



## Prépublications du Département de Mathématiques

Université de La Rochelle  
Avenue Michel Crépeau  
17042 La Rochelle Cedex 1  
<http://www.univ-lr.fr/lab0/lmca/>

The Nehari manifold for systems of nonlinear elliptic equations

Khalid Adriouch et Abdallah El Hamidi

Juin 2005

**Classification:** 35J55; 35J60.

**Mots clés:** Nehari manifold, Ekeland's principle,  $(p, q)$ -Laplacian, Palais-Smale

2005/06

# The Nehari manifold for systems of nonlinear elliptic equations

K. Adriouch and A. El Hamidi

**Keywords.** Nehari manifold, Ekeland's principle,  $(p, q)$ -Laplacian, Palais-Smale

**Mathematics subject Classification.** 35J55; 35J60

## Abstract

This paper deals with existence and multiplicity results of nonlocal positive solutions to the following system

$$\begin{cases} -\Delta_p u = \lambda|u|^{p_1-2}u + (\alpha+1)u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v = \mu|v|^{q-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v, \end{cases}$$

together with Dirichlet or mixed boundary conditions, under some hypotheses on the parameters  $p$ ,  $p_1$ ,  $\alpha$ ,  $\beta$  and  $q$ . More precisely, the system considered corresponds to a perturbed eigenvalue equation combined with a second equation having concave and convex nonlinearities. The study is based on the extraction of Palais-Smale sequences in the Nehari manifold. The behaviour of the energy corresponding to these positive solutions, with respect to the real parameters  $\lambda$  and  $\mu$ , is established.

## 1 Introduction

In this work, we consider the system of quasilinear elliptic equations

$$\begin{cases} -\Delta_p u = \lambda|u|^{p_1-2}u + (\alpha+1)u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v = \mu|v|^{q-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v, \end{cases} \quad (1)$$

together with Dirichlet or mixed boundary conditions

$$\begin{cases} u|_{\Gamma_1} = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu}|_{\Sigma_1} = 0, \\ v|_{\Gamma_2} = 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}|_{\Sigma_2} = 0, \end{cases} \quad (2)$$

where,  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ , with smooth boundary  $\partial\Omega = \overline{\Gamma}_1 \cap \overline{\Sigma}_1$ , where  $\Gamma_i$  are smooth  $(N-1)$ -dimensional submanifolds of  $\partial\Omega$  with positive measures such that  $\Gamma_i \cap \Sigma_i = \emptyset$ ,  $i \in \{1, 2\}$ .  $\Delta_p$  is the  $p$ -Laplacian and  $\frac{\partial}{\partial \nu}$  is the outer normal derivative. It is clear that when  $\Gamma_1 = \Gamma_2 = \partial\Omega$ , one deals with homogeneous Dirichlet boundary conditions.

Our aim here is to establish nonlocal existence and multiplicity results, with respect to the real parameters  $\lambda$  and  $\mu$ , for Problem (1). Along this work, the following assumptions will hold

$$1 < p_1 < p < N, \quad q > 1, \quad \alpha > 1, \quad \beta > 1, \quad (3)$$

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1, \quad (4)$$

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1 \quad \text{and} \quad \frac{\beta+1}{q} < 1, \quad (5)$$

where

$$p^* = \frac{Np}{N-p}, \quad q^* = \frac{Nq}{N-q}$$

are the critical exponents for the  $p$ -Laplacian and  $q$ -Laplacian respectively. These assumptions mean that we are concerned with a subcritical and super-homogeneous system where the first equation is *concave-convex* and the second equation is only a perturbation of an eigenvalue equation. Also, the following assumptions concerning the real parameters  $\lambda$  and  $\mu$  will hold

$$\lambda > 0, \quad \mu < \mu_1,$$

where  $\mu_1$  is the first eigenvalue of  $-\Delta_q$  in  $\Omega$ .

Problem (1), together with (2), is posed in the framework of the Sobolev space  $W = W_{\Gamma_1}^{1,p}(\Omega) \times W_{\Gamma_2}^{1,q}(\Omega)$ , where

$$W_{\Gamma_1}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u|_{\Gamma_1} = 0\}, \quad W_{\Gamma_2}^{1,q}(\Omega) = \{u \in W^{1,q}(\Omega) : u|_{\Gamma_2} = 0\},$$

are respectively the closure of  $C_0^1(\Omega \cap \Gamma_1, \mathbb{R})$  with respect to the norm of  $W^{1,p}(\Omega)$  and  $C_0^1(\Omega \cap \Gamma_2, \mathbb{R})$  with respect to the norm of  $W^{1,q}(\Omega)$ . We can refer the reader to [9] for a complete description of this space in the case  $p = 2$ . Notice that  $\text{meas}(\Gamma_i) > 0$ ,  $i = 1, 2$ , imply that the Poincaré inequality is still available in  $W_{\Gamma_1}^{1,p}(\Omega)$  and  $W_{\Gamma_2}^{1,q}(\Omega)$ , so  $W$  can be endowed with the norm

$$\|(u, v)\| = \|\nabla u\|_p + \|\nabla v\|_q$$

and  $(W, \|\cdot\|)$  is a reflexive and separable Banach space.

Semilinear and quasilinear scalar elliptic equations with concave and convex nonlinearities are widely studied, we can refer the reader to the survey article [5]. For the nonlinear elliptic

systems, we refer to the survey article [13]. In [15], the authors studied the existence of positive solutions to a perturbed eigenvalue problem involving the  $p$ -Laplacian operator. In [6], the authors have generalized the results of [15] to a perturbed eigenvalue system involving  $p$  and  $q$ -Laplacian operators. Recently, in [10] the first author has considered a semilinear elliptic equation with concave and convex nonlinearities, and showed *nonlocal* existence and multiplicity results with respect to the parameter via the extraction of Palais-Smale sequences in the Nehari manifold.

In this paper, we extend this method to the system (1) where one equation contains concave and convex nonlinearities and the other one is simply a perturbation of an eigenvalue equation. We show that Problem (1) has at least two positive solutions when the pair of parameters  $(\lambda, \mu)$  belongs to a subset of  $\mathbb{R}^2$  which will be specified below.

For solutions of (1) we understand critical points of the Euler-Lagrange functional  $I \in C^1(W, \mathbb{R})$  given by

$$I(u, v) = \frac{1}{p}P(u) - \frac{\lambda}{p_1}P_1(u) + \frac{1}{q}(Q(v) - \mu Q_1(v)) - R(u, v),$$

where  $P(u) = \|\nabla u\|_p^p$ ,  $P_1(u) = \|u\|_{p_1}^{p_1}$ ,  $Q(v) = \|\nabla v\|_q^q$ ,  $Q_1(v) = \|v\|_q^q$  and  $R(u, v) = \int_{\Omega} |u|^{\alpha+1}|v|^{\beta+1}dx$ .

Consider the "Nehari" manifold associated to Problem (1) given by

$$\mathcal{N} = \{(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}) / D_1 I(u, v)(u) = D_2 I(u, v)(v) = 0\},$$

where  $D_1 I$  and  $D_2 I$  are the derivatives of  $I$  with respect to the first variable and the second variable respectively.

An interesting and useful characterization of  $\mathcal{N}$ , [15, 19, 24, 10, 7] is the following

$$\mathcal{N} = \{(su, tv) / (s, u, t, v) \in \mathcal{Z}^* \text{ and } \partial_s I(su, tv) = \partial_t I(su, tv) = 0\},$$

where  $\mathcal{Z}^* = (\mathbb{R} \setminus \{0\}) \times (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$  and  $I$  is considered as a functional of four variables  $(s, u, t, v)$  in  $\mathcal{Z} := \mathbb{R} \times W_{\Gamma_1}^{1,p}(\Omega) \times \mathbb{R} \times W_{\Gamma_2}^{1,q}(\Omega)$ . For this reason, we introduce the modified Euler-Lagrange functional  $\tilde{I}$  defined on  $\mathcal{Z}$  by

$$\tilde{I}(s, u, t, v) := I(su, tv).$$

## 2 Preliminary results

In this work, we are interested by nontrivial positive solutions  $u \neq 0$  and  $v \neq 0$  to Problem (1). Since the functional  $\tilde{I}$  is even in  $s$  and  $t$ , we limit our study for  $s > 0$ ,  $t > 0$  and for  $(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$ .

**Lemma 1** For every  $(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$  there exists a unique  $\lambda(u, v) > 0$  such that the real-valued function  $(s, t) \in (0, +\infty)^2 \mapsto \tilde{I}(s, u, t, v)$  has exactly two critical points (resp. one critical point) for  $0 < \lambda < \lambda(u, v)$  (resp.  $\lambda = \lambda(u, v)$ ). This functional has no critical point for  $\lambda > \lambda(u, v)$ .

**Proof.** Let  $(u, v)$  be an arbitrary element in  $(W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$ . Then

$$\tilde{I}(s, u, t, v) = \frac{s^p}{p} P(u) - \frac{\lambda}{p_1} s^{p_1} P_1(u) + \frac{t^q}{q} (Q(v) - \mu Q_1(v)) - s^{\alpha+1} t^{\beta+1} R(u, v).$$

A direct computation gives  $\partial_t \tilde{I}(s, u, t, v) = 0$  if and only if

$$t = t(s) = \left[ (\beta + 1) \frac{R(u, v)}{Q(v) - \mu Q_1(v)} \right]^{\frac{1}{q-(\beta+1)}} s^{\frac{\alpha+1}{q-(\beta+1)}}, \quad (6)$$

and

$$\tilde{I}(s, u, t(s), v) = \frac{s^p}{p} P(u) - \frac{\lambda}{p_1} s^{p_1} P_1(u) - \frac{s^r}{r} A(u, v),$$

where

$$A(u, v) = (\alpha + 1)(\beta + 1)^{\frac{\alpha+1}{q-(\beta+1)}} \frac{R(u, v)^{\frac{q}{q-(\beta+1)}}}{(Q(v) - \mu Q_1(v))^{\frac{\beta+1}{q-(\beta+1)}}}$$

and  $r = \frac{(\alpha+1)q}{q-(\beta+1)}$ . It is easy to verify that  $r > p$ . Now consider the function  $s \in (0, +\infty) \mapsto \tilde{I}(s, u, t(s), v)$  and let us write

$$\partial_s \tilde{I}(s, u, t(s), v) := s^{p_1-1} F_{\lambda, \mu}(s, u, v).$$

where  $F_{\lambda, \mu}(s, u, v) := P(u)s^{p-p_1} - \lambda P_1(u) - A(u, v)s^{r-p_1}$ . The function  $s \in (0, +\infty) \mapsto F_{\lambda, \mu}(s, u, v)$  is increasing on  $(0, \bar{s}_\mu(u, v))$ , decreasing on  $(\bar{s}_\mu(u, v), +\infty)$  and attains its unique maximum for  $s = \bar{s}_\mu(u, v)$ , where

$$\bar{s}_\mu(u, v) = \left[ \frac{p-p_1}{r-p_1} \frac{P(u)}{A(u, v)} \right]^{\frac{1}{r-p}}. \quad (7)$$

So, the function  $s \in (0, +\infty) \mapsto F_{\lambda, \mu}(s, u, v)$  has two positive zeros (resp. one positive zero) if  $F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) > 0$  (resp.  $F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) = 0$ ) and has no zero if  $F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) < 0$ .

On the other hand, a direct computation leads to

$$F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) = \frac{r-p}{r-p_1} \left[ \frac{p-p_1}{r-p_1} \frac{P(u)}{A(u, v)} \right]^{\frac{p-p_1}{r-p_1}} P(u) - \lambda P_1(u).$$

Then,  $F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) > 0$  (resp.  $F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) < 0$ ) if  $\lambda < \lambda(u, v)$  (resp.  $\lambda > \lambda(u, v)$ ) and  $F_{\lambda(u, v), \mu}(\bar{s}_\mu(u, v), u, v) = 0$ , where

$$\lambda(u, v) = \hat{c} \frac{P(u)^{\frac{r-p_1}{r-p}}}{P_1(u) A(u, v)^{\frac{p-p_1}{r-p}}} \quad \text{and} \quad \hat{c} = \frac{r-p}{r-p_1} \left[ \frac{p-p_1}{r-p_1} \right]^{\frac{p-p_1}{r-p}}. \quad (8)$$

Therefore, if  $\lambda \in (0, \lambda(u, v))$ , the function  $s \in (0, +\infty) \mapsto \partial_s \tilde{I}(s, u, t(s), v)$  has two positive zeros denoted by  $s_1(u, v, \lambda, \mu)$  and  $s_2(u, v, \lambda, \mu)$  verifying  $0 < s_1(u, v, \lambda, \mu) < \bar{s}_\mu(u, v) < s_2(u, v, \lambda, \mu)$ . Since  $F_{\lambda, \mu}(s_1(u, v, \lambda, \mu), u, v) = F_{\lambda, \mu}(s_2(u, v, \lambda, \mu), u, v) = 0$ ,  $\partial_s F_{\lambda, \mu}(s, u, v) > 0$  for  $0 < s < \bar{s}_\mu(u, v)$  and  $\partial_s F_{\lambda, \mu}(s, u, v) < 0$  for  $s > \bar{s}_\mu(u, v)$  it follows that

$$\partial_{ss} \tilde{I}(s_1(u, v, \lambda, \mu), u, t(s_1(u, v, \lambda, \mu)), v) > 0, \quad (9)$$

$$\partial_{ss} \tilde{I}(s_2(u, v, \lambda, \mu), u, t(s_2(u, v, \lambda, \mu)), v) < 0. \quad (10)$$

This implies that the real-valued function  $s \in (0, +\infty) \mapsto \tilde{I}(s, u, t(s), v)$  achieves its unique local minimum at  $s = s_1(u, v, \lambda, \mu)$  and its unique local maximum at  $s = s_2(u, v, \lambda, \mu)$ , which ends the proof.  $\square$

Hereafter, we will denote  $t_i(u, v, \lambda, \mu) := t(s_i(u, v, \lambda, \mu))$ ,  $i = 1, 2$ . At this stage, we introduce the characteristic value

$$\hat{\lambda}(\mu) := \inf \{ \lambda(u, v), (u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}) \}.$$

We claim that  $\hat{\lambda}(\mu)$  is great than a positive constant which depends only on  $\mu, p, p_1, q, \alpha, \beta$  and  $\Omega$ . Indeed, using the Hölder inequality, we get

$$R(u, v) \leq |\Omega|^\delta \|u\|_{p^*}^{\alpha+1} \|v\|_{q^*}^{\beta+1},$$

where  $\delta > 1$  is such that  $\frac{1}{p^*} + \frac{1}{q^*} + \frac{1}{\delta} = 1$ . Using the continuous embedding  $W_{\Gamma_2}^{1,q}(\Omega) \subset L^{q^*}(\Omega)$  we get

$$A(u, v) \leq c_1 \frac{P_*(u)^{\frac{r}{p^*}}}{(\mu_1 - \mu)^{\frac{\beta+1}{q-(\beta+1)}}},$$

where  $P_*(u) = \|u\|_{p^*}^{p^*}$  and  $c_1 = c_1(p, p_1, q, \alpha, \beta, \Omega)$ . Using again the continuous embeddings  $W_{\Gamma_1}^{1,p}(\Omega) \subset L^{p_1}(\Omega)$  and  $W_{\Gamma_1}^{1,p}(\Omega) \subset L^{p^*}(\Omega)$  we obtain

$$\lambda(u, v) \geq c_2(\mu_1 - \mu)^{\frac{\beta+1}{q-(\beta+1)} \frac{p-p_1}{r-p}},$$

where  $c_2 = c_2(p, p_1, q, \alpha, \beta, \Omega)$  and then

$$\hat{\lambda}(\mu) \geq c_2(\mu_1 - \mu)^{\frac{\beta+1}{q-(\beta+1)} \frac{p-p_1}{r-p}},$$

which achieves the claim. Now let us introduce

$$\mathcal{D} := \{(\lambda, \mu) \in (0, +\infty) \times (-\infty, \mu_1) : \lambda < \hat{\lambda}(\mu)\}.$$

For every  $(\lambda, \mu) \in \mathcal{D}$ , the functionals  $(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}) \mapsto \tilde{I}(s_i(u, v, \lambda, \mu), u, t_i(u, v, \lambda, \mu), v)$ ,  $i = 1, 2$ , are well defined and one can show easily that they are bounded below. Hence, for every

$(\lambda, \mu) \in \mathcal{D}$ , we define

$$\alpha_1(\lambda, \mu) := \inf\{\tilde{I}(s_1(u, v, \lambda, \mu), u, t_1(u, v, \lambda, \mu), v), (u, v) \in \tilde{W}\} \quad (11)$$

$$\alpha_2(\lambda, \mu) := \inf\{\tilde{I}(s_2(u, v, \lambda, \mu), u, t_2(u, v, \lambda, \mu), v), (u, v) \in \tilde{W}\} \quad (12)$$

where

$$\tilde{W} := (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}).$$

Our aim in the sequel is to show that  $\alpha_1(\lambda, \mu)$  and  $\alpha_2(\lambda, \mu)$  are in fact critical values of the Euler-Lagrange functional  $I$  for every  $(\lambda, \mu) \in \mathcal{D}$ . We start with the following

**Lemma 2** *Let  $(u_n, v_n) \in \tilde{W}$  be a minimizing sequence of (11) (resp. of (12)) and let  $(U_n^1, V_n^1) := (s_1(u_n, v_n, \lambda, \mu)u_n, t_1(u_n, v_n, \lambda, \mu)v_n)$  (resp.  $(U_n^2, V_n^2) := (s_2(u_n, v_n, \lambda, \mu)u_n, t_2(u_n, v_n, \lambda, \mu)v_n)$ ). Then it holds:*

- (i)  $\limsup_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| < \infty$  (resp.  $\limsup_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| < \infty$ ).
- (ii)  $\liminf_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| > 0$  (resp.  $\liminf_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| > 0$ ).

**Proof.** We show the assertion (i), let  $(u_n, v_n) \in \tilde{W}$  be a minimizing sequence of (11). Since  $\partial_s \tilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$  and

$\partial_t \tilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$ , it follows that

$$P(U_n^1) - \lambda P_1(U_n^1) - (\alpha + 1)R(U_n^1, V_n^1) = 0, \quad (13)$$

$$Q(V_n^1) - \mu Q_1(V_n^1) - (\beta + 1)R(U_n^1, V_n^1) = 0. \quad (14)$$

Suppose that there is a subsequence, still denoted by  $(U_n^1, V_n^1)$ , such that

$\lim_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| = \infty$ . We will distinguish three cases:

*Case a)*  $\lim_{n \rightarrow +\infty} \|\nabla U_n^1\|_p = \infty$  and  $\|\nabla V_n^1\|_q$  is bounded. By (14) we get that  $R(U_n^1, V_n^1)$  is bounded. On the other hand, using the continuous embedding  $W_{\Gamma_1}^{1,p}(\Omega) \subset L^{p_1}(\Omega)$ , we have  $P_1(U_n^1) = o_n(P(U_n^1))$ , as  $n$  goes to  $+\infty$ . By (13) we get  $R(U_n^1, V_n^1) = \frac{1}{\alpha+1}(1 + o_n(1))P(U_n^1)$  as  $n$  goes to  $+\infty$  and hence  $\lim_{n \rightarrow +\infty} R(U_n^1, V_n^1) = +\infty$ , which cannot hold true.

*Case b)*  $\lim_{n \rightarrow +\infty} \|\nabla V_n^1\|_q = \infty$  and  $\|\nabla U_n^1\|_p$  is bounded. By (13) we get  $R(U_n^1, V_n^1)$  bounded. If  $0 < \mu < \mu_1$ , using the Sobolev and Young inequalities, for every  $\varepsilon \in (0, 1)$ , there is a positive constant  $C_\varepsilon$  such that

$$\|V_n^1\|_q^q \leq \frac{\varepsilon}{\mu} \|\nabla V_n^1\|_q^q + C_\varepsilon,$$

which gives  $(\beta + 1)R(U_n^1, V_n^1) + \mu C_\varepsilon \geq (1 - \varepsilon)Q(V_n^1)$ . Then  $\lim_{n \rightarrow +\infty} R(U_n^1, V_n^1) = +\infty$ , which is impossible. If  $\mu < 0$ , then  $Q(V_n^1) - \mu Q_1(V_n^1) = (\beta + 1)R(U_n^1, V_n^1) \geq Q(V_n^1)$  so  $\lim_{n \rightarrow +\infty} R(U_n^1, V_n^1) = +\infty$ , which is also impossible.

*Case c)*  $\lim_{n \rightarrow +\infty} \|\nabla U_n^1\|_p = \lim_{n \rightarrow +\infty} \|\nabla V_n^1\|_q = \infty$ . As in the first case, we have

$$R(U_n^1, V_n^1) = \frac{1}{\alpha + 1}(1 + o_n(1))P(U_n^1), \text{ as } n \text{ goes to } +\infty.$$

Then  $I(U_n^1, V_n^1) = \frac{1}{\alpha + 1} \left( \frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1 + o_n(1) \right) P(U_n^1)$  as  $n$  goes to  $+\infty$ . Hence, using the hypothesis (5),  $\lim_{n \rightarrow +\infty} I(U_n^1, V_n^1) = +\infty$ , which is impossible. Consequently,  $\limsup_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| < \infty$ . We show in the same way that  $\limsup_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| < \infty$ .

Now, we show the assertion (ii), let  $(u_n, v_n) \in \widetilde{W}$  be a minimizing sequence of (11). Suppose that there is a subsequence, still denoted by  $(U_n^1, V_n^1)$ , such that  $\lim_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| = 0$ . By (13) we get  $\lim_{n \rightarrow +\infty} I(U_n^1, V_n^1) = 0$  and this can not hold true because  $I(U_n^1, V_n^1) < 0$  for every  $n$ .

Similarly, let  $(u_n, v_n) \in \widetilde{W}$  be a minimizing sequence of (12). Suppose that there is a subsequence, still denoted by  $(U_n^2, V_n^2)$ , such that

$\lim_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| = 0$ . If  $p > \alpha + 1$ , by (10), we have

$$\partial_{ss} I(U_n^2, V_n^2) = (p - 1)P(U_n^2) - \lambda(p_1 - 1)P_1(U_n^2) - \alpha(\alpha + 1)R(U_n^2, V_n^2) < 0$$

Then  $(p - 1)P(U_n^2) - \lambda(p - 1)P_1(U_n^2) - \alpha p R(U_n^2, V_n^2) < 0$ , which implies that  $(p - (\alpha + 1))R(U_n^2, V_n^2) < 0$  and this is impossible. Finally, if  $p \leq \alpha + 1$ , then  $(p - p_1)P(U_n^2) < (\alpha + 1)^2 R(U_n^2, V_n^2)$ . Since  $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$  and  $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$ , then there exist  $\tilde{p}$  and  $\tilde{q}$  satisfying  $p < \tilde{p} < p^*$ ,  $q < \tilde{q} < q^*$  and

$$\frac{\alpha + 1}{\tilde{p}} + \frac{\beta + 1}{\tilde{q}} = 1. \quad (15)$$

Therefore,

$$\begin{aligned} R(U_n^2, V_n^2) &\leq c(\Omega, p, q) \|U_n^2\|_{\tilde{p}}^{\alpha+1} \|V_n^2\|_{\tilde{q}}^{\beta+1} \\ &\leq c'(\Omega, p, q) \|\nabla U_n^2\|_p^{\alpha+1} \|\nabla V_n^2\|_q^{\beta+1} \end{aligned}$$

and consequently,  $(p - p_1) \leq c'(\Omega, p, q)(\alpha + 1)^2 \|\nabla U_n^2\|_p^{\alpha+1-p} \|\nabla V_n^2\|_q^{\beta+1}$  which converges to 0 as  $n$  goes to  $+\infty$ . This contradicts the fact  $p > p_1$ , which ends the proof.  $\square$

### 3 Palais-Smale sequences in the Nehari Manifold

It is interesting to notice that for every  $\gamma > 0$ ,  $\delta > 0$ , it holds

$$\begin{aligned}\tilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \tilde{I}(s, u, t, v), \\ \partial_t \tilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\delta} \partial_t \tilde{I}(s, u, t, v), \\ \partial_s \tilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\gamma} \partial_s \tilde{I}(s, u, t, v), \\ \partial_{ss} \tilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\gamma^2} \partial_{ss} \tilde{I}(s, u, t, v).\end{aligned}$$

This implies that

$$s_1(u, v, \lambda, \mu) = \frac{1}{\gamma} s_1\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall \delta > 0, \quad (16)$$

$$s_2(u, v, \lambda, \mu) = \frac{1}{\gamma} s_2\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall \delta > 0, \quad (17)$$

$$t_1(u, v, \lambda, \mu) = \frac{1}{\delta} t_1\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall \gamma > 0, \quad (18)$$

$$t_2(u, v, \lambda, \mu) = \frac{1}{\delta} t_2\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall \gamma > 0. \quad (19)$$

It follows that

$$\alpha_1(\lambda, \mu) = \inf_{(u, v) \in \mathbb{S}_p \times \mathbb{S}_q} \{\tilde{I}(s_1(u, v, \lambda, \mu), u, t_1(u, v, \lambda, \mu), v)\}, \quad (20)$$

$$\alpha_2(\lambda, \mu) = \inf_{(u, v) \in \mathbb{S}_p \times \mathbb{S}_q} \{\tilde{I}(s_2(u, v, \lambda, \mu), u, t_2(u, v, \lambda, \mu), v)\}, \quad (21)$$

where  $\mathbb{S}_p$  and  $\mathbb{S}_q$  are the unit spheres of  $W_{\Gamma_1}^{1,p}(\Omega)$  and  $W_{\Gamma_2}^{1,q}(\Omega)$  respectively. Make precise that  $\mathbb{S}_p \times \mathbb{S}_q$  is a 2-codimensional and complete submanifold of  $W$ , we will denote it in the sequel by  $\mathbb{S}$ .

**Lemma 3** *Let  $(\lambda, \mu) \in \mathcal{D}$  and let  $(u_n, v_n) \in \mathbb{S}$  be a minimizing sequence of (20) (resp. of (21)).*

*Then  $(s_1(u_n, v_n, \lambda, \mu)u_n, t_1(u_n, v_n, \lambda, \mu)v_n)$ ,*

*(resp.  $(s_2(u_n, v_n, \lambda, \mu)u_n, t_2(u_n, v_n, \lambda, \mu)v_n)$ ) is a Palais-Smale sequence for the functional  $I$ .*

**Proof.** Let  $(\lambda, \mu) \in \mathcal{D}$  and consider a minimizing sequence  $(u_n, v_n) \in \mathbb{S}$  of (20). Let us set

$$U_n = s_1(u_n, v_n, \lambda, \mu)u_n,$$

$$V_n = t_1(u_n, v_n, \lambda, \mu)v_n.$$

The sequence  $(U_n, V_n)$  is clearly bounded in  $W$ . On the other hand, the gradient (resp. the Hessian determinant) of  $\tilde{I}$  with respect to  $s$  and  $t$  at  $(s, t) = (s_1(u_n, v_n, \lambda, \mu), t_1(u_n, v_n, \lambda, \mu))$  is equal to

zero (resp. is strictly negative). So, the implicit function theorem implies that that  $s_1(u_n, v_n, \lambda, \mu)$  and  $t_1(u_n, v_n, \lambda, \mu)$  are  $C^1$  with respect to  $(u, v)$ , since  $\tilde{I}$  is.

We introduce now the functional  $\mathcal{I}$  defined on  $\mathbb{S}$  by

$$\mathcal{I}(u, v) = \tilde{I}(s_1(u, v, \lambda, \mu), u, t_1(u, v, \lambda, \mu), v),$$

then

$$\alpha_1(\lambda, \mu) = \inf_{(u, v) \in \mathbb{S}} \mathcal{I}(u, v) = \lim_{n \rightarrow +\infty} \mathcal{I}(u_n, v_n).$$

Applying the Ekeland variational principle on the complete manifold  $(\mathbb{S}, \|\cdot\|)$  to the functional  $\mathcal{I}$  we get

$$\mathcal{I}'(u_n, v_n)(\varphi_n, \psi_n) \leq \frac{1}{n} \|(\varphi_n, \psi_n)\|, \quad \forall (\varphi_n, \psi_n) \in T_{(u_n, v_n)} \mathbb{S},$$

where  $T_{(u_n, v_n)} \mathbb{S}$  denotes the tangent space to  $\mathbb{S}$  at the point  $(u_n, v_n)$ . Recall that  $T_{(u_n, v_n)} \mathbb{S} = T_{u_n} \mathbb{S}_p \times T_{v_n} \mathbb{S}_q$ , where  $T_{u_n} \mathbb{S}_p$  (resp.  $T_{v_n} \mathbb{S}_q$ ) is the tangent space to  $\mathbb{S}_p$  (resp.  $\mathbb{S}_q$ ) at the point  $u_n$  (resp.  $v_n$ ).

Set

$$A_n := (u_n, v_n, \lambda, \mu), \text{ and } B_n := (s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n).$$

For every  $(\varphi_n, \psi_n) \in T_{u_n} \mathbb{S}_p \times T_{v_n} \mathbb{S}_q$ , one has

$$\mathcal{I}'(u_n, v_n)(\varphi_n, \psi_n) = D_1 \tilde{I}(B_n)(\varphi_n) + D_2 \tilde{I}(B_n)(\psi_n)$$

where

$$\begin{aligned} D_1 \tilde{I}(B_n)(\varphi_n) &= \partial_u s_1(A_n)(\varphi_n) \partial_s \tilde{I}(B_n) + \partial_u \tilde{I}(B_n)(\varphi_n) + \partial_u t_1(A_n)(\varphi_n) \partial_t \tilde{I}(B_n) \\ &= \partial_u \tilde{I}(B_n)(\varphi_n). \end{aligned}$$

Similarly, one has

$$D_2 \tilde{I}(B_n)(\psi_n) = \partial_v \tilde{I}(B_n)(\psi_n).$$

Furthermore, consider the "fiber" maps

$$\begin{aligned} \pi : W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\} &\longrightarrow \mathbb{R} \times \mathbb{S}_p \\ u &\longmapsto \left( \|\nabla u\|_p, \frac{u}{\|\nabla u\|_p} \right) := (\pi_1(u), \pi_2(u)), \\ \tilde{\pi} : W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\} &\longrightarrow \mathbb{R} \times \mathbb{S}_q \\ v &\longmapsto \left( \|\nabla v\|_q, \frac{v}{\|\nabla v\|_q} \right) := (\tilde{\pi}_1(v), \tilde{\pi}_2(v)). \end{aligned}$$

Applying the Hölder inequality we get, for every  $(u, \varphi) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times W_{\Gamma_1}^{1,p}(\Omega)$  and  $(v, \psi) \in (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}) \times W_{\Gamma_2}^{1,q}(\Omega)$ , the following estimates

$$\begin{aligned} |\pi'_1(u)(\varphi)| &\leq \|\nabla \varphi\|_p, \quad |\pi'_2(u)(\varphi)| \leq 2 \frac{\|\nabla \varphi\|_p}{\|\nabla u\|_p}, \\ |\tilde{\pi}'_1(v)(\psi)| &\leq \|\nabla \psi\|_q, \quad |\tilde{\pi}'_2(v)(\psi)| \leq 2 \frac{\|\nabla \psi\|_q}{\|\nabla v\|_q}. \end{aligned}$$

On one hand, from Lemma (2), there is a positive constant  $K$  such that  $s_1(A_n) \geq K$  and  $t_1(A_n) \geq K$ , for every integer  $n$ . On the other hand, for every  $(\varphi, \psi) \in W$ ,

$$\begin{aligned} D_1 I(U_n, V_n)(\varphi) &= \varphi_n^1 \partial_s \tilde{I}(B_n) + \partial_u \tilde{I}(B_n)(\varphi_n^2) + \varphi_n^1 \partial_t \tilde{I}(B_n) \\ &= \partial_u \tilde{I}(B_n)(\varphi_n^2). \end{aligned}$$

where  $\varphi_n^1 = \pi'_1(u_n)(\varphi)$  and  $\varphi_n^2 = \pi'_2(u_n)(\varphi)$ . Then the following estimates hold true:  $|\varphi_n^1| \leq \|\nabla \varphi\|_p$  and  $\|\nabla \varphi_n^2\|_p \leq \frac{2}{K} \|\nabla \varphi\|_p$ . In the same manner, we get

$$\begin{aligned} D_2 I(U_n, V_n)(\psi) &= \psi_n^1 \partial_s \tilde{I}(B_n) + \partial_v \tilde{I}(B_n)(\psi_n^2) + \psi_n^1 \partial_t \tilde{I}(B_n) \\ &= \partial_v \tilde{I}(B_n)(\psi_n^2). \end{aligned}$$

where  $\psi_n^1 = \tilde{\pi}'_1(v_n)(\psi)$  and  $\psi_n^2 = \tilde{\pi}'_2(v_n)(\psi)$ , with the estimates  $|\psi_n^1| \leq \|\nabla \psi\|_q$  and  $\|\nabla \psi_n^2\|_q \leq \frac{2}{K} \|\nabla \psi\|_q$ . Therefore

$$\begin{aligned} |D_1 I(U_n, V_n)(\varphi)| &\leq \frac{1}{n} \|\nabla \varphi_n^2\|_p \\ &\leq \frac{2}{nK} \|\nabla \varphi\|_p \end{aligned}$$

and

$$\begin{aligned} |D_2 I(U_n, V_n)(\psi)| &\leq \frac{1}{n} \|\nabla \psi_n^2\|_q \\ &\leq \frac{2}{nK} \|\nabla \psi\|_q. \end{aligned}$$

We conclude easily that

$$\lim_{n \rightarrow +\infty} \|I'(U_n, V_n)\|_* = 0,$$

where  $I'(U_n, V_n)(\varphi, \psi) = D_1 I(U_n, V_n)(\varphi) + D_2 I(U_n, V_n)(\psi)$  and  $\|\cdot\|_*$  is the norm on the dual space of  $W$ .

The arguments are similar if  $(u_n, v_n) \in \mathbb{S}$  is a minimizing sequence of (21). Hence, the lemma is proved.  $\square$

**Remark.** For every  $(u, v) \in \widetilde{W}$  and  $(\lambda, \mu) \in \mathcal{D}$ , one has  $\widetilde{I}(s, u, t, v) = \widetilde{I}(s, |u|, t, |v|)$ ,  $s_i(|u|, |v|, \lambda, \mu) = s_i(u, v, \lambda, \mu)$ ,  $i \in \{1, 2\}$  and consequently  $t_i(|u|, |v|, \lambda, \mu) = t_i(u, v, \lambda, \mu)$ ,  $i \in \{1, 2\}$ . Therefore, every minimizing sequence  $(u_n, v_n) \in \mathbb{S}_p \times \mathbb{S}_q$  of (20) or (21) can be considered as a sequence satisfying  $u_n \geq 0$  and  $v_n \geq 0$  in  $\Omega$ .

## 4 Positive solutions and the behaviour of their energy

**Theorem 1** Let  $(\lambda, \mu) \in \mathcal{D}$ . Then Problem (1) has at least two nontrivial solutions  $(U^i, V^i)$ ,  $i \in \{1, 2\}$ , such that  $U^i \geq 0$  and  $V^i \geq 0$  in  $\Omega$  and  $U^i \neq 0$ ,  $V^i \neq 0$ , for  $i \in \{1, 2\}$ .

**Proof.** We will use the notations of the previous lemmas. Let  $(\lambda, \mu) \in \mathcal{D}$  and consider a nonnegative minimizing sequence  $(u_n, v_n) \in \mathbb{S}$  of (20). It is known from Lemma (3) that

$$\lim_{n \rightarrow +\infty} I(U_n, V_n) = \alpha_1(\lambda, \mu),$$

$$\lim_{n \rightarrow +\infty} \|I'(U_n, V_n)\|_* = 0$$

and that  $(U_n, V_n)$  is bounded in  $W$ . Passing if necessary to a subsequence, there are  $U^1 \in W_{\Gamma_1}^{1,p}(\Omega)$  and  $V^1 \in W_{\Gamma_2}^{1,q}(\Omega)$  such that

$$\begin{aligned} U_n &\rightharpoonup U^1 \text{ in } W_{\Gamma_1}^{1,p}(\Omega), \\ U_n &\rightarrow U^1 \text{ in } L^{p_1}(\Omega) \text{ and } L^{\tilde{p}}(\Omega), \\ V_n &\rightharpoonup V^1 \text{ in } W_{\Gamma_2}^{1,q}(\Omega), \\ V_n &\rightarrow V^1 \text{ in } L^{q_1}(\Omega) \text{ and } L^{\tilde{q}}(\Omega), \end{aligned}$$

where  $\tilde{p}$  and  $\tilde{q}$  are specified in (15). At this stage, we use the well known inequalities:  $\forall (x, y) \in \mathbb{R}^N$

$$\begin{aligned} |x - y|^\gamma &\leq C(|x|^{\gamma-2}x - |y|^{\gamma-2}y) \cdot (x - y), \text{ if } \gamma \geq 2, \\ |x - y|^2 &\leq C(|x| - |y|)^{2-\gamma} (|x|^{\gamma-2}x - |y|^{\gamma-2}y) \cdot (x - y), \text{ if } \gamma < 2. \end{aligned}$$

where  $\cdot$  denotes the scalar product in  $\mathbb{R}^N$ .

In the case  $p \geq 2$ , we obtain

$$\begin{aligned}
P(U_n - U^1) &\leq C \int_{\Omega} (|\nabla U_n|^{p-2} \nabla U_n - |\nabla V_n|^{p-2} \nabla V_n) \cdot (\nabla U_n - \nabla V_n) \\
&= C(D_1 I(U_n, V_n)(U_n - U^1) - D_1 I(U^1, V^1)(U_n - U^1)) + \\
&\quad C\lambda \int_{\Omega} (|U_n|^{p_1-2} U_n - |U^1|^{p_1-2} U^1) (U_n - U^1) + \\
&\quad C(\alpha + 1) \int_{\Omega} (U_n |U_n|^{\alpha-1} |V_n|^{\beta+1} - U^1 |U^1|^{\alpha-1} |V^1|^{\beta+1}) (U_n - U^1).
\end{aligned}$$

Since  $\lim_{n \rightarrow +\infty} \|I'(U_n, V_n)\|_* = 0$ ,  $(V_n)$  is bounded, and using the fact that  $U_n \rightarrow U^1$  in  $L^{p_1}(\Omega)$  and in  $L^{\tilde{p}}(\Omega)$ ,  $V_n \rightarrow V^1$  in  $L^{\tilde{q}}(\Omega)$ , we conclude, by the Hölder inequality, that  $P(U_n - U^1) \rightarrow 0$ , as  $n$  goes to  $+\infty$ , which means that

$$U_n \rightarrow U^1 \text{ in } W_{\Gamma_1}^{1,p}(\Omega).$$

In the case  $p < 2$ , a direct computation gives

$$\begin{aligned}
\|\nabla U_n - \nabla U^1\|_p^2 &\leq C (\|\nabla U_n\|_p^{2-p} + \|\nabla U^1\|_p^{2-p}) \times \\
&\quad \int_{\Omega} (|\nabla U_n|^{p-2} \nabla U_n - |\nabla U^1|^{p-2} \nabla U^1) \cdot (\nabla U_n - \nabla U^1).
\end{aligned}$$

Since  $\|\nabla U_n - \nabla U^1\|_p$  is bounded, the same arguments used above show that  $U_n \rightarrow U^1$  in  $W_{\Gamma_1}^{1,p}(\Omega)$ , as  $n$  goes to  $+\infty$ . In a similar way we get  $V_n \rightarrow V^1$  in  $W_{\Gamma_2}^{1,q}(\Omega)$ , as  $n$  goes to  $+\infty$ .

Moreover, it is clear that  $(U^1, V^1)$  is a nontrivial solution of Problem (1) verifying  $U^1 \geq 0$  and  $V^1 \geq 0$  in  $\Omega$  and  $U^1 \neq 0$ ,  $V^1 \neq 0$ . On the other hand, there is a subsequence of  $(u_n, v_n)$ , still denoted by  $(u_n, v_n)$  such that

$$\begin{aligned}
U_n := s_1(u_n, v_n, \lambda, \mu) u_n &\rightarrow U^1 \text{ in } W_{\Gamma_1}^{1,p}(\Omega), \\
V_n := t_1(u_n, v_n, \lambda, \mu) v_n &\rightarrow V^1 \text{ in } W_{\Gamma_2}^{1,q}(\Omega).
\end{aligned}$$

According to Lemma (2), let  $(s_1, t_1) \in (0, +\infty)^2$  such that

$$\left\{
\begin{array}{l}
s_1(u_n, v_n, \lambda, \mu) \rightarrow s_1 \text{ in } \mathbb{R}, \\
t_1(u_n, v_n, \lambda, \mu) \rightarrow t_1 \text{ in } \mathbb{R}, \\
u_n \rightarrow u^1 = \frac{U^1}{s_1} \text{ in } W_{\Gamma_1}^{1,p}(\Omega), \\
v_n \rightarrow v^1 = \frac{V^1}{t_1} \text{ in } W_{\Gamma_2}^{1,q}(\Omega),
\end{array}
\right.$$

with  $u^1 = \frac{U^1}{s_1} \in \mathbb{S}_p$ ,  $v^1 = \frac{V^1}{t_1} \in \mathbb{S}_q$ ,  $s_1 = s_1(u^1, v^1, \lambda, \mu)$  and  $t_1 = t_1(u^1, v^1, \lambda, \mu)$ . Therefore,  $\partial_{ss} \tilde{I}(s_1(u^1, v^1, \lambda, \mu), u^1, t_1(u^1, v^1, \lambda, \mu), v^1) > 0$ .

Proceeding in the same manner with a nonnegative minimizing sequence  $(\tilde{u}_n, \tilde{v}_n) \in \mathbb{S}$  of (21), we obtain a second nontrivial solution  $(U^2, V^2)$  of (1) verifying  $U^2 \geq 0$  and  $V^2 \geq 0$  in  $\Omega$  and  $U^2 \neq 0, V^2 \neq 0$ .

Now, we have to show that  $(U^1, V^1) \neq (U^2, V^2)$ . Let  $(s_2, t_2) \in (0, +\infty)^2$  such that

$$\left\{ \begin{array}{ll} s_2(\tilde{u}_n, \tilde{v}_n, \lambda, \mu) & \longrightarrow s_2 \text{ in } \mathbb{R}, \\ t_2(\tilde{u}_n, \tilde{v}_n, \lambda, \mu) & \longrightarrow t_2 \text{ in } \mathbb{R}, \\ \tilde{u}_n & \longrightarrow u^2 = \frac{U^2}{s_2} \text{ in } W_{\Gamma_1}^{1,p}(\Omega), \\ \tilde{v}_n & \longrightarrow v^2 = \frac{V^2}{t_2} \text{ in } W_{\Gamma_2}^{1,q}(\Omega), \end{array} \right.$$

with  $u^2 = \frac{U^2}{s_2} \in \mathbb{S}_p$ ,  $v^2 = \frac{V^2}{t_2} \in \mathbb{S}_q$ ,  $s_2 = s_2(u^2, v^2, \lambda, \mu)$  and  $t_2 = t_2(u^2, v^2, \lambda, \mu)$ . Therefore,  $\partial_{ss}\tilde{I}(s_2(u^2, v^2, \lambda, \mu), u^2, t_2(u^2, v^2, \lambda, \mu), v^2) < 0$ . Hence  $(U^1, V^1) \neq (U^2, V^2)$ , which ends the proof.

□

In the sequel, for every  $(\lambda, \mu) \in \mathcal{D}$ , the functions  $(u^1, v^1)$  and  $(u^2, v^2)$  will be denoted by  $(u^1(\lambda, \mu), v^1(\lambda, \mu))$  and  $(u^2(\lambda, \mu), v^2(\lambda, \mu))$  respectively. Similarly, the solutions  $(U^i, V^i)$ ,  $i \in \{1, 2\}$ , will be denoted by  $(U^i(\lambda, \mu), V^i(\lambda, \mu))$ ,  $i \in \{1, 2\}$ .

**Theorem 2** *Let  $(\lambda, \mu) \in \mathcal{D}$ . Then*

$$(i) \quad I(U^1, V^1) < 0 \text{ for } \lambda \in ]0, \hat{\lambda}(\mu)[,$$

$$(ii) \quad \begin{cases} I(U^2, V^2) > 0 & \text{for } \lambda \in ]0, \lambda_0(\mu)[, \\ I(U^2, V^2) < 0 & \text{for } \lambda \in ]\lambda_0(\mu), \hat{\lambda}(\mu)[, \end{cases}$$

where

$$\lambda_0(\mu) := \frac{p_1}{r} \left( \frac{r}{p} \right)^{\frac{r-p_1}{r-p}} \hat{\lambda}(\mu).$$

**Proof.** In this proof,  $\mu$  will be fixed in  $(-\infty, \mu_1)$ , so we will omit the dependence on  $\mu$  in the expressions which will follow. However, the dependence on  $\lambda$  will be specified. In particular, the Euler-Lagrange functional  $I$  will be denoted by  $I_\lambda$ .

(ii) Let  $(u, v)$  be an arbitrary element of  $\widetilde{W}$ . We denote

$$\widetilde{I}_\lambda(s, u, t(s), v) = \frac{s^p}{p} P(u) - \frac{\lambda}{p_1} s^{p_1} P_1(u) - \frac{s^r}{r} A(u, v),$$

and write

$$\widetilde{I}_\lambda(s, u, t(s), v) = s^{p_1} \widetilde{G}_\lambda(s, u, v),$$

where

$$\tilde{G}_\lambda(s, u, v) = s^{p-p_1} \frac{P(u)}{p} - \lambda \frac{P_1(u)}{p_1} - s^{r-p_1} \frac{A(u, v)}{r}.$$

It follows that

$$\partial_s \tilde{I}_\lambda(s, u, t(s), v) = p_1 s^{p_1-1} \tilde{G}_\lambda(s, u, v) + s^{p_1} \partial_s \tilde{G}_\lambda(s, u, v),$$

with

$$\partial_s \tilde{G}_\lambda(s, u, v) = s^{p-p_1-1} \left\{ \frac{p-p_1}{p} P(u) - \frac{r-p_1}{r} s^{r-p} A(u, v) \right\}.$$

The real valued function  $s \mapsto \tilde{G}_\lambda(s, u, v)$  is increasing on  $]0, s_0(u, v)[$ , decreasing on  $]s_0(u, v), +\infty[$  and attains its unique maximum for  $s = s_0(u, v)$ , where

$$s_0(u, v) = \left( \frac{r}{p} \right)^{\frac{1}{r-p}} \bar{s}_\mu(u, v), \quad (22)$$

and  $\bar{s}_\mu(u, v)$  is defined in (7). On the other hand, a direct computation gives

$$\tilde{G}_\lambda(s_0(u, v), u, v) = \left( \frac{p-p_1}{r-p_1} \frac{P(u)}{A(u, v)} \right)^{\frac{r-p_1}{r-p}} R(u, v) - \lambda P_1(u).$$

Similarly,  $\tilde{G}_\lambda(s_0(u, v), u, v) > 0$  (resp.  $\tilde{G}_\lambda(s_0(u, v), u, v) < 0$ ) if  $\lambda < \lambda_0(u, v)$  (resp.  $\lambda > \lambda_0(u, v)$ ) and  $\tilde{G}_{\lambda_0(u, v)}(s_0(u, v), u, v) = 0$ , where

$$\lambda_0(u, v) = \frac{p_1}{r} \left( \frac{r}{p} \right)^{\frac{r-p_1}{r-p}} \lambda(u, v), \quad (23)$$

with  $\lambda(u, v)$  given by (8). Thus, we get

$$\begin{cases} \tilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) > 0 & \text{if } \lambda < \lambda_0(u, v), \\ \tilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) = 0 & \text{if } \lambda = \lambda_0(u, v), \\ \tilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) < 0 & \text{if } \lambda > \lambda_0(u, v). \end{cases} \quad (24)$$

First, since the function

$$\begin{aligned} ]0, 1[ &\longrightarrow \mathbb{R} \\ t &\longmapsto \frac{\ln t}{1-t} \end{aligned}$$

is increasing, then for every real numbers  $x, y$  such that  $0 < x < y < 1$ , one has

$$\ln \left[ \frac{1}{x} \right] > \frac{1-x}{1-y} \ln \left[ \frac{1}{y} \right] = \ln \left[ \left( \frac{1}{y} \right)^{\frac{1-x}{1-y}} \right],$$

and consequently

$$0 < x \left( \frac{1}{y} \right)^{\frac{1-x}{1-y}} < 1.$$

In the particular case  $x = p_1/r$  and  $y = p/r$  we get

$$0 < \frac{p_1}{r} \left( \frac{r}{p} \right)^{\frac{r-p_1}{r-p}} < 1,$$

and therefore  $0 < \lambda_0(u, v) < \lambda(u, v)$ .

Moreover, for every  $(u, v) \in \widetilde{W}$ , one has  $\widetilde{G}_{\lambda_0(u, v)}(s, u, v) < 0$  for  $s \in ]0, +\infty[ \setminus \{s_0(u, v)\}$  and  $\widetilde{G}_{\lambda_0(u, v)}(s_0(u, v), u, v) = 0$ . Hence, the real valued function  $s \mapsto \widetilde{I}_{\lambda_0(u, v)}(s, u, t(s), v)$ ,  $(s > 0)$ , attains its unique maximum at  $s = s_0(u, v)$  and we obtain the following interesting identity

$$s_2(u, v, \lambda_0(u, v), \mu) = s_0(u, v). \quad (25)$$

We will set

$$t_0(u, v) := t_2(u, v, \lambda_0(u, v), \mu).$$

On the other hand, it is clear that the functional  $\lambda_0(u, v)$  is weakly lower semi-continuous on  $\widetilde{W}$ . Thus, the value

$$\widehat{\lambda}_0 := \inf_{(u, v) \in \widetilde{W}} \lambda_0(u, v) \quad (26)$$

is achieved on  $\widetilde{W}$ . Since  $\lambda_0(u, v)$  is 0-homogeneous in  $u$  and  $v$ , we can assume that there is some  $(u^*, v^*) \in \mathbb{S}_p \times \mathbb{S}_q$  such that  $\widehat{\lambda}_0 = \lambda_0(u^*, v^*)$ .

Now, let  $\lambda$  be such that  $0 < \lambda < \widehat{\lambda}_0$ . Then, for every  $(u, v) \in \widetilde{W}$  one has  $0 < \lambda < \lambda_0(u, v)$  and consequently  $\widetilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) > 0$  holds from (24). But,  $s \mapsto \widetilde{I}_\lambda(s, u, t(s), v)$ ,  $(s > 0)$  attains its unique maximum for  $s = s_2(u, v, \lambda)$ , hence  $\widetilde{I}_\lambda(s_2(u, v, \lambda), u, t_2(u, v, \lambda), v) > 0$ , for every  $(u, v) \in \widetilde{W}$ . In particular, we have

$$\widetilde{I}_\lambda(s_2(u^2(\lambda), v^2(\lambda), \lambda), u^2(\lambda), t_2(u^2(\lambda), v^2(\lambda), \lambda), v^2(\lambda)) > 0,$$

i.e.  $I_\lambda(U^2(\lambda), V^2(\lambda)) > 0$ .

If  $\lambda = \widehat{\lambda}_0$ , then

$$\begin{aligned} I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0), V^2(\widehat{\lambda}_0)) &= \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), u^2(\widehat{\lambda}_0), t_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), v^2(\widehat{\lambda}_0)) \\ &= \inf_{(u, v) \in \mathbb{S}_p \times \mathbb{S}_q} \widetilde{I}_{\widehat{\lambda}_0}(s_2(u, v, \widehat{\lambda}_0), u, t_2(u, v, \widehat{\lambda}_0), v) \\ &\leq \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*, v^*), u^*, t_2(u^*, v^*), v^*) \\ &= \widetilde{I}_{\lambda_0(u^*, v^*)}(s_0(u^*, v^*), u^*, t_0(u^*, v^*), v^*) \\ &= 0 \end{aligned}$$

which implies that  $I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0), V^2(\widehat{\lambda}_0)) \leq 0$ . In addition, it is known from (24) that

$$\begin{aligned} \widetilde{I}_{\widehat{\lambda}_0}(s_0(u, v), u, t_0(u, v), v) &\geq 0, \\ \widetilde{I}_{\widehat{\lambda}_0}(s_1(u, v, \widehat{\lambda}_0), u, t_1(u, v, \widehat{\lambda}_0), v) &< 0, \end{aligned}$$

for every  $(u, v) \in \widetilde{W}$ . Then

$$s_0(u, v) > s_1(u, v, \widehat{\lambda}_0), \quad \forall (u, v) \in \widetilde{W}.$$

It follows that

$$\begin{aligned} \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), u^2(\widehat{\lambda}_0), t_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), v^2(\widehat{\lambda}_0)) &\geq \\ \widetilde{I}_{\widehat{\lambda}_0}(s_0(u^1(\widehat{\lambda}_0), v^1(\widehat{\lambda}_0)), u^1(\widehat{\lambda}_0), t_0(u^1(\widehat{\lambda}_0), v^1(\widehat{\lambda}_0)), v^1(\widehat{\lambda}_0)) &\geq 0. \end{aligned}$$

Hence,

$$\begin{aligned} \widetilde{I}_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0), V^2(\widehat{\lambda}_0)) &= \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), u^2(\widehat{\lambda}_0), t_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), v^2(\widehat{\lambda}_0)) \\ &= 0. \end{aligned}$$

Finally, assume that  $\widehat{\lambda}_0 < \lambda < \widehat{\lambda}$ . Since, for every  $s \in ]0, +\infty[$  and  $(u, v) \in \widetilde{W}$ , the real valued function  $\lambda \mapsto \widetilde{I}_\lambda(s, u, t(s), v)$  is decreasing, it follows that

$$\widetilde{I}_\lambda(s, u, t(s), v) < \widetilde{I}_{\widehat{\lambda}_0}(s, u, t(s), v), \quad \text{for every } s > 0 \text{ and } (u, v) \in \widetilde{W}. \quad (27)$$

In addition, we have

$$\begin{aligned} \widetilde{I}_\lambda(s_2(u^2(\lambda), v^2(\lambda), \lambda), u^2(\lambda), t_2(u^2(\lambda), v^2(\lambda), \lambda), v^2(\lambda)) &= \\ \inf_{(u, v) \in \mathbb{S}_p \times \mathbb{S}_q} \widetilde{I}_\lambda(s_2(u, v, \lambda), u, t_2(u, v, \lambda), v) &\leq \\ \widetilde{I}_\lambda(s_2(u^*, v^*, \lambda), u^*, t_2(u^*, v^*, \lambda), v^*) &< \\ \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*, v^*, \lambda), u^*, t_2(u^*, v^*, \lambda), v^*) \end{aligned}$$

where the last inequality follows from (27). Moreover, the real valued function  $s \mapsto \widetilde{I}_{\widehat{\lambda}_0}(s, u^*, t(s), v^*)$ , ( $s > 0$ ), achieves its unique maximum at  $s = s_0(u^*, v^*)$ . Thus,

$$\begin{aligned} \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*, v^*, \lambda), u^*, t_2(u^*, v^*, \lambda), v^*) &\leq \widetilde{I}_{\widehat{\lambda}_0}(s_0(u^*, v^*), u^*, t_0(u^*, v^*), v^*) \\ &= \widetilde{I}_{\lambda_0(u^*, v^*)}(s_0(u^*, v^*), u^*, t_0(u^*, v^*), v^*) \\ &= 0. \end{aligned}$$

Hence  $\widetilde{I}_\lambda(s_2(u^2(\lambda), v^2(\lambda), \lambda), u^2(\lambda), t_2(u^2(\lambda), v^2(\lambda), \lambda), v^2(\lambda)) < 0$ , which ends the proof.  $\square$

The following result shows the subtle link existing between the characteristic value  $\widehat{\lambda}_0$  defined by (26) and Problem (1).

**Theorem 3** *If  $(u, v)$  is a solution of (26) then  $(s_0(u, v)u, t_0(u, v)v)$  is a solution of the system (1) when  $\lambda = \widehat{\lambda}_0$ .*

**Proof.** Let  $(u, v)$  be a solution of (26). In order to simplify the notations, we set  $U := s_0(u, v)u$  and  $V := t_0(u, v)v$ . Thus, for  $\lambda = \widehat{\lambda}_0 = \lambda_0(u, v)$  we have:

$$I_{\widehat{\lambda}_0, \mu}(U, V) = \frac{s_0(u, v)^p}{p} P(u) - \widehat{\lambda}_0 \frac{s_0(u, v)^{p_1}}{p_1} P_1(u) - \frac{s_0(u, v)^r}{r} A(u, v)$$

and for every  $\varphi \in W_0^{1,p}(\Omega)$ :

$$D_1 I_{\widehat{\lambda}_0, \mu}(U, V)(\varphi) = \frac{1}{p} P'(U)(\varphi) - \frac{\widehat{\lambda}_0}{p_1} P'_1(U)(\varphi) - \frac{1}{r} D_1 A(U, V)(\varphi),$$

where

$$\begin{cases} P'(U)(\varphi) &= s_0(u, v)^{p-1} P'(u)(\varphi), \\ P'_1(U)(\varphi) &= s_0(u, v)^{p_1-1} P'_1(u)(\varphi), \\ D_1 A(U, V)(\varphi) &= s_0(u, v)^{r-1} D_1 A(u, v)(\varphi). \end{cases}$$

We calculate now,

$$\begin{aligned} \widehat{\lambda}_0 P'_1(U)(\varphi) &= \lambda_0(u, v) s_0(u, v)^{p_1-1} P'_1(u)(\varphi) \\ &= \frac{p_1}{r} \left( \frac{r}{p} \right)^{\frac{r-p_1}{r-p}} \left( \frac{p-p_1}{r-p_1} \right)^{\frac{p-p_1}{r-p}} \frac{P(u)}{P_1(u)} \left( \frac{P(u)}{A(u, v)} \right)^{\frac{p-1}{r-p}} \\ &\quad \times \frac{r-p}{r-p_1} \left( \frac{r}{p} \right)^{\frac{p_1-1}{r-p}} \left( \frac{p-p_1}{r-p_1} \frac{P(u)}{A(u, v)} \right)^{\frac{p_1-1}{r-p}} P'_1(u)(\varphi) \\ &= \frac{r-p}{r-p_1} \frac{p_1}{r} \left( \frac{p-p_1}{r-p_1} \right)^{\frac{p-1}{r-p}} \left( \frac{r}{p} \right)^{\frac{p-1}{r-p}} \frac{P(u)}{P_1(u)} \frac{P(u)}{A(u, v)}^{\frac{p-1}{r-p}} P'_1(u)(\varphi) \\ &= \frac{r-p}{r-p_1} \frac{p_1}{p} \frac{P(u)}{P_1(u)} \left( \left( \frac{r}{p} \right)^{\frac{1}{r-p}} \left( \frac{p-p_1}{r-p_1} \right)^{\frac{1}{r-p}} \frac{P(u)}{A(u, v)} \right)^{p-1} P'_1(u)(\varphi) \\ &= \frac{p_1}{p} \frac{r-p}{r-p_1} P(u) s_0(u, v)^{p-1} \frac{P'_1(u)(\varphi)}{P_1(u)}. \end{aligned}$$

In addition, one has

$$\begin{aligned} D_1 A(U, V)(\varphi) &= s_0(u, v)^{r-1} D_1(u, v)(\varphi) \\ &= \left( \frac{r}{p} \frac{p-p_1}{r-p_1} \frac{P(u)}{A(u, v)} \right)^{\frac{p-1}{r-p}} \frac{r}{p} \frac{p-p_1}{r-p_1} \frac{P(u)}{A(u, v)} D_1 A(u, v)(\varphi) \\ &= \frac{r}{p} \frac{p-p_1}{r-p_1} P(u) s_0(u, v)^{p-1} \frac{D_1 A(u, v)(\varphi)}{A(u, v)}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} D_1 I_{\widehat{\lambda}_0, \mu}(U, V)(\varphi) &= \left[ \frac{P'(u)(\varphi)}{P(u)} - \frac{r-p}{r-p_1} \frac{P'_1(u)(\varphi)}{P_1(u)} - \frac{p-p_1}{r-p_1} \frac{D_1 A(u, v)(\varphi)}{A(u, v)} \right] \\ &\quad \times \frac{P(u) s_0(u, v)^{p-1}}{p} \\ &= K \left( \frac{r-p_1}{r-p} \frac{P'(u)(\varphi)}{P(u)} - \frac{P'_1(u)(\varphi)}{P_1(u)} - \frac{p-p_1}{r-p} \frac{D_1 A(u, v)(\varphi)}{A(u, v)} \right), \end{aligned}$$

where  $K := \frac{r-p}{r-p_1} \frac{P(u)}{p} s_0(u, v)^{p-1}$ . On the other hand, a direct computation gives:

$$D_1 \lambda_0(u, v)(\varphi) = \widehat{\lambda}_0 \left( \frac{r-p_1}{r-p} \frac{P'(u)(\varphi)}{P(u)} - \frac{P'_1(u)(\varphi)}{P_1(u)} - \frac{p-p_1}{r-p} \frac{D_1 A(u, v)(\varphi)}{A(u, v)} \right),$$

which is equal to zero by assumption. Hence  $D_1 I_{\widehat{\lambda}_0, \mu}(U, V)(\varphi) = 0$  since it is proportional to  $D_1 \lambda_0(u, v)(\varphi)$ .

Moreover, for every  $\psi \in W_0^{1,q}(\Omega)$ , we get

$$D_2 \lambda_0(u, v)(\psi) = -\frac{p-p_1}{r-p_1} \lambda_0(u, v) \frac{D_2 A(u, v)(\psi)}{A(u, v)},$$

which is also equal to zero by assumption. This implies that  $D_2 A(u, v)(\psi) = 0$ , since  $\lambda_0(u, v) = \widehat{\lambda}_0 \neq 0$ . Then

$$D_2 I_{\widehat{\lambda}_0, \mu}(U, V)(\psi) = -\frac{s_0(u, v)^r}{r} D_2 A(u, v)(\psi) = 0.$$

which implies that  $(s_0(u, v)u, t_0(u, v)v)$  is well a solution of the problem (1) with  $\lambda = \widehat{\lambda}_0$ .  $\square$

## Acknowledgments

The authors are very grateful to Prof. Claudianor O. Alves for the interesting discussions and references on this subject.

## References

- [1] A. Ahammou, *Positive radial solutions of nonlinear elliptic systems*. New York J. Math. 7 (2001), 267–280.
- [2] C. O. Alves, D. G. de Figueiredo, *Nonvariational elliptic systems*. Current developments in partial differential equations (Temuco, 1999). Discrete Contin. Dyn. Syst. 8 (2002), no. 2, 289–302.
- [3] C. O. Alves, D. C. de Moraes Filho, M. A. S. Souto, *On systems of elliptic equations involving subcritical or critical Sobolev exponents*. Nonlinear Anal. T. M. A. 42 (2000) no. 5, 771–787.
- [4] A. Ambrosetti, H. Brézis, G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*. J. Funct. Anal. 122 (1994), no. 2, 519–543.
- [5] A. Ambrosetti, J. García Azorero, I. Peral, *Existence and multiplicity results for some nonlinear elliptic equations: a survey*. Rend. Mat. Appl. (7) 20 (2000), 167–198.

- [6] Y. Bozhkov, E. Mitidieri, *Existence of multiple solutions for quasilinear systems via fibering method*. J. Differential Equations 190 (2003), no. 1, 239–267.
- [7] K. J. Brown, Y. Zhang, *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*. J. Differential Equations 193 (2003), no. 2, 481–499.
- [8] P. Clément, D. G. de Figueiredo, E. Mitidieri, *Positive solutions of semilinear elliptic systems*. Comm. Partial Differential Equations 17 (1992), no. 5-6, 923–940.
- [9] E. Colorado, I. Peral, *Semilinear elliptic problems with mixed Dirichlet-Neumann boundary conditions*. J. Funct. Anal. 199 (2003), no. 2, 468–507.
- [10] A. El Hamidi, *Multiple solutions with changing sign energy to a nonlinear elliptic equation*. Commun. Pure Appl. Anal. Vol 3, No 2 (2004) 253–265.
- [11] A. El Hamidi, *Existence results to elliptic systems with nonstandard growth conditions*. J. Math. Anal. Appl. 300 (2004), no. 1, 30–42.
- [12] I. Ekeland, *On the Variational Principle*. J. Math. Anal. Appl. 47 (1974) 324–353.
- [13] D. G. de Figueiredo, *Nonlinear elliptic systems*. An. Acad. Brasil. Ciênc. 72 (2000), no. 4, 453–469.
- [14] D. G. de Figueiredo, P. Felmer, *On superquadratic elliptic systems*. Trans. Amer. Math. Soc. 343 (1994), no. 1, 99–116.
- [15] P. Drabek, S. Pohozaev, *Positive solutions for the  $p$ -Laplacian: application of the fibering method*. Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 703–726.
- [16] J. Mawhin, M. Willem, *Critical Point Theory and Hamilton Systems*. Springer Verlag, Berlin (1989)
- [17] Z. Nehari, *On a class of nonlinear second-order differential equations*. Trans. Amer. Math. Soc. 95 (1960) 101–123.
- [18] M. Struwe, *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*. Springer-Verlag, (1996)
- [19] G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponent*. Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (1992), no. 3, 281–304.

- [20] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*. Reg. Conf. Ser. Math. 65 (1986), 1–100.
- [21] M. Squassina, *An eigenvalue problem for elliptic systems*. New York J. Math. 6 (2000), 95–106
- [22] J. Vélin, *Existence results for some nonlinear elliptic system with lack of compactness*. Nonlinear Anal. 52 (2003), no. 3, 1017–1034.
- [23] J. Vélin, F. de Thélin, *Existence and nonexistence of nontrivial solutions for some nonlinear elliptic systems*. Rev. Mat. Univ. Complut. Madrid 6 (1993), no. 1, 153–194.
- [24] M. Willem, *Minimax theorems*. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, (1996)

Khalid Adriouch  
 Laboratoire de Math. & Applications,  
 Université de la Rochelle,  
 17042 La Rochelle, France.  
 e-mail: kadriouc@univ-lr.fr

Abdallah El Hamidi  
 Laboratoire de Math. & Applications,  
 Université de la Rochelle,  
 17042 La Rochelle, France.  
 e-mail: aelhamid@univ-lr.fr

Liste des prépublications

99-1 Monique Jeanblanc et Nicolas Privault. A complete market model with Poisson and Brownian components. A paraître dans *Proceedings of the Seminar on Stochastic Analysis, Random Fields and Applications*, Ascona, 1999.

99-2 Laurence Cherfils et Alain Miranville. Generalized Cahn-Hilliard equations with a logarithmic free energy. A paraître dans *Revista de la Real Academia de Ciencias*.

99-3 Jean-Jacques Prat et Nicolas Privault. Explicit stochastic analysis of Brownian motion and point measures on Riemannian manifolds. *Journal of Functional Analysis* **167** (1999) 201-242.

99-4 Changgui Zhang. Sur la fonction  $q$ -Gamma de Jackson. A paraître dans *Aequationes Math.*

99-5 Nicolas Privault. A characterization of grand canonical Gibbs measures by duality. A paraître dans *Potential Analysis*.

99-6 Guy Wallet. La variété des équations surstables. A paraître dans *Bulletin de la Société Mathématique de France*.

99-7 Nicolas Privault et Jiang-Lun Wu. Poisson stochastic integration in Hilbert spaces. *Annales Mathématiques Blaise Pascal*, **6** (1999) 41-61.

99-8 Augustin Fruchard et Reinhard Schäfke. Sursabilité et résonance.

99-9 Nicolas Privault. Connections and curvature in the Riemannian geometry of configuration spaces. *C. R. Acad. Sci. Paris, Série I* **330** (2000) 899-904.

99-10 Fabienne Marotte et Changgui Zhang. Multisommabilité des séries entières solutions formelles d'une équation aux  $q$ -différences linéaire analytique. A paraître dans *Annales de l'Institut Fourier*, 2000.

99-11 Knut Aase, Bernt Øksendal, Nicolas Privault et Jan Ubøe. White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance. *Finance and Stochastics*, **4** (2000) 465-496.

00-01 Eric Benoît. Canards en un point pseudo-singulier noeud. A paraître dans *Bulletin de la Société Mathématique de France*.

00-02 Nicolas Privault. Hypothesis testing and Skorokhod stochastic integration. *Journal of Applied Probability*, **37** (2000) 560-574.

00-03 Changgui Zhang. La fonction théta de Jacobi et la sommabilité des séries entières  $q$ -Gevrey, I. *C. R. Acad. Sci. Paris, Série I* **331** (2000) 31-34.

00-04 Guy Wallet. Déformation topologique par changement d'échelle.

00-05 Nicolas Privault. Quantum stochastic calculus for the uniform measure and Boolean convolution. A paraître dans *Séminaire de Probabilités XXXV*.

00-06 Changgui Zhang. Sur les fonctions  $q$ -Bessel de Jackson.

00-07 Laure Coutin, David Nualart et Ciprian A. Tudor. Tanaka formula for the fractional Brownian motion. A paraître dans *Stochastic Processes and their Applications*.

00-08 Nicolas Privault. On logarithmic Sobolev inequalities for normal martingales. *Annales de la Faculté des Sciences de Toulouse* **9** (2000) 509-518.

01-01 Emanuelle Augeraud-Veron et Laurent Augier. Stabilizing endogenous fluctuations by fiscal policies; Global analysis on piecewise continuous dynamical systems. A paraître dans *Studies in Nonlinear Dynamics and Econometrics*.

01-02 Delphine Boucher. About the polynomial solutions of homogeneous linear differential equations depending on parameters. A paraître dans *Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation: ISSAC 99, Sam Dooley Ed., ACM, New York 1999*.

01-03 Nicolas Privault. Quasi-invariance for Lévy processes under anticipating shifts.

01-04 Nicolas Privault. Distribution-valued iterated gradient and chaotic decompositions of Poisson jump times functionals.

01-05 Christian Houdré et Nicolas Privault. Deviation inequalities: an approach via covariance representations.

01-06 Abdallah El Hamidi. Remarques sur les sentinelles pour les systèmes distribués

02-01 Eric Benoît, Abdallah El Hamidi et Augustin Fruchard. On combined asymptotic expansions in singular perturbation.

02-02 Rachid Bebbouchi et Eric Benoît. Equations différentielles et familles bien nées de courbes planes.

02-03 Abdallah El Hamidi et Gennady G. Laptev. Nonexistence of solutions to systems of higher-order semilinear inequalities in cone-like domains.

02-04 Hassan Lakhel, Youssef Ouknine, et Ciprian A. Tudor. Besov regularity for the indefinite Skorohod integral with respect to the fractional Brownian motion: the singular case.

02-05 Nicolas Privault et Jean-Claude Zambrini. Markovian bridges and reversible diffusions with jumps.

02-06 Abdallah El Hamidi et Gennady G. Laptev. Existence and Nonexistence Results for Reaction-Diffusion Equations in Product of Cones.

02-07 Guy Wallet. Nonstandard generic points.

02-08 Gilles Bailly-Maitre. On the monodromy representation of polynomials.

02-09 Abdallah El Hamidi. Necessary conditions for local and global solvability of nondiagonal degenerate systems.

02-10 Abdallah El Hamidi et Amira Obeid. Systems of Semilinear higher order evolution inequalities on the Heisenberg group.

03-01 Abdallah El Hamidi et Gennady G. Laptev. Non existence de solutions d'inéquations semilinéaires dans des domaines coniques.

03-02 Eris Benoît et Marie-Joëlle Rochet. A continuous model of biomass size spectra governed by predation and the effects of fishing on them.

03-03 Catherine Stenger: On a conjecture of Wolfgang Wasow concerning the nature of turning points.

03-04 Christian Houtré et Nicolas Privault. Surface measures and related functional inequalities on configuration spaces.

03-05 Abdallah El Hamidi et Mokhtar Kirane. Nonexistence results of solutions to systems of semilinear differential inequalities on the Heisenberg group.

03-06 Uwe Franz, Nicolas Privault et René Schott. Non-Gaussian Malliavin calculus on real Lie algebras.

04-01 Abdallah El Hamidi. Multiple solutions to a nonlinear elliptic equation involving Paneitz type operators.

04-02 Mohamed Amara, Amira Obeid et Guy Vallet. Relaxed formulation and existence result of the degenerated elliptic small disturbance model.

04-03 Hippolyte d'Albis et Emmanuelle Augeraud-Veron. Competitive Growth in a Life-cycle Model: Existence and Dynamics

04-04 Sadja Aït-Mokhtar: Third order differential equations with fixed critical points.

04-05 Mokhtar Kirane et Nasser-eddine Tatar. Asymptotic Behavior for a Reaction Diffusion System with Unbounded Coefficients.

04-06 Mokhtar Kirane, Eric Nabana et Stanislav I. Pohozaev. Nonexistence of Global Solutions to an Elliptic Equation with a Dynamical Boundary Condition.

04-07 Khaled M. Furati, Nasser-eddine Tatar and Mokhtar Kirane. Existence and asymptotic behavior for a convection Problem.

04-08 José Alfredo López-Mimbela et Nicolas Privault. Blow-up and stability of semilinear PDE's with gamma generator.

04-09 Abdallah El Hamidi. Multiple solutions with changing sign energy to a nonlinear elliptic equation.

04-10 Sadja Aït-Mokhtar: A singularly perturbed Riccati equation.

04-11 Mohamed Amara, Amira Obeid et Guy Vallet. Weighted Sobolev spaces for a degenerated nonlinear elliptic equation.

04-12 Abdallah El Hamidi. Existence results to elliptic systems with nonstandard growth conditions.

04-13 Eric Edo et Jean-Philippe Furter: Some families of polynomial automorphisms.

04-14 Laurence Cherfils et Yavdat Il'yasov. On the stationary solutions of generalized reaction diffusion equations with  $p$  &  $q$ -Laplacian.

04-15 Jean-Christophe Breton et Youri Davydov. Local limit theorem for supremum of an empirical processes for i.i.d. random variables.

04-16 Jean-Christophe Breton, Christian Houtré et Nicolas Privault. Dimension free and infinite variance tail estimates on Poisson space.

04-17 Abdallah El Hamidi et Gennady G. Laptev. Existence and nonexistence results for higher-order semilinear evolution inequalities with critical potential.

05-01 Mokhtar Kirane et Nasser-eddine Tatar. Nonexistence of Solutions to a Hyperbolic Equation with a Time Fractional Damping.

05-02 Mokhtar Kirane et Yamina Laskri. Nonexistence of Global Solutions to a Hyperbolic Equation with a Time Fractional Damping.

05-03 Mokhtar Kirane, Yamina Laskri et Nasser-eddine Tatar. Critical Exponents of Fujita Type for Certain Evolution Equations and Systems with Spatio-Temporal Fractional Derivatives.

05-04 Abdallah El Hamidi et Jean-Michel Rakotoson. Compactness and quasilinear problems with critical exponents

05-05 Claudianor O. Alves et Abdallah El Hamidi. Nehari manifold and existence of positive solutions to a class of quasilinear problems.

05-06 Khalid Adriouch et Abdallah El Hamidi. The Nehari manifold for systems of nonlinear elliptic equations.