $(\xi_1^n, \dots, \xi_n^n)$ for the triangular array of order statistics of $(\xi_1, \dots, \xi_n)$.

Localization

The study of argmax of $G_{n,c}x$ and $G_{\infty,c}x$ for $x = \zeta_n$ or W_Φ^0 has some importance in this proof. In order, to ensure the argmax occur in finite points, we use the following localization procedure, consisting in the choice for P_∞-almost all x in Theorem A of the open ball $V(x)$ and $\epsilon > 0$.

Since P_∞ almost surely $\#\text{argmax}\ (x) = 1$ and argmax $(x) \in \mathbb{R}$ (i.e. the maximum does not occur at $\pm\infty$, see [T]), for such an x, define

$$\epsilon := \epsilon(x) = \frac{\sqrt{2\pi}}{4} \sup_{t\in\mathbb{R}} x(t), \qquad V := V(x) = \{y \in \mathbb{D}_0 \mid d_{0,\mathbb{R}}(x, y) < \epsilon\}.$$

Then for $\omega \in \Omega(x) := (W_\Phi^0)^{-1}\{V(x)\}$, $W_\Phi^0(\omega) \in V(x)$ and $d_{0,\mathbb{R}}(W_\Phi^0(\omega), x) < \epsilon$, ensuring $G_{\infty,c}(W_\Phi^0(\omega))$ take positive values, so that its argmax does not occur at $\pm\infty$.

Since from the Skorokhod's representation Theorem $\zeta_n \to W_\Phi^0$ in $\mathbb{D}_0(\mathbb{R})$ a.s. when $n \to +\infty$, for n (randomly) large enough, $\zeta_n(\omega) \in V(x)$ for $x \in \Omega(x)$ and $G_{\infty,c}(\zeta_n(\omega))$ take positive values, so that its argmax does not occur at $\pm\infty$.

The next sections are devoted to the study of conditions (i)–(v) of Theorem A.

1 First Point

In our setting, point (i) of Theorem A, expressed in terms of the process ζ_n, rewrites:

$$\forall\varepsilon > 0, \quad \mathbb{P}\{d_{0,\mathbb{R}}(G_{n,c}(\zeta_n), G_{\infty,c}(\zeta_n)) > \varepsilon\} \to 0, \quad n \to \infty \tag{1.1}$$

where $d_{0,\mathbb{R}}$ stands for the Skorokhod's metric on $\mathbb{D}_0(\mathbb{R})$. Note that for $t \in [0, 1]$:

$$\zeta_n \circ \Psi(t) \;=\; \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(\mathbf{1}_{]-\infty,\Psi(t)]}(\xi_i) - \Phi(\Psi(t))\Big)$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbf{1}_{[0,t]}(\Phi(\xi_i)) - t\Big), \tag{1.2}$$

$$\begin{aligned} G_{n,c}\zeta_n \circ \Psi(t) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbf{1}_{]-\infty,\Psi(t)]}(\xi_i + c/\sqrt{n}) - \Phi(\Psi(t))\Big) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbf{1}_{[0,t]}(\Phi(\xi_i + c/\sqrt{n})) - t\Big). \end{aligned} \tag{1.3}$$

We have

$$\begin{aligned} & d_{0,\mathbb{R}}(G_{n,c}(\zeta_n), G_{\infty,c}(\zeta_n)) \\ &= d_0(G_{n,c}(\zeta_n)\circ\Psi, G_{\infty,c}(\zeta_n)\circ\Psi) \\ &= \inf_{\lambda\in\Lambda}\left(\|G_{n,c}(\zeta_n)(\Psi(\lambda t)) - G_{\infty,c}(\zeta_n)(\Psi(t))\|_\infty + \|\lambda t - t\|_\infty\right) \\ &= \inf_{\lambda\in\Lambda}\Big\{\Big\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbf{1}_{[0,\lambda t]}(\Phi(\xi_i + c/\sqrt{n})) - \lambda t - \mathbf{1}_{[0,t]}(\Phi(\xi_i)) + t\Big) + \\ &\qquad\qquad + c\Phi'(\Psi(t))\Big\|_\infty + \|\lambda t - t\|_\infty\Big\}. \end{aligned} \tag{1.4}$$

We estimate $\inf_{\lambda\in\Lambda}$ with the appropriate λ_n that cancels the indicator terms. To this way, take λ_n the piecewise linear function in Λ given by

$$\lambda_n\Phi(\xi_i) = \Phi(\xi_i + \frac{c}{\sqrt{n}}), \quad i = 1,\ldots,n, \quad \text{and} \quad \lambda_n(0) = 0, \quad \lambda_n(1) = 1.$$

For this choice of λ_n, we have for any $1 \le i \le n$:

$$\mathbf{1}_{[0,\lambda_n t]}(\Phi(\xi_i + c/\sqrt{n})) = \mathbf{1}_{[0,t]}(\Phi(\xi_i)). \tag{1.5}$$

Moreover, $\|\lambda_n t - t\|_\infty$ is obviously reached at some $\Phi(\xi_i)$ and is computed as follows:

$$\begin{aligned} \|\lambda_n t - t\|_\infty &= \sup_{1\le i\le n}|\lambda_n\Phi(\xi_i) - \Phi(\xi_i)| \\ &= \sup_{1\le i\le n}|\Phi(\xi_i + c/\sqrt{n}) - \Phi(\xi_i)| \\ &\le c/\sqrt{n}. \end{aligned}$$

since Φ is obviously 1-Lipschitz. So that, a.s.

$$\|\lambda_n t - t\|_\infty \longrightarrow 0, \quad n \to \infty. \tag{1.6}$$

Using (1.1), (1.4), (1.5), (1.6), it is enough, by now, to derive for all $\varepsilon > 0$:

$$\mathbb{P}\left\{\sqrt{n}\left\|\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'(\Psi(t))\right\|_\infty > \varepsilon\right\} \longrightarrow 0, \quad n \to \infty.$$

First, note that $(\Phi'\circ\Psi)(t) = e^{-\Psi(t)^2/2}/\sqrt{2\pi} \to 0$ as $t \to 0$ or 1 and fix, $[\alpha, 1-\alpha] \subset]0,1[$ such that $(\Phi'\circ\Psi)(t) < \varepsilon/(2\delta)$ for $t \in]0,1[\setminus[2\alpha, 1-2\alpha]$. Writing $A_n(t)$ for $\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'(\Psi(t))$, we have:

$$\mathbb{P}\left\{\sqrt{n}\sup_{t\in[0,1]}|A_n(t)| > \varepsilon\right\} \le \mathbb{P}\left\{\sqrt{n}\sup_{t\in[0,\alpha]}|A_n(t)| > \varepsilon\right\} \tag{1.7}$$

$$+\mathbb{P}\left\{\sqrt{n}\sup_{t\in[\alpha,1-\alpha]}|A_n(t)|>\varepsilon\right\}+\mathbb{P}\left\{\sqrt{n}\sup_{t\in[1-\alpha,1]}|A_n(t)|>\varepsilon\right\}.$$

Clearly for $t\in[\Phi(\xi_i^n),\Phi(\xi_{i+1}^n)]$, the supremum of $|\lambda_n t-t|$ is reached at $\Phi(\xi_i^n)$ or at $\Phi(\xi_{i+1}^n)$, and is equal to

$$\Phi(\xi_i^n+c/\sqrt{n})-\Phi(\xi_i^n)=\frac{c}{\sqrt{n}}\Phi'(\theta_{n,i})\simeq\frac{c}{\sqrt{n}}\Phi'(\xi_i^n)=\frac{c}{\sqrt{n}}\Phi'\circ\Psi(\Phi(\xi_i^n))$$

or to the same for index $i+1$. But for $t\in[0,\alpha]$, $c\in[0,\delta]$ and n large enough:

$$\begin{aligned}
&\sqrt{n}\sup_{t\in[0,\alpha]}\left|\lambda_n t-t-\frac{c}{\sqrt{n}}\Phi'(\Psi(t))\right|\\
&\quad\le\sup_{t\in[0,\alpha]}\sqrt{n}|\lambda_n t-t|+\delta\sup_{t\in[0,\alpha]}|\Phi'(\Psi(t))|\\
&\quad\le\sup_{\{i\,|\,\Phi(\xi_i^n)\text{ or }\Phi(\xi_{i-1}^n)\in[0,\alpha]\}}\sqrt{n}|\lambda_n\Phi(\xi_i^n)-\Phi(\xi_i^n)|+\varepsilon/2\\
&\quad<\sup_{\{i\,|\,\Phi(\xi_i^n)\in[0,2\alpha]\}}\sqrt{n}|\Phi(\xi_i^n+\frac{c}{\sqrt{n}})-\Phi(\xi_i^n)|+\varepsilon/2
\end{aligned}\tag{1.8}$$

where in the later inequality we bound the sup over the range $\{i\mid\Phi(\xi_i^n)\text{ or }\Phi(\xi_{i-1}^n)\in[0,\alpha]\}$ by those over $\{i\mid\Phi(\xi_i^n)\in[0,2\alpha]\}$ using

$$\sup_{1\le i\le n}\{\Phi(\xi_{i+1}^n)-\Phi(\xi_i^n)\}\to 0$$

following from forthcoming (1.13) so that for n large enough we have $|\Phi(\xi_{i+1}^n)-\Phi(\xi_i^n)|\le\alpha$.

Now Taylor-Lagrange formula yields $\tilde\theta_{n,i}\in[\xi_i^n,\xi_i^n+c/\sqrt{n}]$ such that:

$$\sqrt{n}\left|\Phi(\xi_i^n+\frac{c}{\sqrt{n}})-\Phi(\xi_i^n)\right|=\left|c\Phi'(\xi_i^n)+\frac{c^2}{\sqrt{n}}\Phi''(\tilde\theta_{n,i})\right|\le\delta|\Phi'(\xi_i^n)|+O(1/\sqrt{n})\tag{1.9}$$

since $\Phi''(x)=-xe^{-x^2/2}/\sqrt{2\pi}$ is bounded. But, since we consider indices i for which $\Phi(\xi_i^n)<2\alpha$, we have $\Phi'(\xi_i^n)=\Phi'\circ\Psi(\Phi(\xi_i^n))\le\varepsilon/(2\delta)$, so that for n large enough, (1.8) and (1.9) give:

$$\sup_{t\in[0,\alpha]}\left|\lambda_n t-t-\frac{c}{\sqrt{n}}\Phi'(\Psi(t))\right|\ <\ \varepsilon+O(1/\sqrt{n})$$

and the probability $\mathbb{P}\left\{\sqrt{n}\sup_{t\in[0,\alpha]}\left|\lambda_n t-t-\frac{c}{\sqrt{n}}\Phi'(\Psi(t))\right|>\varepsilon\right\}$ is zero for n large enough as should be the probability of an empty set. The same being true for the probability relative to the sup over $\{t\in[1-\alpha,1]\}$, it remains to deal with the term relative to the sup over $\{t\in[\alpha,1-\alpha]\}$. To this way, note first that for $t=\Phi(\xi_i^n)$,

$$\lambda_n t-t-\frac{c}{\sqrt{n}}\Phi'(\Psi(t))=\Phi(\xi_i^n+c/\sqrt{n})-\Phi(\xi_i^n)-c/\sqrt{n}\Phi'(\xi_i^n)\simeq(c/\sqrt{n})^2\Phi''(\xi_i^n).$$

Let us do it more precisely. The extremum of $\lambda_n t - t$ over $[\Phi(\xi_i^n), \Phi(\xi_{i+1}^n)]$ is reached at $\Phi(\xi_i^n)$ or at $\Phi(\xi_{i+1}^n)$. Using again Taylor-Lagrange formula, there is some $\tilde{\theta}_{n,i} \in]\Phi(\xi_i^n), \Phi(\xi_{i+1}^n)[$ such that

$$\Phi'(\Psi(t)) = \Phi'(\Psi(\Phi(\xi_i^n))) + (t - \Phi(\xi_i^n))(\Phi' \circ \Psi)'(\tilde{\theta}_{n,i}).$$

It follows with previous notation

$$A_n(t) \quad = \quad \lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n) - \frac{c}{\sqrt{n}}(t - \Phi(\xi_i^n))(\Phi' \circ \Psi)'(\tilde{\theta}_{n,i})$$

and taking supremum over $[\Phi(\xi_i^n), \Phi(\xi_{i+1}^n)]$ yields:

$$\begin{aligned} &\sqrt{n}\,\|A_n(t)\|_{\infty,i} \\ &\leq \sqrt{n}\left\|\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n)\right\|_{\infty,i} + \sqrt{n}\left\|\frac{c}{\sqrt{n}}(t - \Phi(\xi_i^n))(\Phi' \circ \Psi)'(\tilde{\theta}_i^n)\right\|_{\infty,i} \end{aligned} \tag{1.10}$$

where we note $\|\cdot\|_{\infty,i} = \|\cdot\|_{\infty,[\Phi(\xi_i^n),\Phi(\xi_{i+1}^n)]} = \sup\limits_{x\in[\Phi(\xi_i^n),\Phi(\xi_{i+1}^n)]}\{\cdot\}$. For the first term in the right-hand side of (1.10), the supremum is reached

- either at $\Phi(\xi_i^n)$ and is thus equal to

$$\sqrt{n}\left|\lambda_n\Phi(\xi_i^n) - \Phi(\xi_i^n) - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n)\right| = \sqrt{n}\left(\frac{c}{\sqrt{n}}\right)^2 |\Phi''(\hat{\theta}_{n,i})| = O(1/\sqrt{n}) \tag{1.11}$$

  with $\hat{\theta}_{n,i} \in [\xi_i^n, \xi_i^n + c/\sqrt{n}]$ and this goes to zero uniformly (since $\Phi''(x) = -xe^{-x^2/2}/\sqrt{2\pi}$ is a bounded function).

- either at $\Phi(\xi_{i+1}^n)$ and is equal to

$$\begin{aligned} &\sqrt{n}\left|\lambda_n\Phi(\xi_{i+1}^n) - \Phi(\xi_{i+1}^n) - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n)\right| \\ &\quad = \sqrt{n}\left|\Phi(\xi_{i+1}^n + c/\sqrt{n}) - \Phi(\xi_{i+1}^n) - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n)\right| \\ &\quad = \sqrt{n}\left|\frac{c}{\sqrt{n}}\Phi'(\xi_{i+1}^n) + \frac{c^2}{n}\Phi''(\hat{\theta}_{n,i+1}) - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n)\right| \\ &\quad = c|\Phi'(\xi_{i+1}^n) - \Phi'(\xi_i^n)| + \frac{c^2}{\sqrt{n}}|\Phi''(\hat{\theta}_{n,i+1})| \\ &\quad = c|\Phi' \circ \Psi(\Phi(\xi_{i+1}^n)) - \Phi' \circ \Psi(\Phi(\xi_i^n))| + O(1/\sqrt{n}). \end{aligned} \tag{1.12}$$

But, $[\alpha, 1-\alpha] \subset \bigcup\limits_i [\Phi(\xi_i^n), \Phi(\xi_{i+1}^n)]$ where the union is taken for those indices i for which $\Phi(\xi_i^n) \in [\alpha/2, 1-\alpha/2]$. The sup over $[\alpha, 1-\alpha]$ is thus bounded by the greatest of the sup's over $[\Phi(\xi_i^n), \Phi(\xi_{i+1}^n)]$. Now, in order to compute

$$\sup_{t\in[\alpha,1-\alpha]}\left|\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'(\Psi(t))\right|,$$

considering (1.12) and the uniform continuity of $\Phi' \circ \Psi$ on $[\alpha/2, 1-\alpha/2]$, we are left with the study of $\sup_{1\le i\le n} |\Phi(\xi_{i+1}^n - \Phi(\xi_i^n)|$ for which we dispose of the Glivenko-Cantelli Theorem: since $U_i^n := \Phi(\xi_i^n)$ is uniformly distributed on $[0,1]$, the variables U_i^n are the uniform order statistics. But $F_U(t) = t$, $t \in [0,1]$ and

$$F_U(U_i^n) = U_i^n, \quad F_n(U_i^n) = i/n,$$

whence

$$\begin{aligned}
&\sup_{1\le i\le n} |\Phi(\xi_{i+1}^n) - \Phi(\xi_i^n)| = \sup_{1\le i\le n} |U_{i+1}^n - U_i^n| = \sup_{1\le i\le n} |F_U(U_{i+1}^n) - F_U(U_i^n)| \\
&\le \sup_{1\le i\le n} |F_U(U_{i+1}^n) - F_n(U_{i+1}^n)| + \sup_{1\le i\le n} |F_n(U_{i+1}^n) - F_n(U_i^n)| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \sup_{1\le i\le n} |F_n(U_i^n) - F_U(U_i^n)| \\
&\le 2 \sup_{t\in[0,1]} |F_n(t) - F_U(t)| + \sup_{1\le i\le n} |(i+1)/n - i/n| \longrightarrow 0, \quad \text{a.s.}
\end{aligned}$$

using Glivenko-Cantelli Theorem. So that, for uniform random variables on $[0,1]$

$$\sup_{1\le i\le n} |U_{i+1}^n - U_i^n| \longrightarrow 0, \quad n\to\infty \quad \text{a.s.} \tag{1.13}$$

and now using (1.12) and uniform continuity of $\Phi' \circ \Psi$ on $[\alpha/2, 1-\alpha/2]$

$$\sqrt{n} \sup_{i \mid \Phi(\xi_i^n)\in[\frac{\alpha}{2}, 1-\frac{\alpha}{2}]} \left| \lambda_n \Phi(\xi_{i+1}^n) - \Phi(\xi_{i+1}^n) - \frac{c}{\sqrt{n}} \Phi'(\xi_i^n) \right| \longrightarrow 0, \quad n\to\infty \quad \text{a.s.}$$

since it is true in the both situations by (1.11), (1.12). To finish with point (i), there is still the second summand of (1.10) to deal with, but it is an easy matter since

$$(\Phi' \circ \Psi)'(t) = \Psi'(t)\Phi''(\Psi(t)) = \frac{\Phi''(\Psi(t))}{\Phi'(\Psi(t))} = -\Psi(t)$$

is bounded for $t \in [\Phi(\xi_i^n), \Phi(\xi_{i+1}^n)] \subset [\frac{\alpha}{2}, 1-\frac{\alpha}{2}]$ and since the convergence of $\sup_{1\le i\le n} |\Phi(\xi_{i+1}^n) - \Phi(\xi_i^n)|$ to zero has just been derived in (1.13).

Finally, (1.10) goes to zero uniformly with respect to all i for which $\Phi(\xi_i^n) \in [\alpha/2, 1-\alpha/2]$ so that

$$\mathbb{P}\left\{ \sqrt{n} \sup_{t\in[\alpha, 1-\alpha]} \left| \lambda_n t - t - \frac{c}{\sqrt{n}} \Phi'(\Psi(t)) \right| \ge \varepsilon \right\} \longrightarrow 0.$$

Gathering all the intermediate results from (1.7), point (i) is satisfied.

2 Point (ii)

In our setting, this point is straightforward. The function $G_{\infty,c}$ is obviously continuous since we have chosen $G_{\infty,c}x = x - c\Phi'$. And

$$d_{0,\mathbb{R}}(G_{\infty,c}x, x) = d(G_{\infty,c}x \circ \Psi, x\circ\Psi) \le \|G_{\infty,c}x \circ \Psi - x \circ \Psi\|_\infty = c\|\Phi' \circ \Psi\|_\infty \to 0,$$

as $c \to 0$, uniformly in $x \in \mathbb{D}_0(\mathbb{R})$ since Φ' is bounded.

3 Point (iii)

The purpose of this section is to justify

$$\lim_{c\to 0}\overline{\lim}_{n\to\infty}\|P_nG_{n,c}^{-1}-P_n\|=0.$$

But from expressions (0.1), (0.6) we can write

$$P_n=\mathcal{L}(\Theta_n(\xi_1,\ldots,\xi_n)),\quad P_nG_{n,c}^{-1}=\mathcal{L}(\Theta_n(\xi_1+c/\sqrt{n},\ldots,\xi_n+c/\sqrt{n})),$$

where $\Theta_n:\mathbb{R}^n\longrightarrow\mathbb{D}(\mathbb{R})$ is given by

$$\Theta_n(x_1,\ldots,x_n)(t)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\mathbf{1}_{]-\infty,t]}(x_i)-\Phi(t)\right).$$

We derive now

$$\begin{aligned}\|P_nG_{n,c}^{-1}-P_n\| &\le \|\mathcal{L}(\xi_1+c/\sqrt{n},\ldots,\xi_n+c/\sqrt{n})-\mathcal{L}(\xi_1,\ldots,\xi_n)\|\\ &\le \int_{\mathbb{R}^n}\left|\prod_{i=1}^{n}p(x_i-\frac{c}{\sqrt{n}})-\prod_{i=1}^{n}p(x_i)\right|dx_1\cdots dx_n. \end{aligned}\tag{3.1}$$

Now, we can apply Lemma 20.1 in [DLS] (to $l(t)=t$) in order to get from (3.1):

$$\|P_nG_{n,c}^{-1}-P_n\|\le c,$$

from which point (iii) follows immediately.

4 Point (iv)

In this section, we study point (iv) of Theorem A, that is for all $\delta\in(0,\epsilon]$:

$$\int_V\|\lambda_{[0,\delta]}\varphi_{n,z}^{-1}-\lambda_{[0,\delta]}\varphi_{\infty,z}^{-1}\|\;P_n(dz)\longrightarrow 0,\quad n\to\infty,\tag{4.1}$$

where $\varphi_{n,z}(c)=\sup_{t\in\mathbb{R}}(G_{n,c}z(t))$, $n\in\mathbb{N}\cup\{\infty\}$, $c\in(0,\delta]$ and V is some $V(x)$ defined in the localization procedure.

Since $P_n\Rightarrow P_\infty$, (4.1) is obtained if we show that for P_∞-almost all z, the convergence $z_n\to z$ implies

$$\|\lambda_{[0,\delta]}\varphi_{n,z_n}^{-1}-\lambda_{[0,\delta]}\varphi_{\infty,z_n}^{-1}\|\to 0\tag{4.2}$$

(see the remark after Theorem 18.4 in [DLS]). To derive such convergence in variation, we dispose of the following result of Davydov in [D] for one-dimensional image measures:

Proposition 1 *Let $f_n:[0,1]\to\mathbb{R}$, $n\in\mathbb{N}\cup\{\infty\}$ be a sequence of absolutely continuous functions such that*

- $f_n(0)\to f_\infty(0)$,

- $f'_n \to f'_\infty$ *in* $L^1([0,1])$,
- $f'_\infty \neq 0$ *a.e.*

Then $\lambda_{[0,1]} f_n^{-1} \xrightarrow{var} \lambda_{[0,1]} f_\infty^{-1}$.

Unfortunately, Proposition 1 can not be applied directly in our setting because (4.2) is concerned with the asymptotic distance in variation of two sequences of measures, whereas Proposition 1 deals with a single converging sequence. Hence, we introduce $\lambda_{[0,\delta]}\varphi^{-1}_{\infty,W^0_\Phi}$ and split the study of (4.2) into those of

$$\|\lambda_{[0,\delta]}\varphi^{-1}_{\infty,\zeta_n} - \lambda_{[0,\delta]}\varphi^{-1}_{\infty,W^0_\Phi}\| \longrightarrow 0, \tag{4.3}$$
$$\|\lambda_{[0,\delta]}\varphi^{-1}_{n,\zeta_n} - \lambda_{[0,\delta]}\varphi^{-1}_{\infty,W^0_\Phi}\| \longrightarrow 0 \tag{4.4}$$

for both of which Prop. 1 (or with weaker hypothesis, see Proposition 2) is usable. In this section $\|\cdot\|$ stands for the variation norm computed on $\Omega(x)$, the sub-probability space related to $V(x)$ in the localization procedure. To this way, we study the following functions and their derivatives

$$\varphi_{n,\zeta_n}(c) = \sup_{t\in\mathbb{R}} G_{n,c}\zeta_n(t) \tag{4.5}$$
$$\varphi_{\infty,\zeta_n}(c) = \sup_{t\in\mathbb{R}} G_{\infty,c}\zeta_n(t) \tag{4.6}$$
$$\varphi_{\infty,W^0_\Phi}(c) = \sup_{t\in\mathbb{R}} G_{\infty,c}W^0_\Phi(t) \tag{4.7}$$

with the tranformations $G_{n,c}$, $n \in \mathbb{N}\cup\{\infty\}$, given in (0.6) and (0.7).

4.1 Derivatives

In order to use Proposition 1 (or Prop. 2), the first step is to compute the derivatives of φ_{n,ζ_n}, φ_{∞,ζ_n}, $\varphi_{\infty,W^0_\Phi}$. This is the purpose of this section.

4.1.1 Derivative of φ_{n,ζ_n}

First, ζ_n reaches its maximum at some a.s. unique ξ_i (unique since almost surely the ξ_i's are all distinct). Similarly, $G_{n,c}\zeta_n$ reaches its maximum at some else $\xi_i + c/\sqrt{n}$. We can suppose again unicity of this max for almost all c, since if $G_{n,c}\zeta_n(\xi_i^n + c/\sqrt{n}) = G_{n,c}\zeta_n(\xi_j^n + c/\sqrt{n})$, we have

$$\begin{aligned}\frac{i}{\sqrt{n}} - \sqrt{n}\Phi(\xi_i^n + \frac{c}{\sqrt{n}}) &= \frac{j}{\sqrt{n}} - \sqrt{n}\Phi(\xi_j^n + \frac{c}{\sqrt{n}}) \\ \frac{j-i}{n} &= \Phi(\xi_j^n + \frac{c}{\sqrt{n}}) - \Phi(\xi_i^n + \frac{c}{\sqrt{n}})\end{aligned}$$

which is possible for at most two distinct values of c, as follows from an elementary study of Φ. So that,

$$\#\{c \in [0,\delta] \mid \exists (i,j), \frac{i}{\sqrt{n}} - \sqrt{n}\Phi(\xi_i + \frac{c}{\sqrt{n}}) = \frac{j}{\sqrt{n}} - \sqrt{n}\Phi(\xi_j + \frac{c}{\sqrt{n}})\} < \infty$$

and finally for almost all c, $\#\text{argmax } G_{n,c}\zeta_n = 1$.

We compute now the derivative of φ_{n,ζ_n} at almost all c.

Assume $\#\text{argmax } \{G_{n,c}\zeta_n\} = 1$ and note i_0 the corresponding index, realizing $\text{argmax } G_{n,c}\zeta_n = \xi_{i_0} + c/\sqrt{n}$. For d near enough c, $G_{n,d}\zeta_n$ reaches its maximum for the same index i_0 in $\xi_{i_0} + d/\sqrt{n}$, so that

$$\begin{aligned}
\varphi'_{n,\zeta_n}(c) &= \lim_{d\to c} \frac{\varphi_{n,\zeta_n}(d) - \varphi_{n,\zeta_n}(c)}{d-c} \\
&= -\lim_{d\to c} \sqrt{n} \frac{\Phi(\xi_{i_0} + d/\sqrt{n}) - \Phi(\xi_{i_0} + c/\sqrt{n})}{d-c} \\
&= -\Phi'(\xi_{i_0} + c/\sqrt{n}) \\
&= -\Phi'(\text{argmax } (G_{n,c}\zeta_n)). \qquad (4.8)
\end{aligned}$$

4.1.2 Derivative of φ_{∞,ζ_n}

We study the derivative of φ_{∞,ζ_n}, given in (4.6). First, rewrite

$$\varphi_{\infty,\zeta_n}(c) = \sup_{t\in\mathbb{R}} \big(\zeta_n(t) - c\Phi'(t)\big) = \sup_{t\in\mathbb{R}} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \big\{ \mathbf{1}_{]-\infty,t]}(\xi_i) - \Phi(t) \big\} - c\Phi'(t) \right).$$

For $t \in [\xi_i^n, \xi_{i+1}^n]$, $G_{\infty,c}\zeta_n(t) = \frac{i}{\sqrt{n}} - \sqrt{n}\Phi(t) - c\Phi'(t)$. But since the jumps of $G_{\infty,c}\zeta_n(t)$ are non negative and since

$$\big(-\sqrt{n}\Phi(t) - c\Phi'(t)\big)' = -\sqrt{n}\Phi'(t) - c\Phi''(t) = \Phi'(t)(-\sqrt{n} + ct),$$

the function $G_{\infty,c}\zeta_n$ is increasing for $t \geq \sqrt{n}/c$ (it is obtained adding positive jumps to an increasing function). Since, its limit as $t \to +\infty$ is 0, its sup for $t \geq \sqrt{n}/c$ must be reached in $+\infty$ and equals to 0.

Note that the sup can not occur at $\pm\infty$ since we consider $\omega \in \Omega(x)$ for which we have seen in the localization procedure that, at least for n large enough, $G_{\infty,c}\zeta_n$ reached its maximum in a finite point.

But, on the other hand, for $t \leq \sqrt{n}/c$, the expression $-\sqrt{n}\Phi(t) - c\Phi'(t)$ is decreasing and one obtains the function $G_{\infty,c}\zeta_n$ adding jumps $+1/\sqrt{n}$ at $\xi_i \leq \sqrt{n}/c$ to a decreasing function. For $t \leq \sqrt{n}/c$, local maxima are located in the ξ_i's and are all distinct for at least almost all c since if $G_{\infty,c}\zeta_n(\xi_i^n) = G_{\infty,c}\zeta_n(\xi_j^n)$, we have

$$\begin{aligned}
&\frac{i}{\sqrt{n}} - \sqrt{n}\Phi(\xi_i^n) - c\Phi'(\xi_i^n) = \frac{j}{\sqrt{n}} - \sqrt{n}\Phi(\xi_j^n) - c\Phi'(\xi_j^n) \\
\iff \quad & c = \frac{\frac{j-i}{n} - (\Phi(\xi_i^n) - \Phi(\xi_j^n))}{\Phi'(\xi_j^n) - \Phi'(\xi_i^n)} \sqrt{n}.
\end{aligned}$$

So that, finally, for $c \notin \bigcup_{1\leq i\neq j\leq n} \left\{ \frac{\frac{j-i}{n} - (\Phi(\xi_i^n) - \Phi(\xi_j^n))}{\Phi'(\xi_j^n) - \Phi'(\xi_i^n)} \sqrt{n} \right\}$, the maximum over $\mathbb{R}$ of $G_{\infty,c}\zeta_n$ is reached necessarily once at some ξ_i^n and equals

$$G_{\infty,c}\zeta_n(\xi_i^n) = \frac{i}{\sqrt{n}} - \sqrt{n}\Phi(\xi_i^n) - c\Phi'(\xi_i^n).$$

For almost all c, we can thus define without ambiguity argmax $(G_{\infty,c}\zeta_n)$.

For such a c, note $t_0 = \xi^n_{i_0}$ the corresponding argmax, so that

$$\varphi_{\infty,c}(\zeta_n) = G_{\infty,c}\zeta_n(t_0) = \frac{i_0}{\sqrt{n}} - \sqrt{n}\Phi(t_0) - c\Phi'(t_0).$$

For $c_p \to c$, note also $t_p \in \text{argmax}\,\{G_{\infty,c_p}\zeta_n\}$. Since t_p belongs to the finite set $\{\xi^n_1, \ldots, \xi^n_n\}$, for any subsequence $(t_{p'})_{p'}$, extract further $(t_{p''})_{p''}$ with $t_{p''} \to t_\infty$. First, we show that necessarily $t_\infty = t_0$ so that the whole sequence $(t_p)_p$ converges to t_0. To this way, in order to simplify notation, write henceforth t_p for $t_{p''}$.

Note that $(\zeta_n(t_p))_p$ must converge to $\zeta_n(t_\infty)$. Indeed, it is clear whenever ζ_n is continuous at t_∞ or at least (by right-continuity) if $(t_p)_p$ decreases to t_∞. Else $(t_p)_p$ increases to a jump-time t_∞ of ζ_n and in this case $\zeta_n(t_p) \to \zeta_n(t_\infty^-) = \zeta_n(t_\infty) - 1/\sqrt{n}$ as $p \to \infty$ and for p large enough: $\zeta_n(t_p) < \zeta_n(t_\infty) - 1/(2\sqrt{n})$. It follows

$$\zeta_n(t_p) - c_p\Phi'(t_p) < \zeta_n(t_\infty) - c_p\Phi'(t_\infty) + \underbrace{c_p(\Phi'(t_\infty) - \Phi'(t_p))}_{\text{goes to } 0, \text{ as } p\to\infty} - \frac{1}{2\sqrt{n}}.$$

And for p large enough:

$$\zeta_n(t_p) - c_p\Phi'(t_p) < \zeta_n(t_\infty) - c_p\Phi'(t_\infty)$$

which contradicts $t_p \in \text{argmax}\,(G_{\infty,c_p}\zeta_n)$. Finally, $(\zeta_n(t_p))_p$ must converge to $\zeta_n(t_\infty)$. Now, since

$$G_{\infty,c_p}\zeta_n(t_p) = \zeta_n(t_p) - c_p\Phi'(t_p) \geq \zeta_n(t) - c_p\Phi'(t), \quad \forall t,$$

taking limit in $p \to \infty$ yields

$$\zeta_n(t_\infty) - c\Phi'(t_\infty) \geq \zeta_n(t) - c\Phi'(t), \quad \forall t.$$

We can thus rewrite $G_{\infty,c}\zeta_n(t_\infty) \geq G_{\infty,c}\zeta_n(t)$, so that $t_\infty \in \text{argmax}\,\{G_{\infty,c}\zeta_n\} = \{t_0\}$. This justifies the convergence of the whole sequence $(t_p)_p$ to t_0.

We have $\varphi_{\infty,\zeta_n}(c_p) = G_{\infty,c_p}\zeta_n(t_p) \geq G_{\infty,c_p}\zeta_n(t_0)$, so that

$$\varphi_{\infty,\zeta_n}(c_p) - \varphi_{\infty,\zeta_n}(c) \geq G_{\infty,c_p}\zeta_n(t_0) - G_{\infty,c}\zeta_n(t_0) = (c - c_p)\Phi'(t_0).$$

And similarly, $\varphi_{\infty,\zeta_n}(c) \geq G_{\infty,c}\zeta_n(t_p)$ implies

$$\varphi_{\infty,\zeta_n}(c_p) - \varphi_{\infty,\zeta_n}(c) \leq G_{\infty,c_p}\zeta_n(t_p) - G_{\infty,c}\zeta_n(t_p) = (c - c_p)\Phi'(t_p)$$

and it follows

$$(c - c_p)\Phi'(t_0) \leq \varphi_{\infty,\zeta_n}(c_p) - \varphi_{\infty,\zeta_n}(c) \leq (c - c_p)\Phi'(t_p),$$

from which we derive for almost all c:

$$\varphi'_{\infty,\zeta_n}(c) = -\Phi'(t_0) = -\Phi'(\text{argmax}\,(G_{\infty,c}\zeta_n)). \tag{4.9}$$

4.1.3 Derivative of $\varphi_{\infty,W_\Phi^0}$

The first step, in this section, consists in deriving $\#\text{argmax }\{G_{\infty,c}W_\Phi^0\}=1$ for almost all c. Then similar computations as for φ_{∞,ζ_n} allow to derive the expression of the derivative.

For $c=0$, $G_{\infty,0}W_\Phi^0=W_\Phi^0$ has almost surely an unique argmax.

Since $\lim_{t\to\pm\infty}\Phi'(t)=0$ and $\Phi'\in L^2(\mathbb{R})$ is absolutely continuous, Φ' is an admissible direction for the process W_Φ^0. It follows $\mathcal{L}(W_\Phi^0-c\Phi')\ll\mathcal{L}(W_\Phi^0)$. Whence $W_\Phi^0-c\Phi'$ has also an unique argmax since

$$1=\mathbb{P}\big\{\#\text{argmax }(W_\Phi^0)=1\big\}=\mathbb{P}_{W_\Phi^0}\big\{x\mid\#\text{argmax }(x)=1\big\}$$

implies

$$\mathbb{P}\big\{\#\text{argmax }(W_\Phi^0-c\Phi')=1\big\}=\mathbb{P}_{W_\Phi^0-c\Phi'}\big\{x\mid\#\text{argmax }(x)=1\big\}=1$$

where $\big\{x\mid\#\text{argmax }(x)=1\big\}$ is measurable since we can rewrite it as for example

$$\bigcap_{n\in\mathbb{N}^*}\ \bigcup_{\substack{t,s\in\mathbb{Q}\\ t<s\\ |t-s|\le 1/n}}\left\{x\in\mathbb{D}\mid\sup_{u\in[t,s]\cap\mathbb{Q}}x(u)>\sup_{u\in[t,s]^c\cap\mathbb{Q}}x(u)\right\},$$

which is measurable in $\mathcal{C}(\mathbb{R})$. Since $(\omega,c)\mapsto G_{\infty,c}W_\Phi^0(\omega)=W_\Phi^0(\omega)-c\Phi'$ is bimeasurable, using Fubini Theorem, we derive that almost surely $\#\text{argmax }\{G_{\infty,c}W_\Phi^0\}=1$ for almost all c.

The only point to revise to apply the computations of φ_{∞,ζ_n} in this case is the fact that for $c_p\to c$ and $t_p\in\text{argmax }G_{\infty,c_p}W_\Phi^0$ we still have the convergence $t_p\to t_0:=\text{argmax }G_{\infty,c}W_\Phi^0$. To this way, for any subsequence $(t_{p'})_{p'}$, we have extracted in Section 4.1.2 a further subsequence $(t_{p''})_{p''}$ converging to some t_∞ and we have derived $t_\infty=t_0$. The extraction of the convergent sequence $(t_{p''})_{p''}$ was straightforward in Section 4.1.2 since t_p lay in a finite set $\{\xi_1^n,\dots,\xi_n^n\}$. This point has to be revised in the present case: let c_0 be such that the related argmax t_0 is unique and $I(t_0)=[t_0-1,t_0+1]$ be a neighborhood of t_0. We have

$$\sup_{t\notin I(t_0)}\{W_\Phi^0(t)-c_0\Phi'(t)\}<\sup_{t\in\mathbb{R}}\{W_\Phi^0(t)-c_0\Phi'(t)\}=W_\Phi^0(t_0)-c_0\Phi'(t_0).$$

Note $\alpha=\sup_{t\in\mathbb{R}}\{W_\Phi^0(t)-c_0\Phi'(t)\}-\sup_{t\notin I(t_0)}\{W_\Phi^0(t)-c_0\Phi'(t)\}>0$. For $c\in]c_0-\frac{\sqrt{2\pi}}{3}\alpha,c_0+\frac{\sqrt{2\pi}}{3}\alpha[$ (where $1/\sqrt{2\pi}=\sup_{t\in\mathbb{R}}\Phi(t)$) and $t\notin I(t_0)$, we have

$$W_\Phi^0(t)-c\Phi'(t)=W_\Phi^0(t)-c_0\Phi'(t)+(c_0-c)\Phi'(t)<\sup_{t\in\mathbb{R}}\{W_\Phi^0(t)-c_0\Phi'(t)\}-2\alpha/3,$$

whereas

$$\begin{aligned}W_\Phi^0(t_0)-c\Phi'(t_0)&=W_\Phi^0(t_0)-c_0\Phi'(t_0)+(c_0-c)\Phi'(t_0)\\&>\sup_{t\in\mathbb{R}}\{W_\Phi^0(t)-c_0\Phi'(t)\}-\alpha/3,\end{aligned}$$

so that for all $t\notin I(t_0)$, $W_\Phi^0(t)-c\Phi'(t)<W_\Phi^0(t_0)-c\Phi'(t_0)$ and

$$\sup_{t\in\mathbb{R}}(W_\Phi^0(t)-c\Phi'(t))=\sup_{t\in I(t_0)}(W_\Phi^0(t)-c\Phi'(t)).$$

For p large enough the argmax t_p of $G_{\infty,c_p}W^0_\Phi$ thus lies in $I(t_0)$ and we can extract from any subsequence $(t_{p'})$ some further converging subsequence $(t_{p''})$. Finally, we finish the computation as in Section 4.1.2 and obtain

$$\varphi'_{\infty,W^0_\Phi}(c) = -\Phi'(\text{argmax } (G_{\infty,c}W^0_\Phi)). \tag{4.10}$$

4.2 Convergences of image measures in (4.3) and (4.4)

4.2.1 The case of $\lambda_{[0,\delta]}\varphi^{-1}_{\infty,\zeta_n}$

In order to derive (4.3), we shall use Proposition 1. Remember we are working on a probability space given by the Skorokhod's representation Theorem on which the weak convergence (0.2) is strengthened into $\zeta_n \to W^0_\Phi$ almost surely for the uniform metric, that is

$$\|\zeta_n - W^0_\Phi\|_{\infty,\mathbb{R}} \longrightarrow 0, \quad n \to \infty. \tag{4.11}$$

Since they are obviously 1-Lipschitzian, the functions

$$\begin{array}{rclcrcl} \varphi_{\infty,\zeta_n}(c) & = & \sup_{t\in\mathbb{R}} G_{\infty,c}\zeta_n(t) & \qquad & \varphi_{\infty,W^0_\Phi}(c) & = & \sup_{t\in\mathbb{R}} G_{\infty,c}W^0_\Phi(t) \\ & = & \sup_{t\in\mathbb{R}}(\zeta_n(t) - c\Phi'(t)), & & & = & \sup_{t\in\mathbb{R}}(W^0_\Phi(t) - c\Phi'(t)), \end{array}$$

are absolutely continuous, with the derivatives computed in (4.9), (4.10) in Sections 4.1.2, 4.1.3. For the other points of Proposition 1:

First, since $\zeta_n \to W^0_\Phi$ uniformly, we have $\varphi_{\infty,\zeta_n}(c) \longrightarrow \varphi_{\infty,W^0_\Phi}(c)$, $\forall c$.
Second, since

$$\begin{array}{rcl} \varphi'_{\infty,\zeta_n}(c) & = & -\Phi'(\text{argmax } (G_{\infty,c}\zeta_n)), \\ \varphi'_{\infty,W^0_\Phi}(c) & = & -\Phi'(\text{argmax } (G_{\infty,c}W^0_\Phi)), \end{array}$$

and the function Φ' is continuous and bounded, it is enough to derive

$$\text{argmax } (G_{\infty,c}\zeta_n) \longrightarrow \text{argmax } (G_{\infty,c}W^0_\Phi), \quad n \to +\infty. \tag{4.12}$$

But we dispose of the following elementary result:

Lemma 1 *Let f_n and f be real functions such that $f_n \to f$ uniformly and $\#$argmax $\{f\} = 1$. Then for any sequence $(t_n)_n$ with $t_n \in$ argmax $\{f_n\}$, we have*

$$t_n \to \textit{argmax } f, \quad n \to \infty.$$

Moreover, on the set of functions whose maximum is reached only once, argmax is a continuous function.

Proof: Indeed, let $t_n \in$ argmax $\{f_n\}$ and $t_0 =$ argmax f, and note $V_0 = [t_0 - 1, t_0 + 1]$ be some compact neighborhood of t_0. We have

$$\sup_{t\notin V_0} f(t) < \sup_{t\in\mathbb{R}} f(t).$$

Note $\alpha = \sup_{t\in\mathbb{R}} f(t) - \sup_{t\notin V_0} f(t)$, and choose n_0 such that for $n \geq n_0$, $\|f_n - f\|_\infty \leq \alpha/3$. For any $t \notin V_0$, we have $f_n(t) < f(t) + \alpha/3 \leq \sup_{s\notin V_0} f(s) + \alpha/3$. So that

$$\sup_{t\notin V_0} f_n(t) < \sup_{t\notin V_0} f(t) + \alpha/3 = \sup_{t\in\mathbb{R}} f(t) - 2\alpha/3.$$

On the other hand, $f_n(t_0) > f(t_0) - \alpha/3 = \sup_{t\in\mathbb{R}} f(t) - \alpha/3$. From the definition of α, it follows $\sup_{t\notin V_0} f_n(t) < f_n(t_0)$ for any $n \geq n_0$ so that argmax $\{f_n\} \subset V_0$.

For any subsequence $(t_{n'})_{n'}$, we can thus extract some further $(t_{n''})_{n''}$ converging to t_∞, $f_{n''}(t_{n''}) \geq f_{n''}(t)$ yields as $n'' \to \infty$, $f(t_\infty) \geq f(t)$ by uniform convergence. We deduce $t_\infty = t_0$, and since any subsequence of $(t_{n'})_{n'}$ has some further subsequence converging to t_0, the whole sequence converges to t_0. ∎

Since the following argmax are unique, applying Lemma 1 yields,

$$\text{argmax}\ \{\zeta_n - c\Phi'\} \longrightarrow \text{argmax}\ \{W_\Phi^0 - c\Phi'\},$$

for almost all c, from which we deduce easily the convergence in $L^1([0,\delta])$ of the derivatives (4.9) to (4.10), using dominated convergence.

Finally, $\varphi'_{\infty,W_\Phi^0}(c) = -\Phi'(\text{argmax}\ G_{\infty,c}W_\Phi^0) \neq 0$ a.e. since argmax $G_{\infty,c}W_\Phi^0$ is finite and Φ' vanishes only at $\pm\infty$ when $\omega \in \Omega(x)$.

We can thus apply Proposition 1 and derive (4.3): almost surely on $\Omega(x)$,

$$\lambda_{[0,\delta]}\varphi^{-1}_{\infty,\zeta_n} \xrightarrow{var} \lambda_{[0,\delta]}\varphi^{-1}_{\infty,W_\Phi^0}, \quad n \to \infty. \tag{4.13}$$

4.2.2 The case of $\lambda_{[0,\delta]}\varphi^{-1}_{\infty,\zeta_n}$

We study in this section (4.4), for which we shall still use the almost sure uniform convergence $\zeta_n \to W_\Phi^0$, obtained from the Skorokhod's representation theorem. The functions

$$\varphi_{n,\zeta_n}(c) = \sup_{t\in\mathbb{R}} G_{n,c}\zeta_n(t), \qquad \varphi_{\infty,W_\Phi^0}(c) = \sup_{t\in\mathbb{R}} G_{\infty,c}W_\Phi^0(t)$$

are absolutely continuous, with derivatives given by (4.8) and (4.10) respectively. The case of $\varphi_{\infty,W_\Phi^0}$ has been derived in Section 4.2.1.

For φ_{n,ζ_n}, observe first that φ_{n,ζ_n} can be made explicit on the subset where argmax $G_{n,c}\zeta_n$ is identified: that is

$$\varphi_{n,\zeta_n}(c) = \sum_{i=1}^{n}\left[\frac{i}{\sqrt{n}} - \sqrt{n}\Phi(\xi_i^n + c/\sqrt{n})\right]\mathbf{1}_{A_i}(c), \tag{4.14}$$

with $A_i := \{c \in [0,\delta] \mid \text{argmax}\ G_{n,c}\zeta_n = \xi_i^n + c/\sqrt{n}\}$. Since for $c \in A_i$ and d near enough c, $G_{n,d}$ reaches also its maximum at $\xi_i^n + d/\sqrt{n}$, we have $d \in A_i$, so that A_i is an open set and is thus a countable union of open intervals. The union is in fact finite as the following argument shows. Note that

$$G_{n,c}\zeta_n(\xi_i^n + c/\sqrt{n}) > G_{n,c}\zeta_n(\xi_j^n + c/\sqrt{n}) \tag{4.15}$$

holds *iff*

$$\Phi(\xi_i^n + \frac{c}{\sqrt{n}}) - \Phi(\xi_j^n + \frac{c}{\sqrt{n}}) < \frac{i-j}{n}.$$

But the function $c \mapsto \Phi(\xi_i^n + \frac{c}{\sqrt{n}}) - \Phi(\xi_j^n + \frac{c}{\sqrt{n}})$ is non-decreasing (resp. non-increasing) for $c \geq -\sqrt{n}\frac{\xi_i^n+\xi_j^n}{2}$ and non-increasing (resp. non-decreasing) for $c \leq -\sqrt{n}\frac{\xi_i^n+\xi_j^n}{2}$ when $i < j$ (resp. $i > j$). From this elementary study, it follows:

- when $i < j$ and (4.15) holds for c_1 and c_2, it holds also for any $c \in [c_1, c_2]$.
- when $i > j$ and (4.15) holds for c_1 and c_2, then it holds for any $c \in [c_1, c_2] \setminus I_j$ where I_j is some subinterval that may be empty.

The finiteness of the open union of A_i comes as follows: c_1, c_2 are in A_i *iff* (4.15) holds at c_1, c_2 for any $j \in \{1, \dots, n\} \setminus \{i\}$. But from the previous argument, (4.15) still holds for any $j \in \{1, \dots, n\} \setminus \{i\}$ and for any $c \in [c_1, c_2] \setminus (I_1 \cup \dots \cup I_n)$. This is possible only if the open union of A_i is finite. From now on, we write this union $A_i = \cup_{j=1}^{p_i} I_{i,j}$.

Then,

- since $[0, \delta]$ can be splited as follows

$$[0, \delta] = \overline{\cup_{i=1}^{n} A_i} = \overline{\bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p_i}} I_{i,j}} = \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p_i}} \overline{I_{i,j}},$$

- since the restrictions of φ_{n,ζ_n} on those intervals are absolutely continuous, see expression (4.14),
- and since the values of φ_{n,ζ_n} coincide on the common frontier of two intervals $I_{i,j}$,

the absolute continuity of φ_{n,ζ_n} comes from the following elementary lemma

Lemma 2 *Let $a_1 < \dots < a_p$, and $f : [a_1, a_p] \to \mathbb{R}$ such that $f|_{[a_i, a_{i+1}]} = f_i$, where f_i is an absolutely continuous function on $[a_i, a_{i+1}]$ with derivatives g_i and $f_i(a_{i+1}) = f_{i+1}(a_{i+1})$. Then f is absolutely continous on $[a_1, a_p]$*

$$f(x) = f(a_1) + \int_{a_1}^{x} g(t)\, dt$$

with derivative $g(x) = \sum_{i=1}^{n-1} g_i(x) \mathbf{1}_{[a_i, a_{i+1}]}(x)$.

Since the last hypothesis of Proposition 1 can not be obtained (at least easily) for the function φ_{n,ζ_n} of (4.4), we use the following version of Prop. 1, whose weaker last hypothesis can be satisfied for φ_{n,ζ_n}. The proof of this proposition can be found in [BD, p. 44–45].

Proposition 2 *Let for $n \in \mathbb{N} \cup \{\infty\}$, $f_n : (\Omega \times [0, \delta], \mathcal{F} \times \mathcal{B}([0, \delta]), \mathbb{P} \otimes \lambda) \longrightarrow \mathbb{R}$, $\Omega^* \in \mathcal{F}, \Omega^* \subset \Omega$ be such that*

1. *$\forall \omega \in \Omega^*$, $\exists N_1(\omega)$, $\forall n \geq N_1(\omega)$, $f_n(\omega, \cdot)$ is absolutely continuous;*
2. *$f_n(\omega, 0) \xrightarrow{\mathbb{P}} f_\infty(\omega, 0)$ on Ω^*;*
3. *$f_n(\omega, \delta) \xrightarrow{\mathbb{P}} f_\infty(\omega, \delta)$ on Ω^*;*
4. *$\forall \omega \in \Omega^*$, $\exists N_4(\omega)$, $\forall n \geq N_4(\omega)$, $\frac{\partial}{\partial c} f_n(\omega, c) > 0$ λ-a.e. for $c \in (0, \delta)$;*

5. $\frac{\partial}{\partial c} f_n(\omega, c) \xrightarrow{\mathbb{P}\otimes\bar{\lambda}} \frac{\partial}{\partial c} f_\infty(\omega, c)$ *on* Ω^*.

Then on Ω^*

$$\|\lambda_{[0,\delta]} f_n(\omega, \cdot)^{-1} - \lambda_{[0,\delta]} f_\infty(\omega, \cdot)^{-1}\| \xrightarrow{\mathbb{P}} 0.$$

First for $c = 0$, $\varphi_{n,\zeta_n}(0) = \sup_{t\in\mathbb{R}} \zeta_n(t)$ and $\varphi_{\infty,W_\Phi^0}(0) = \sup_{t\in\mathbb{R}} W_\Phi^0(t)$. Since, $\zeta_n \to W_\Phi^0$ uniformly, we still have $\varphi_{n,\zeta_n}(0) \longrightarrow \varphi_{\infty,W_\Phi^0}(0)$.

For the point 3, we dispose of $\zeta_n \longrightarrow W_\Phi^0$ uniformly and of point (i) of Theorem A, yet justified in Section 1:

$$\mathbb{P}\text{-}\lim_{n\to\infty} d_{0,\mathbb{R}}(G_{n,c}\zeta_n, G_{\infty,c}\zeta_n) = 0, \quad \forall c.$$

Since

$$\begin{aligned} d_{0,\mathbb{R}}(G_{n,c}\zeta_n, G_{\infty,c}W_\Phi^0) &\leq d_{0,\mathbb{R}}(G_{n,c}\zeta_n, G_{\infty,c}\zeta_n) + d_{0,\mathbb{R}}(G_{\infty,c}\zeta_n, G_{\infty,c}W_\Phi^0) \\ &\leq d_{0,\mathbb{R}}(G_{n,c}\zeta_n, G_{\infty,c}\zeta_n) + \|\zeta_n - W_\Phi^0\|_\infty, \end{aligned}$$

we derive

$$\mathbb{P}\text{-}\lim_{n\to\infty} d_{0,\mathbb{R}}(G_{n,c}\zeta_n, G_{\infty,c}W_\Phi^0) = 0 \tag{4.16}$$

and we can obtain the same for the uniform norm as follows: first, remember by definition of the Skorokhod metric $d_{0,\mathbb{R}}$ on $\mathbb{R}$ that:

$$d_{0,\mathbb{R}}(G_{n,c}\zeta_n, G_{\infty,c}W_\Phi^0) = d_0(G_{n,c}\zeta_n \circ \Psi, G_{\infty,c}W_\Phi^0 \circ \Psi), \tag{4.17}$$

where d_0 stands for the standard Skorokhod metric on $\mathbb{D}([0,1])$. Note that $G_{\infty,c}W_\Phi^0 \circ \Psi = W_U^0 - c\Phi' \circ \Psi$ where W_U^0 is a standard Brownian bridge (on $[0,1]$) and $\Phi' \circ \Psi$ can be extended as a continuous function on $[0,1]$, with 0 on the edges. So that $G_{\infty,c}W_\Phi^0 \circ \Psi$ is an uniformly continuous function. By definition of d_0, choose now $\lambda_n \in \Lambda([0,1])$ such that

$$\begin{aligned} \|G_{n,c}\zeta_n \circ \Psi &- G_{\infty,c}W_\Phi^0 \circ \Psi \circ \lambda_n\|_{\infty,[0,1]} + \|\lambda_n - id\|_{\infty,[0,1]} \\ &\leq 2d_0(G_{n,c}\zeta_n \circ \Psi, G_{\infty,c}W_\Phi^0 \circ \Psi) \end{aligned}$$

and derive for any $t \in [0,1]$:

$$\begin{aligned} &|G_{n,c}\zeta_n \circ \Psi(t) - G_{\infty,c}W_\Phi^0 \circ \Psi(t)| \\ &\leq |G_{n,c}\zeta_n \circ \Psi(t) - G_{\infty,c}W_\Phi^0 \circ \Psi(\lambda_n t)| + |G_{\infty,c}W_\Phi^0 \circ \Psi(\lambda_n t) - G_{\infty,c}W_\Phi^0 \circ \Psi(t)| \\ &\leq 2d_n + w_{|\lambda_n t - t|}(G_{\infty,c}W_\Phi^0 \circ \Psi) \end{aligned}$$

where we note d_n for $d_0(G_{n,c}\zeta_n \circ \Psi, G_{\infty,c}W_\Phi^0 \circ \Psi)$ and where $w_\delta(x)$ stands for the module of (uniform) continuity of a function $x \in \mathcal{C}(\mathbb{R})$. We obtain

$$\|G_{n,c}\zeta_n \circ \Psi - G_{\infty,c}W_\Phi^0 \circ \Psi\|_{\infty,[0,1]} \leq 2d_n + w_{2d_n}(G_{\infty,c}W_\Phi^0 \circ \Psi). \tag{4.18}$$

Now since $G_{\infty,c}W_\Phi^0 \circ \Psi$ is unifomly continuous, for all $\varepsilon > 0$, there is $\alpha > 0$ such that $0 < \delta \leq \alpha$ implies $w_\delta(G_{\infty,c}W_\Phi^0 \circ \Psi) \leq \varepsilon$, so that

$$\mathbb{P}\big\{w_{2d_n}(G_{\infty,c}W_\Phi^0 \circ \Psi) > \varepsilon\big\} \leq \mathbb{P}\{d_n > \alpha/2\}$$

and $w_{2d_n}(G_{\infty,c}W^0_\Phi \circ \Psi)$ converges to 0 in probability. We deduce finally from (4.16) and (4.18)

$$\|G_{n,c}\zeta_n - G_{\infty,c}W^0_\Phi\|_{\infty,\mathbb{R}} = \|G_{n,c}\zeta_n \circ \Psi - G_{\infty,c}W^0_\Phi \circ \Psi\|_{\infty,[0,1]} \xrightarrow{\mathbb{P}} 0, \quad n \to \infty$$

that is for all $\varepsilon > 0$,

$$\mathbb{P}\{\|G_{n,c}\zeta_n - G_{\infty,c}W^0_\Phi\|_{\infty,\mathbb{R}} \geq \varepsilon\} \longrightarrow 0, \quad n \to \infty$$

from which point 3 is now derived.

The point 4 follows from the expression (4.8) of φ'_{n,ζ_n} ensuring the non-degeneracy a.e. of the derivatives when $\omega \in \Omega(x)$.

Next, to see the convergence of the derivatives, we establish first the convergence

$$\text{argmax } (G_{n,c}\zeta_n) \longrightarrow \text{argmax } (G_{\infty,c}W^0_\Phi), \quad n \to \infty.$$

But, we have just derived $\mathbb{P}\text{-}\lim_{n\to\infty} \|G_{n,c}\zeta_n - G_{\infty,c}W^0_\Phi\|_\infty = 0$. Since both

$$\begin{aligned} G_{\infty,c}W^0_\Phi(t) &= W^0_\Phi(t) - c\Phi'(t) \\ G_{n,c}\zeta_n(t) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \left[\mathbf{1}_{]-\infty,t]}(\xi_i + c/\sqrt{n}) - \Phi(t)\right] \end{aligned}$$

are bimeasurable as functions of (ω, c), Fubini Theorem yields

$$(\lambda \otimes \mathbb{P})\ \{\|G_{n,c}\zeta_n - G_{\infty,c}W^0_\Phi\|_\infty \geq \varepsilon\} \longrightarrow 0, \quad n \to \infty.$$

Lemma 3 *Let X_n and X be B-valued random variables, where B is a Polish space. If $X_n \xrightarrow{\mathbb{P}} X$ in B and $f : B \longrightarrow \mathbb{R}$ is continuous. Then $f(X_n) \xrightarrow{\mathbb{P}} f(X)$.*

¿From Lemmas 1 and 3, we derive

$$\text{argmax } G_{n,c}\zeta_n \xrightarrow{\lambda\otimes\mathbb{P}} \text{argmax } G_{\infty,c}W^0_\Phi,$$

and since Φ' is also continuous, we have also

$$\Phi'(\text{argmax } G_{n,c}\zeta_n) \xrightarrow{\lambda\otimes\mathbb{P}} \Phi'(\text{argmax } G_{\infty,c}W^0_\Phi),$$

that is $\varphi'_{n,\zeta_n} \longrightarrow \varphi'_{n,W^0_\Phi}$ in measure $\lambda \otimes \mathbb{P}$.

Proposition 2 finally applies and yields a weakened version of (4.4), that is

$$\|\lambda_{[0,\delta]}\varphi^{-1}_{n,\zeta_n} - \lambda_{[0,\delta]}\varphi^{-1}_{\infty,W^0_\Phi}\| \xrightarrow{\mathbb{P}_{\Omega(x)}} 0, \tag{4.19}$$

where $\mathbb{P}_{\Omega(x)}$ stands for the restriction of $\mathbb{P}$ on $\Omega(x)$.

4.3 Conclusion for point (iv)

Gathering convergences (4.13), (4.19), we achieve deriving point (iv). First, by dominated convergence, it is an easy matter to derive from (4.13)

$$E\big[\|\lambda\varphi^{-1}_{\infty,\zeta_n} - \lambda\varphi^{-1}_{\infty,W^0_\Phi}\|\mathbf{1}_{\Omega(x)}\big] \to 0, \quad n \to \infty.$$

Second, the weakened convergence (4.19) yields:

$$\begin{aligned} &E\big[\|\lambda\varphi^{-1}_{n,\zeta_n} - \lambda\varphi^{-1}_{\infty,W^0_\Phi}\|\mathbf{1}_{\Omega(x)}\big] \\ &\quad\le E\big[\|\lambda\varphi^{-1}_{n,\zeta_n} - \lambda\varphi^{-1}_{\infty,W^0_\Phi}\|\mathbf{1}_{\{\|\lambda\varphi^{-1}_{n,\zeta_n} - \lambda\varphi^{-1}_{\infty,W^0_\Phi}\|\le\varepsilon\}}\mathbf{1}_{\Omega(x)}\big] \\ &\qquad\qquad + E\big[\|\lambda\varphi^{-1}_{n,\zeta_n} - \lambda\varphi^{-1}_{\infty,W^0_\Phi}\|\mathbf{1}_{\{\|\lambda\varphi^{-1}_{n,\zeta_n} - \lambda\varphi^{-1}_{\infty,W^0_\Phi}\|>\varepsilon\}}\mathbf{1}_{\Omega(x)}\big] \\ &\quad\le \varepsilon + 2\mathbb{P}\{\omega\in\Omega(x) \mid \|\lambda\varphi^{-1}_{n,\zeta_n} - \lambda\varphi^{-1}_{\infty,W^0_\Phi}\| > \varepsilon\}. \end{aligned}$$

So that from (4.19),

$$\overline{\lim}_n E\big[\|\lambda\varphi^{-1}_{n,\zeta_n} - \lambda\varphi^{-1}_{\infty,W^0_\Phi}\|\mathbf{1}_{\Omega(x)}\big] \le \varepsilon,$$

then, letting ε go to 0, we get $\lim_{n\to\infty} E\big[\|\lambda\varphi^{-1}_{n,\zeta_n} - \lambda\varphi^{-1}_{\infty,W^0_\Phi}\|\mathbf{1}_{\Omega(x)}\big] = 0$. Finally,

$$\begin{aligned} &\overline{\lim}_n E\big[\|\lambda\varphi^{-1}_{n,\zeta_n} - \lambda\varphi^{-1}_{\infty,\zeta_n}\|\mathbf{1}_{\Omega(x)}\big] \\ &\quad\le \overline{\lim}_n E\big[\|\lambda\varphi^{-1}_{n,\zeta_n} - \lambda\varphi^{-1}_{\infty,\zeta_n}\|\mathbf{1}_{\Omega(x)}\big] + \overline{\lim}_n E\big[\|\lambda\varphi^{-1}_{\infty,\zeta_n} - \lambda\varphi^{-1}_{\infty,W^0_\Phi}\|\mathbf{1}_{\Omega(x)}\big] \\ &\quad= 0. \end{aligned}$$

Condition (iv) of Theorem A is finally fulfilled.

5 Point (v) of Theorem A

The purpose of this section is to derive the continuity of $z \in V := V(x) \mapsto \lambda_{[0,\delta]}\varphi^{-1}_{\infty,z}$ P_∞-almost everywhere. Once more, for $z_n \to z$, we intend to apply Proposition 1 to

$$\varphi_{\infty,z_n}(c) = \sup_{t\in\mathbb{R}}\left(z_n(t) - c\Phi'(t)\right), \quad \varphi_{\infty,z}(c) = \sup_{t\in\mathbb{R}}\left(z(t) - c\Phi'(t)\right).$$

To this way, first of all, we have to choose z_n and z such that for almost all c, #argmax $(z - c\Phi')$ = #argmax $(z - c\Phi') = 1$ to compute as in Section 4.1.3, the derivatives

$$\varphi'_{\infty,z_n}(c) = -\Phi'(\text{argmax }(z_n - c\Phi')), \quad \varphi'_{\infty,z}(c) = -\Phi'(\text{argmax }(z - c\Phi')) \tag{5.1}$$

and to be able to apply either Prop. 1 (or at least Prop. 2) to derive convergence in variation of related image measures. Such a condition for z_n can not be ensured. Anyway, we can remove this condition replacing (v) by a weaker condition (v'):

"the application $z \in \mathcal{X}_0 \mapsto \lambda\varphi^{-1}_{\infty,z}$ is continuous where $\mathcal{X}_0$ is a measurable subset of $\mathcal{X}$ satisfying $P_n(\mathcal{X}_0) = P_\infty(\mathcal{X}_0) = 1$".

Indeed, a carefull reading of the proof of Theorem A ([DLS, Th. 18.4]) shows that (v) is used to derive condition (ii) in Theorem 18.3 necessary to apply the following result to the P_∞-continuous function $z \mapsto \lambda\varphi^{-1}_{\infty,z}$.

Lemma 4 ([DLS, Th. 2.3]) *Let h be a measurable mapping from a separable space $\mathcal{X}$ to a Banach space $(B, \|\cdot\|)$. We assume that h is bounded with respect to the norm and continuous P_∞-almost everywhere. If $P_n \Rightarrow P_\infty$, then*

$$\left\| \int_{\mathcal{X}} hdP_n - \int_{\mathcal{X}} hdP_\infty \right\| \to 0, \quad n \to \infty.$$

But the same holds true if for a measurable subset $\mathcal{X}_0 \subset \mathcal{X}$, with $P_n(\mathcal{X}_0) = P_\infty(\mathcal{X}_0) = 1$, the restriction $h_{|\mathcal{X}_0}$ is continuous: first, the restricted probabilities over $\mathcal{X}_0$ weakly converge: $P_{n|\mathcal{X}_0} \Rightarrow P_{\infty|\mathcal{X}_0}$, since for a closed subset F of $\mathcal{X}$:

$$\begin{aligned} \overline{\lim}_n P_{n|\mathcal{X}_0}(F) &= \overline{\lim}_n P_n(F \cap \mathcal{X}_0) = \overline{\lim}_n P_n(F) \\ &\leq P_\infty(F) = P_\infty(F \cap \mathcal{X}_0) = P_{\infty|\mathcal{X}_0}(F). \end{aligned}$$

Then, lemma 4 applies to the restricted measures and yields

$$\int hdP_{n|\mathcal{X}_0} \xrightarrow{B} \int hdP_{\infty|\mathcal{X}_0}, \quad n \to +\infty.$$

But since

$$\int hdP_{n|\mathcal{X}_0} = \int_{\mathcal{X}_0} hdP_n = \int hdP_n \text{ and } \int hdP_{\infty|\mathcal{X}_0} = \int_{\mathcal{X}_0} hdP_\infty = \int hdP_\infty,$$

the conclusion of Lemma 4 still holds in this setting. Finally, Theorem A holds also with (v) replaced by (v') and it is enough to satisfy (v').

Take $\mathcal{X}_0$ to be the subset of function in $\mathbb{D}(\mathbb{R})$ whose supremum is reached only once. We have $P_n(\mathcal{X}_0) = P_\infty(\mathcal{X}_0) = 1$ and we derive (v') applying Proposition 1:

- Since $z_n \to z$ in $\mathbb{D}(\mathbb{R})$, the first point is easily satisfied.
- In order to obtain the convergence of the derivatives (5.1) in $L^1([0,\delta])$, it is enough to prove for c such that $\#\text{argmax } \{z - c\Phi'\} = 1$, that

  $$\text{argmax } (z_n - c\Phi') \to \text{argmax } (z - c\Phi').$$

  But since z can be taken continuous and z_n converge to z in $\mathbb{D}$, the convergence holds also uniformly and the convergence of relative argmax follows from Lemma 1 since the asympotic argmax is unique.
- $\varphi'_{\infty,z}(c)$ is given by (5.1) and is thus non zero since Φ' vanishes only at $\pm\infty$ whereas argmax $(z - c\Phi')$ is necessarily finite when $z \in V(x)$.

Finally Proposition 1 applies and yields

$$\lambda_{[0,\delta]}\varphi^{-1}_{\infty,z_n} \xrightarrow{var} \lambda_{[0,\delta]}\varphi^{-1}_{\infty,z}$$

whenever $z_n \to z$ in $\mathcal{X}_0$.

Point (v') is thus also satisfied.

Finally, Theorem A applies and proves Theorem 1.

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