Prépublications du Département de Mathématiques

Université de La Rochelle
Avenue Michel Crépeau
17042 La Rochelle Cedex 1
http://www.univ-lr.fr/labo/lmca

Locally Finite Polynomial Endomorphisms

J-P. Furter et S. Maubach

Octobre 2005

Classification: 14R10, 17B40.
Mots clés: Polynomial automorphisms, affine algebraic geometry, derivations.

2005/08

Locally Finite Polynomial Endomorphisms

Jean-Philippe FURTER,
Dpt. of Math., Univ. of La Rochelle,
av. M. Crépeau, 17 000 La Rochelle, FRANCE
email: jpfurter@univ-lr.fr

Stefan MAUBACH.
Dpt. of Math., Radboud University Nijmegen (RUN),
postbus 9010, 6500 GL Nijmegen, THE NETHERLANDS
email: s.maubach@science.ru.nl

Corresponding author: Jean-Philippe FURTER.
Fax : 33 (0)5 46 45 82 40.

Abstract.

In this paper, we study polynomial endomorphisms F of $\mathbb{C}^N$ which are locally finite in the following sense: the vector space generated by $r \circ F^n$ $(n \geq 0)$ is finite dimensional for each $r \in \mathbb{C}[x_1, \ldots, x_N]$. We show that such endomorphisms are very similar to the linear ones: they satisfy the Jacobian Conjecture, have vanishing polynomials (we can even define their minimal and characteristic polynomials) and the invertible ones admit a Dunford decomposition in "semisimple" and "unipotent" factors. We also point out some connections with linear recurrent sequences and derivations. Finally, we give particular attention to the special cases where F is nilpotent and where $N = 2$.

Keywords.

Polynomial automorphisms, affine algebraic geometry, derivations.

Mathematics Subject Classification (2000).

14R10, 17B40.

INTRODUCTION.

This paper is devoted to the study of polynomial endomorphisms F of $\mathbb{C}^N$ satisfying the following equivalent assertions (see th. 1.1.): (i) $\dim \underset{n\geq 0}{Span}\ F^n < +\infty$;
(ii) $\sup_{n\geq 0} \deg F^n < +\infty$; (iii) $\dim \underset{n\geq 0}{Span}\ r\circ F^n < +\infty$ for each $r \in \mathbb{C}[x_1,\ldots,x_N]$.

Such polynomial endomorphisms are called locally finite (LF for short) since condition (ii) exactly means that the linear endomorphism $r \mapsto r \circ F$ is LF in a more usual sense (see [10] and def. 1.2 below). However, the most intuitive way of understanding them is probably via condition (i) which means that they satisfy a relation of the shape $p(F) = 0$ where $p \in \mathbb{C}[T]$ is nonzero. One of our motivation for studying these endomorphisms is the Jacobian Conjecture. It is the natural generalization of the well-known theorem asserting that a finite dimensional linear endomorphism is invertible if and only if its determinant is a nonzero constant. For linear endomorphisms, the determinant is connected with the last coefficient of the characteristic polynomial. Furthermore, the characteristic polynomial is a vanishing polynomial by Cayley-Hamilton. Does this connection extend to polynomial endomorphisms ? How interesting this may all sound, the fact is that many (heuristically "almost all") polynomial endomorphisms are *not* LF. Indeed, it is worth noticing that LF endomorphisms constitute a subset of the so called dynamically trivial endomorphisms, i.e. endomorphisms whose dynamical degree $dd(F) := \lim_{n\to\infty} (\deg F^n)^{\frac{1}{n}}$ is equal to one (for automorphisms, it is equivalent to saying that the topological entropy is zero, see [11] and [30]). Nevertheless, surprisingly many polynomial endomorphisms are LF:

1. Affine endomorphisms are LF;
2. Triangular and elementary maps are LF. We recall that an elementary map is of the shape $(x_1,\ldots,x_{L-1},x_L+p,x_{L+1},\ldots,x_N)$, where $p \in \mathbb{C}[x_1,\ldots,\widehat{x_L},\ldots,x_N]$;
3. The Nagata automorphism $F := (x-2yw-zw^2, y+zw, z) \in Aut(\mathbb{C}^3)$ where $w = xz+y^2$ is LF. Indeed, it is a zero of $p(T) = (T-1)^3$. This actually means that $F^3-3F^2+3F-I = 0$ which is not the same equality as $(F-I)^3 = 0$ (since F is not linear!);
4. Recently, in [4], de Bondt used so-called quasi-translations as the main tool to obtain strong new results. These quasi-translations are defined as maps of the shape $I+H$ whose inverse is $I-H$. It is not very difficult to check that F is a quasi-translation if and only if F is a zero of $(T-1)^2$;
5. Automorphisms of finite order (i.e. maps satisfying $F^k = I$ for some $k \geq 1$) are LF. However, it is still unknown whether or not these maps are linear up to conjugation;
6. If D is a locally finite derivation (including the locally nilpotent case), then $\exp D$ is a LF automorphism (see II.2). The following natural question seems interesting: is the converse true, i.e. is any LF automorphism the exponential of a LF derivation?
7. Nilpotent endomorphisms are LF.

So, even though "very few" endomorphisms are LF, they constitute an important subclass and this paper is a first systematic survey on them. Let us note that it has only been proven recently by Shestakov and Umirbaev that the Nagata automorphism is not tame (see [27] and [28]). This shows incidentally that LF and dynamically trivial endomorphisms are not trivial! At this level, the search for generators of the automorphisms group is wide open! In [10], van den Essen asks if the automorphism group is generated by exponentials of locally nilpotent derivations. Less ambitiously, we can now ask if it is generated by LF automorphisms.

Our paper is divided into four sections. In section I, we define the minimal polynomial (see def. 1.1), prove an extension of the Cayley-Hamilton theorem (see th. 1.2) and relate the theory of LF polynomial endomorphisms to the theory of linear recurrent sequences (see prop. 1.3). In section II, we study the case of automorphisms. We give a Dunford decomposition (see th. 2.1) and explain some (possible) connections with LF derivations. In section III, we show that if F is a nilpotent polynomial endomorphism of $\mathbb{C}^N$, then $F^N = 0$ (see th. 3.1). In section IV, we have a close look at the dimension two. We can use the amalgamated structure of the automorphisms group and everything is getting simpler. Let F be a LF polynomial endomorphism of $\mathbb{C}^2$ satisfying $F(0) = 0$. If $d = \deg F$, we define an explicit vanishing polynomial of degree $\dfrac{d(d+3)}{2}$. Furthermore, we show that the minimal polynomial of F has degree at most $d+1$.

I. GENERALITIES.

1. LF ENDOMORPHISMS.

Let us denote by $\mathbb{A}^N = \mathbb{C}^N$ the complex affine space of dimension N and by $End = End(\mathbb{A}^N)$ the set of polynomial endomorphisms of $\mathbb{A}^N$. As usual, we identify an element F of End to the N-uple of its coordinate functions $F = (F_1, \ldots, F_N)$ where each F_L belongs to the ring $\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_N]$ of regular functions on $\mathbb{A}^N$. We set $\deg F = \max\limits_{1 \leq L \leq N} \deg F_L$. Let us denote by $F^\# : \mathbb{C}[X] \to \mathbb{C}[X]$, $r \mapsto r \circ F$, the $\mathbb{C}$-algebra morphism associated to F. To simplify the notations, we use the indeterminates x, y, z instead of the x_L when $N \leq 3$.
Let us recall that a (complex) near-algebra A is a linear space on which a composition is defined such that (i) A forms a semigroup under composition; (ii) composition is right distributive with respect to addition (i.e. $(a+b) \circ c = a \circ c + b \circ c$ for all $a, b, c \in A$); (iii) $\lambda(a \circ b) = (\lambda a) \circ b$ for all $a, b \in A$ and $\lambda \in \mathbb{C}$. If $a \in A$, we set $\mathcal{I}_a := \{p \in \mathbb{C}[T],\ p(a) = 0\}$. Since $\mathcal{I}_a$ is a vector subspace of $\mathbb{C}[T]$ which is stable by multiplication by T, it is clear that $\mathcal{I}_a$ is an ideal of $\mathbb{C}[T]$.

Example 1.1. If l belongs to the algebra $\mathcal{L}(V)$ of linear endomorphisms of a vector space V, it is well known that $\mathcal{I}_l$ is an ideal of $\mathbb{C}[T]$. If W is a subspace which is stable by l and if $l_{||W} \in \mathcal{L}(W)$ denotes the induced endomorphism, let us note that $\mathcal{I}_l \subset \mathcal{I}_{l_{||W}}$.

Example 1.2. If F belongs to the near-algebra $End(\mathbb{A}^N)$, $\mathcal{I}_F$ is an ideal of $\mathbb{C}[T]$, but since $F^\# \in \mathcal{L}(\mathbb{C}[X])$, $\mathcal{I}_{F^\#}$ is also an ideal of $\mathbb{C}[T]$. In general we do not have $\mathcal{I}_F = \mathcal{I}_{F^\#}$ (see th. 2.2), but only $\mathcal{I}_{F^\#} \subset \mathcal{I}_{(F^\#)_{||W}} = \mathcal{I}_F$, where $W = \text{Span}\left((F^\#)^n(x_L)\right)_{n \in \mathbb{N},\ 1 \leq L \leq N.}$ Indeed, $p(F) = 0 \iff \forall L,\ x_L \circ p(F) = 0$, i.e. $p(F^\#)\,(x_L) = 0 \iff W \subset \text{Ker } p(F^\#)$.

Definition 1.1. If a belongs to a near-algebra A and if $\mathcal{I}_a \neq 0$, we define the minimal polynomial μ_a of a as the (unique) monic polynomial generating the ideal $\mathcal{I}_a$.

We now recall a few things on LF linear endomorphisms. If l is a linear endomorphism of a vector space V, let us denote by $\mathcal{F}(l)$ the set of finite dimensional subspaces W of V such that $l(W) \subset W$.

Definition 1.2. A linear endomorphism l is LF if it satisfies the following equivalent assertions (see [10]): (i) $\dim \underset{n \geq 0}{Span}\ l^n(v) < +\infty$ for each $v \in V$; (ii) $V = \bigcup\limits_{W \in \mathcal{F}(l)} W$;

(iii) any finite dimensional subspace of V is included into some $W \in \mathcal{F}(l)$.

In other words: l is LF if it is an (inductive) limit of finite dimensional linear endomorphisms. Indeed, it is uniquely determined by $l_{||W}$, $W \in \mathcal{F}(l)$. Therefore, most definitions made in the finite dimensional case extend to the LF case (see [10]):

Definition 1.3. A LF endomorphism l is *semisimple* (resp. *unipotent*, resp. *locally nilpotent*) if $l_{||W}$ is semisimple (resp. unipotent, resp. nilpotent) for each $W \in \mathcal{F}(l)$.

By applying the additive Jordan decomposition to each $l_{||W}$, we obtain the additive Jordan decomposition for l: there exist unique LF endomorphisms l_s, l_n such that:
(i) $l = l_s + l_n$ with $l_s \circ l_n = l_n \circ l_s$; (ii) l_s is semisimple ; (iii) l_n is locally nilpotent.
In the same way, we obtain the multiplicative Jordan decomposition (or Dunford decomposition) in the invertible case: there exist unique LF endomorphisms l_s, l_u such that: (i) $l = l_s \circ l_u = l_u \circ l_s$; (ii) l_s is semisimple ; (iii) l_u is unipotent.

Theorem 1.1. Let $F \in End$. The three following assertions are equivalent:
(i) $\mathcal{I}_F \neq \{0\}$; (ii) $\sup\limits_{n \geq 0} \deg F^n < +\infty$; (iii) $F^{\#}$ is LF.

Proof. (i) $\Longrightarrow$ (ii). If $F^d = a_{d-1}F^{d-1} + \ldots + a_0 F^0$, an easy induction would show that $F^n \in \mathrm{Span}(F^0, \ldots, F^{d-1})$ (for each $n \geq 0$), so that $\deg F^n \leq C := \max\limits_{0 \leq k \leq d-1} \deg F^k$.
(ii) $\Longrightarrow$ (iii). If $r \in \mathbb{C}[X]$ and $\deg F^n \leq C$ for any n, then $\deg r \circ F^n \leq \deg r \times C$, so that $\dim \underset{n \geq 0}{Span}\ r \circ F^n < +\infty$.
(iii) $\Longrightarrow$ (i). If W is as in ex. 1.2, then $\dim W < +\infty$, so that $\mathcal{I}_{(F^{\#})_{||W}} \neq \{0\}$. $\square$

Definition 1.4. A polynomial endomorphism F is LF if it satisfies (i)-(iii) of th. 1.1.

As in the linear case, we can show the following result.

Proposition 1.1. If $F \in End$ is LF, the five following assertions are equivalent:
(i) F is an automorphism; (ii) F is injective; (iii) F is surjective;
(iv) $\mu_F(0) \neq 0$; (v) Jac $F \neq 0$ (where Jac F is the Jacobian determinant of F).

Proof. (i) and (ii) are equivalent even if F is not LF (see prop. 17.9.6 p. 80 in [15] for the original idea, but the precise result is proven in [2], [5], [3], [8] and [24]). (i) $\Longrightarrow$ (iii) and (i) $\Longrightarrow$ (v) are obvious. Let us prove (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (ii) and (v) $\Longrightarrow$ (i).
(iii) $\Longrightarrow$ (iv). If we had $\mu_F(0) = 0$, then $p(T) := \mu_F(T)\, T^{-1} \in \mathbb{C}[T]$ and $p(F) \circ F = 0$. Since F is onto, this would imply $p(F) = 0$ contradicting the definition of μ_F.
(iv) $\Longrightarrow$ (ii). If $\mu_F(0) \neq 0$, there exists $p \in \mathbb{C}[T]$ such that $p(T)T \equiv 1 \bmod \mu_F(T)$, so that $p(F) \circ F = I$ and F is injective.
(v) $\Longrightarrow$ (i). If F is not an automorphism, we have $\mu_F(0) = 0$ and we have seen that $p(F) \circ F = 0$ where $p(T) := \mu_F(T)\, T^{-1} \in \mathbb{C}[T]$. Since $p(F) \neq 0$ (by definition of μ_F), there exists some nonzero component $r \in \mathbb{C}[X]$ of the endomorphism $p(F)$. We have $r(F_1, \ldots, F_N) = 0$, which shows that $F_1, \ldots, F_N$ are algebraically dependant over $\mathbb{C}$. This last condition is equivalent to Jac $F = 0$ (see [23] and [14]). $\square$

Corollary 1.1. If F is LF, then Jac F is a constant.

Corollary 1.2. If F is LF, then the Jacobian conjecture holds for F, i.e. F is an automorphism if and only if Jac F is a nonzero constant.

2. THE CHARACTERISTIC POLYNOMIAL.

If F is a finite dimensional linear endomorphism, the theorem of Cayley-Hamilton shows us that $\chi_F(F) = 0$ where χ_F is the (classical) characteristic polynomial of F. Let us note that this characteristic polynomial χ_F is given by a closed formula. If F is a LF polynomial endomorphism, we would like to find a closed formula giving a polynomial χ_F such that $\chi_F(F) = 0$. The next result gives us a partial answer since it allows us to find a vanishing polynomial of F depending only on the linear part $\mathcal{L}(F)$ of F and on $\sup_{n\in\mathbb{N}} \deg F^n$. However, there remains the problem of computing $\sup_{n\in\mathbb{N}} \deg F^n$.

Theorem 1.2. Let $F \in End(\mathbb{A}^N)$ be such that $F(0) = 0$ and $d := \sup_{n\in\mathbb{N}} \deg F^n < +\infty$. If $(\lambda_L)_{1\leq L\leq N}$ are the eigenvalues of $\mathcal{L}(F)$ and if we set $\lambda^\alpha := \prod_L \lambda_L^{\alpha_L}$ and $|\alpha| := \sum_L \alpha_L$ for $\alpha = (\alpha_L)_L \in \mathbb{N}^N$ then $\prod_{\substack{\alpha\in\mathbb{N}^N\\ 0<|\alpha|\leq d}} (T - \lambda^\alpha)$ is a vanishing polynomial of F.

Our proof will use the next two lemmas. We recall a few facts about symmetric powers (for more details, see chap. 3, § 6 in [6], app. 2 in [9] or any book dealing with multilinear algebra). If E is a vector space with basis $e_1, \ldots, e_N$, the k-th symmetric power of E, denoted by $\mathrm{Sym}^k E$, is naturally isomorphic to the vector space whose elements are the k-homogeneous polynomials in the indeterminates $e_1, \ldots, e_N$. Since any element of E can be thought of as a 1-homogeneous polynomial in the indeterminates $e_1, \ldots, e_N$, we have $E \simeq \mathrm{Sym}^1 E$. In the same way, $a_1 \ldots a_k$ can be seen as an element of $\mathrm{Sym}^k E$ where all a_L belong to E. Finally, if $u : E \to F$ is a linear map, $\mathrm{Sym}^k u : \mathrm{Sym}^k E \to \mathrm{Sym}^k F$ is the unique linear map sending $a_1 \ldots a_k \in \mathrm{Sym}^k E$ to $u(a_1) \ldots u(a_k) \in \mathrm{Sym}^k F$.

Lemma 1.1. Let E be a finite dimensional complex vector space and let $u \in \mathcal{L}(E)$. If we write the characteristic polynomial of u under the shape $\chi(u, E) = \prod_{1\leq L\leq N} (T - \lambda_L)$, then the characteristic polynomial of the k-th symmetric power $\mathrm{Sym}^k u \in \mathcal{L}(\mathrm{Sym}^k E)$ is the polynomial $\chi(\mathrm{Sym}^k u, \mathrm{Sym}^k E) = \prod_{\substack{\alpha\in\mathbb{N}^N\\ |\alpha|=k}} (T - \lambda^\alpha)$.

Proof. It is a classical result. Let us prove it anyway for the sake of completeness. Let $(e_1, \ldots, e_N)$ be a basis of E such that the matrix of u in this basis is an upper triangular matrix $\begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix}$, i.e. $\forall\, L,\ u(e_L) - \lambda_L e_L \in \mathrm{Span}(e_M)_{M<L}$.

If $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, let us set $e^\alpha := e_1^{\alpha_1} \ldots e_N^{\alpha_N} \in \mathrm{Sym}^{|\alpha|} E$. Let $M := \{e^\alpha,\ \alpha \in \mathbb{N}^N\}$ be the set of all monomials in $e_1, \ldots, e_N$ and let us endow M with any monomial order $\prec$ such that $e_1 \prec e_2 \prec \ldots \prec e_N$ (we say that $\prec$ is a monomial order if $m_1 \prec m_2$ implies $m_1 \prec mm_1 \prec mm_2$ for any $m, m_1, m_2 \in M$ with $m \neq 1$, see [9]). We could for example take the orders $\prec_1$ or $\prec_2$ defined by
$e^\alpha \prec_1 e^\beta \iff \alpha_L < \beta_L$ for the last integer L such that $\alpha_L \neq \beta_L$ and
$e^\alpha \prec_2 e^\beta \iff \alpha_L > \beta_L$ for the first integer L such that $\alpha_L \neq \beta_L$
where $\alpha = (\alpha_1, \ldots, \alpha_N),\ \beta = (\beta_1, \ldots, \beta_N) \in \mathbb{N}^N$.

It is well known that $M_k := \{e^\alpha, |\alpha| = k\}$ is a basis of $\mathrm{Sym}^k E$. Furthermore, since $\prec$ is a monomial order: $\forall\, e^\alpha \in M_k,\ \mathrm{Sym}^k u(e^\alpha) - \lambda^\alpha e^\alpha \ \in\ \mathrm{Span}(e^\beta)_{e^\beta \in M_k}$ and $e^\beta \prec e^\alpha$.
The matrix of $\mathrm{Sym}^k u$ in the basis e^α where the e^α are taken with the order $\prec$ is upper triangular with the λ^α on the diagonal. $\square$

We will omit the proof of the following usual result.

Lemma 1.2. Let E be a finite dimensional complex vector space and let $u \in \mathcal{L}(E)$ be a linear endomorphism of E. Let us assume that $E = E_1 \supset E_2 \supset \ldots \supset E_d \supset E_{d+1} = \{0\}$ is a filtration of E by subspaces which are stable by u (i.e. $u(E_k) \subset E_k$). If $\chi(u, E)$ denotes the characteristic polynomial of u and if $\chi(u, E_k/E_{k+1})$ denotes the characteristic polynomial of the endomorphism induced by u on E_k/E_{k+1}, then

$$\chi(u, E) \ = \ \prod_{1 \le k \le d} \chi(u, E_k/E_{k+1}).$$

Proof of th. 1.2. If W is defined as in ex. 1.2, then $W \in \mathcal{F}(F^\#)$ and $\chi(F^\#, W)$ is a vanishing polynomial of F. Let $\mathcal{M}$ be the maximal ideal of $\mathbb{C}[X]$ generated by $x_1, \ldots, x_N$. Since $F(0) = 0$, we have $F^\#(\mathcal{M}) \subset \mathcal{M}$, so that $F^\#(\mathcal{M}^k) \subset \mathcal{M}^k$ (for $k \ge 0$). If we set $W_k := W \cap \mathcal{M}^k$ (for $1 \le k \le d+1$), then W_k is stable by $F^\#$ and we have the filtration: $W = W_1 \supset W_2 \supset \ldots \supset W_d \supset W_{d+1} = \{0\}$. By lemma 1.2, we have $\chi(F^\#, W) = \prod_{1 \le k \le d} \chi(F^\#, W_k/W_{k+1})$. But, there is a natural embedding of $W_k/W_{k+1} = W \cap \mathcal{M}^k \ / \ W \cap \mathcal{M}^{k+1}$ in $\mathcal{M}^k/\mathcal{M}^{k+1}$, so that $\chi(F^\#, W_k/W_{k+1})$ divides $\chi(F^\#, \mathcal{M}^k/\mathcal{M}^{k+1})$. Let us denote by $u_k \in \mathcal{L}(\mathcal{M}^k/\mathcal{M}^{k+1})$ the linear endomorphism induced by $F^\#$ on $\mathcal{M}^k/\mathcal{M}^{k+1}$. If $k = 1$, $\mathcal{M}/\mathcal{M}^2$ is classically called the cotangent space at the origin of the affine space $\mathbb{A}^N$. The dual map of u_1 is naturally identified to the differential at the origin of the map $F : \mathbb{A}^N \to \mathbb{A}^N$, which is itself identified to the linear part $\mathcal{L}(F)$ of F, so that $\chi(F^\#, \mathcal{M}/\mathcal{M}^2) = \prod_{1 \le L \le N} (T - \lambda_L)$. If $k \ge 1$ is any integer, $\mathcal{M}^k/\mathcal{M}^{k+1}$ is naturally isomorphic to $\mathrm{Sym}^k(\mathcal{M}/\mathcal{M}^2)$ and u_k is naturally identified to $\mathrm{Sym}^k u_1$. Therefore, by lemma 1.1, we have $\chi(F^\#, \mathcal{M}^k/\mathcal{M}^{k+1}) = \prod_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| = k}} (T - \lambda^\alpha)$. $\square$

3. LINEAR RECURRENT SEQUENCES.

We now introduce the language of linear recurrent sequences (LRS for short), because they are a nice tool for some proofs (see section IV). Let V be any complex vector space. The set of sequences $u : \mathbb{N} \to V$ will be denoted by $V^{\mathbb{N}}$. If $p = p(T) = \sum_k p_k\, T^k \in \mathbb{C}[T]$, we define $p(u) \in V^{\mathbb{N}}$ by the formula $\forall\, n \in \mathbb{N},\ \big(p(u)\big)(n) = \sum_k p_k\, u(n+k)$.

The theory of LRS relies on the next result (see [7]).

Proposition 1.2. Let $u = u(n)_{n \in \mathbb{N}} \in V^{\mathbb{N}}$ and let p be a nonzero polynomial of $\mathbb{C}[T]$. If $p(T) = \alpha \prod_{1 \le k \le c} (T - \omega_k)^{r_k}$ is the decomposition into irreducible factors of p, then the two following assertions are equivalent: (i) $p(u) = 0$; (ii) there exist $q_1, \ldots, q_c \in V[T]$ with

$\deg q_k \ \le r_k - 1$ such that $\forall\ n,\ u(n) \ = \ \sum_{1\le k\le c} \omega_k^n q_k(n) \qquad (*).$

Remarks. 1. The vector space $V[T]$ is the set of polynomials in T with coefficients in V alias the set of "polynomial" maps from $\mathbb{C}$ to V.
2. The expression (*) is called an exponential-polynomial. We say that u is polynomial (resp. of exponential type) if $c = 1$ and $\omega_1 = 1$ (resp. all the q_k are constant).
3. In the case where u is of exponential-type, we will sometimes be more precise and say that u is of Ω-exponential type, where $\Omega := \{\omega_1, \ldots, \omega_c\}$. If u (resp. u') is a complex sequence of Ω (resp. Ω')-exponential type, then it is obvious that $u + v$ (resp. uv) is of $\Omega\cup\Omega'$ (resp. $\Omega.\Omega'$)-exponential type. In particular, if $u_1, \ldots, u_e$ are of Ω exponential type, then $u_1u_2\ldots u_e$ is of Ω^e-exponential type, where $\Omega^e = \underbrace{\Omega.\Omega.\ldots.\Omega}_{e}$. Therefore, if u_1, u_2 are of Ω-exponential type and if $q(x, y) \in \mathbb{C}[x, y]$ is such that $q(0, 0) = 0$ and $\deg q \ \le e$, then the sequence $q(u_1, u_2)$ is of $\bigcup_{1\le k\le e} \Omega^k$-exponential type.

Using prop. 1.2, it is clear that if $u \in V^{\mathbb{N}}$, then $\mathcal{I}_u := \{p \in \mathbb{C}[T],\ p(u) = 0\}$ is an ideal of $\mathbb{C}[T]$.

Definition 1.5. We say that $u \in V^{\mathbb{N}}$ is a LRS if $\mathcal{I}_u \neq \{0\}$. In this case, we define the minimal polynomial of u as the (unique) monic polynomial μ_u generating the ideal $\mathcal{I}_u$.

Remarks. 1. The LRS are classically complex sequences, but we found it convenient to extend their definition to the case of vector spaces.
2. A LRS is polynomial (resp. of exponential type) if and only if its minimal polynomial is of the shape $(T - 1)^m$ (resp. has only single roots).
3. Let E be a finite dimensional vector space and let $F \in \mathcal{L}(E)$ be a linear endomorphism of E. It is a classical fact that F is unipotent (resp. semisimple) if and only if the sequence $(F^n)_{n\in\mathbb{N}}$ is polynomial (resp. of exponential type). We will later on generalize this definition to the case of LF polynomial endomorphisms.

Proposition 1.3. If $F \in End$ and $u := (F^n)_{n\in\mathbb{N}} \in End^{\mathbb{N}}$, then $\mathcal{I}_F = \mathcal{I}_u$. In particular, F is LF if and only if u is a LRS. If it is the case, we have $\mu_F = \mu_u$.

Proof. If $p = \sum_k p_k T^k \in \mathbb{C}[T], \quad \sum_k p_k\ F^k = 0 \Longleftrightarrow \forall\ n \in \mathbb{N},\ \sum_k p_k\ F^{k+n} = 0.$ $\square$

Remark. If $F \in End$ is LF, then $(F^n(a))_{n\in\mathbb{N}}$ is a LRS for any $a \in \mathbb{A}^N$, but the converse is false: take $F = (xy, y) \in End(\mathbb{A}^2)$. If $\mathbb{C}(X) := \mathbb{C}(x_1, \ldots, x_N)$ and $K := \{r \in \mathbb{C}(X),\ r \circ F = r\}$, it is shown in [13] that the following assertions are equivalent :
(i) $(F^n(a))_{n\in\mathbb{N}}$ is a LRS for any a; (ii) $p(F) = 0$ for some nonzero $p \in K[T]$.

II. LF AUTOMORPHISMS.

1. DUNFORD DECOMPOSITION.

Proposition 2.1. If $F \in End$ is LF, the following assertions are equivalent:
(i) $F^{\#}$ is unipotent; (ii) $\mu_F = (T - 1)^m$ for some $m \ge 0$;
(iii) the sequence $(F^n)_{n\in\mathbb{N}}$ is polynomial.

If $F(0) = 0$, these assertions are still equivalent to the following one:
(iv) the linear map $\mathcal{L}(F)$ is unipotent.

Proof. (i) $\Longrightarrow$ (ii). Let W be as in ex. 1.2. Since $F^{\#}_{||W}$ is unipotent, its characteristic polynomial is equal to $\chi(F^{\#}, W) = (T-1)^{\dim W}$ and it is a vanishing polynomial of F.
(ii) $\Longleftrightarrow$ (iii) is obvious from the theory of LRS.
(iii) $\Longrightarrow$ (i) Let $W \in \mathcal{F}(F^{\#})$. We want to show that $F^{\#}_{||W}$ is unipotent.
But for all $w \in W$, the sequence $n \mapsto \big(F^{\#}\big)^n(w)$ is polynomial since $\big(F^{\#}\big)^n(w) = w \circ F^n$. This implies that the sequence $n \mapsto \Big(F^{\#}_{||W}\Big)^n$ is polynomial and this means that $F^{\#}_{||W}$ is unipotent (see rem. 3 following def. 1.5).

Let us now assume that $F(0) = 0$.
(iii) $\Longrightarrow$ (iv). Since $F(0) = 0$, we have $\mathcal{L}(F^n) = \mathcal{L}(F)^n$ and since the sequence $(F^n)_{n\in\mathbb{N}}$ is polynomial, the sequence$(\mathcal{L}(F)^n)_{n\in\mathbb{N}}$ also, so that $\mathcal{L}(F)$ is unipotent.
(iv) $\Longrightarrow$ (ii). We know that the characteristic polynomial of $\mathcal{L}(F)$ is equal to $(T-1)^N$. Therefore, by th. 1.2, F admits a vanishing polynomial of the shape $(T-1)^p$. $\square$

Definition 2.1. If F satisfies (i)-(iii) of prop. 2.1, we say that F is unipotent.

Example. If the Nagata automorphism is LF, it has to be unipotent by prop. 2.1. It is indeed the case because one checks easily that its minimal polynomial is $(T-1)^3$.

If $F(0) \neq 0$, let us show by two examples that (i)-(iii) and (iv) are independant. We take $N = 2$. If $F = (F_1, F_2) \in End(\mathbb{A}^2)$ and $a \in \mathbb{A}^2$, $F'(a)$ will denote the Jacobian matrix of F at the point a and we will identify $\mathcal{L}(F)$ and $F'(0)$. Let us set $a := (1,1) \in \mathbb{A}^2$ and let us consider the group H of all automorphisms φ of $\mathbb{A}^2$ such that $\varphi(0) = 0$, $\varphi'(0) = I$ and $\varphi(a) = a$. If $\varphi \in H$, it is clear that $\varphi'(a) \in SL_2$ since $\det \varphi'(a) = \det \varphi'(0) = 1$. Let us show that the group-morphism $m : H \to SL_2$, $\varphi \mapsto \varphi'(a)$ is onto. If we set $\alpha_u := (x + uy^2(y-1), y)$ and $\beta_u := (x, y + ux^2(x-1)) \in H$ for each $u \in \mathbb{C}$, then $m(\alpha_u) = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$ and $m(\beta_u) = \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix}$. Since SL_2 is generated by these matrices, we actually obtain $m(H) = SL_2$. If G is any automorphism of $\mathbb{A}^2$ such that $G(0) = a$ and if φ is any element of H, then $F := F_{(G,\varphi)} := \varphi^{-1} \circ G \circ \varphi$ satisfies $F'(0) = \varphi'(a)^{-1} G'(0) \varphi'(0) = \varphi'(a)^{-1} G'(0)$ and the equality $F^n = \varphi^{-1} \circ G^n \circ \varphi$ shows that F is unipotent if and only if G is unipotent.

First example. If $G := (x+1, y+1)$ and $\varphi \in H$, then $F := F_{(G,\varphi)}$ is unipotent and $F'(0) = \varphi'(a)^{-1}$. Therefore, if we choose φ such that $m(\varphi) = \varphi'(a)$ is not unipotent, then $\mathcal{L}(F) = F'(0)$ will not be unipotent. We can just take $\varphi := \alpha_1 \circ \beta_1$, because

$$\varphi'(a) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ is not unipotent.}$$

Second example. If $G := (1-x, 1-y)$ and $\varphi \in H$, then $F := F_{(G,\varphi)}$ is not unipotent and $F'(0) = -\varphi'(a)^{-1}$. Therefore, if we choose φ such $-m(\varphi) = -\varphi'(a)$ is unipotent, then $\mathcal{L}(F) = F'(0)$ will be unipotent. We can just take $\varphi := (\alpha_2 \circ \beta_{-1})^2$, because

$$\varphi'(a) = \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right)^2 = -I.$$

The next result could be proven in the same way we prove prop. 2.1:

Proposition and definition 2.2. If F satisfies the following equivalent assertions, we say that F is semisimple: (i) $F^{\#}$ is semisimple; (ii) μ_F has single roots;
(iii) the sequence $(F^n)_{n\in\mathbb{N}}$ is of exponential type.

Remark. If F is semisimple and $F(0) = 0$, one could show that $\mathcal{L}(F)$ is semisimple. The converse is false even if $F(0) = 0$ (take the Nagata automorphism).

We can now state the Dunford decomposition for LF polynomial automorphisms.

Theorem 2.1. Let F be a LF polynomial automorphism of $\mathbb{A}^N$, then there exist unique LF polynomial automorphisms F_s and F_u such that
(i) $F = F_s \circ F_u = F_u \circ F_s$; (ii) F_s is semisimple ; (iii) F_u is unipotent.

The proof is a direct consequence of the following result applied to $F^{\#}$:

Lemma 2.1. If l is a LF automorphism of a $\mathbb{C}$-algebra A, then its semisimple and unipotent parts (l_s and l_u) are algebra-morphisms.

Proof. Let $a, b \in A$. We want to show that $l_s(ab) = l_s(a)l_s(b)$ and $l_u(ab) = l_u(a)l_u(b)$. Let $W \in \mathcal{F}(l)$ be such that a, b and $ab \in W$. Let $H \subset GL(W)$ be the closed subgroup defined by $H := \{h \in GL(W),\ h(ab) = h(a)h(b)\}$. Since $l_{||W} \in H$, by the classical Dunford decomposition for linear algebraic groups (see [16]), we know that the semisimple and unipotent parts of $l_{||W}$ still belong to H. $\square$

Lemma 2.2. If a unipotent automorphism F of $\mathbb{A}^N$ satisfies $\mathcal{I}_{F^{\#}} \neq \{0\}$, then $F = I$.

Proof. Let $r \in \mathbb{C}[X]$. Since the sequence $n \mapsto (F^{\#})^n(r)$ is polynomial, its minimal polynomial is of the shape $\mu_r = (T-1)^{m_r}$, where $m_r \geq 0$ is an integer.
However, since $I_{F^{\#}} \neq \{0\}$, the sequence $n \mapsto (F^{\#})^n$ is a LRS with minimal polynomial μ. The polynomial μ is the least common multiple of the μ_r ($r \in \mathbb{C}[X]$). This shows that $\mu = (T-1)^m$, where $m = \max\limits_r m_r$. Let us show by contradiction that $m = 1$. Otherwise, let $r \in \mathbb{C}[X]$ be such that $m_r = m \geq 2$. This means that the sequence $n \mapsto (F^{\#})^n(r)$ is polynomial of degree $m-1$. Therefore, the sequence $n \mapsto (F^{\#})^n(r^2)$ is polynomial of degree $2(m-1)$, showing that $m_{r^2} = 2m-1 > m$. This is impossible. $\square$

Theorem 2.2. The only automorphisms F of $\mathbb{A}^N$ such that $\mathcal{I}_{F^{\#}} \neq \{0\}$ are the automorphisms of finite order.

Proof. If $F^k = I$, we clearly have $(F^{\#})^k = I$ and $T^k - 1 \in \mathcal{I}_{F^{\#}}$.
Let us now assume that F is an automorphism of $\mathbb{A}^N$ such that $\mathcal{I}_{F^{\#}} \neq \{0\}$. Let F_s (resp. F_u) be its semisimple (resp. unipotent) part. If l is a linear endomorphism, let $\mathcal{E}(l)$ be the set of its eigenvalues. Since $\mathcal{E}(F^{\#})$ is a finite subset of $\mathbb{C}^*$ (because $\mathcal{I}_{F^{\#}} \neq \{0\}$) which is stable by multiplication (because $F^{\#}$ is an algebra-morphism), it is a finite subgroup of $\mathbb{C}^*$, so that it is equal to some $U_k := \{z \in \mathbb{C},\ z^k = 1\}$. However, $\mathcal{E}(F_s^{\#}) = \mathcal{E}(F^{\#})$, so that $(F_s)^k = I$. The automorphism $G := F^k = (F_u)^k$ is unipotent and satisfies $\mathcal{I}_{G^{\#}} \neq \{0\}$. By lemma 2.2, we have $G = I$. $\square$

2. DERIVATIONS.

We begin to note that the exponential of a LF linear endomorphism $l : V \to V$ is well defined by $(\exp l)_{||W} := \exp\ l_{||W}$, $W \in \mathcal{F}(l)$. We observe that $\exp l$ is LF.

Lemma 2.3 (i) the exponential defines a surjective map from the LF linear endomorphisms of V to the LF linear automorphisms of V;
(ii) the exponential defines a bijective map from the locally nilpotent linear endomorphisms to the LF unipotent automorphisms.

Proof. If V is finite dimensional, it is well known. If V is any vector space, (ii) is a direct consequence of the finite dimensional case. The assertion (i) is more complicated. It is easy to show that the exponential of a LF endomorphism is an automorphism. Let us rather prove that if l is a LF automorphism, then there exists a LF endomorphism m such that $\exp m = l$. Let $l = l_s \circ l_u$ be the Dunford decomposition of l.
If $\lambda \in \mathbb{C}$, the characteristic space N_λ of l is defined by $N_\lambda := \bigcup_{k\in\mathbb{N}} \mathrm{Ker}(l - \lambda\, I)^k$. Since l is a LF automorphism, it is easy to prove that $V = \bigoplus_{\lambda\in\mathbb{C}^*} N_\lambda$. Furthermore, it is well known that $l_{s||N_\lambda} = \lambda\, I_{N_\lambda}$. For each $\lambda \in \mathbb{C}^*$, let us choose $\ln\lambda \in \mathbb{C}$ such that $\exp(\ln\lambda) = \lambda$ (of course, the map $\ln\ :\ \mathbb{C}^* \to \mathbb{C}$ is not continuous !).
There exists a unique endomorphism $m_s \in \mathcal{L}(V)$ such that $m_{s||N_\lambda} = (\ln\lambda)\, I_{N_\lambda}$, $\lambda \in \mathbb{C}^*$. It is clear that m_s is a LF (semisimple) endomorphism such that $\exp m_s = l_s$. Also, since l_u is unipotent, by (ii), there exists a unique locally nilpotent endomorphism m_u such that $\exp m_u = l_u$.
Since $l = l_s \circ l_u = \exp m_s \circ \exp m_u$, in order to see that $l = \exp(m_s + m_u)$ it remains to show that m_s and m_u commute (in particular, if m_s and m_u commute, $m_s + m_u$ will still be LF !). But this is clear, because for each $\lambda \in \mathbb{C}^*$ we have $m_s(N_\lambda) \subset N_\lambda$, $m_u(N_\lambda) \subset N_\lambda$ and $m_{s||N_\lambda} = (\ln\lambda)\, I_{N_\lambda}$ so that $m_{s||N_\lambda}$ commutes with any endomorphism of N_λ ! $\square$

Let us recall that a derivation of $\mathbb{C}[X]$ is an operator of the shape $D = \sum_{1\le L\le N} a_L \dfrac{\partial}{\partial x_L}$ where the a_L belong to $\mathbb{C}[X]$ (see [10]). It turns out that if D is a LF derivation of $\mathbb{C}[X]$, then $\exp D$ is a (LF) algebra-automorphism of $\mathbb{C}[X]$. Therefore, there exists a (LF) polynomial automorphism F of $\mathbb{A}^N$ such that $F^\# = \exp D$. One often writes (improperly) $F = \exp D$ and we have of course $F = ((\exp D)(x_1), \ldots, (\exp D)(x_N))$.

If we assume furthermore that D is locally nilpotent, then we know that $F^\#$ is a (LF) unipotent linear automorphism, which means that F is unipotent. Conversely, if F (and therefore $F^\#$) is unipotent, we know that there exists a unique locally nilpotent linear endomorphism D of $\mathbb{C}[X]$ such that $\exp D = F^\#$. Moreover, D must be a derivation. Indeed, for any locally nilpotent linear endomorphism l of a $\mathbb{C}$-algebra A, the two following assertions are equivalent (see ex. 6, p. 50 of [10]):
(i) $\exp l$ is an algebra-morphism ; (ii) l is a derivation.
Hence, we have shown the following result.

Theorem 2.3. The exponential defines a bijective map from the locally nilpotent derivations of $\mathbb{C}[X]$ to the unipotent polynomial automorphisms of $\mathbb{A}^N$.

Example. Since the Nagata automorphism is unipotent (see the remark following def. 2.1), it is the exponential of a locally nilpotent derivation (see [29]).

If F is any LF polynomial automorphism of $\mathbb{A}^N$, there still exists a LF linear endomorphism D such that $F^\# = \exp D$ (by lemma 2.3), but D does not need to be a derivation

! However, there exist infinitely many D such that $F^{\#} = \exp D$ and one can ask our main question.

Question 2.1. Is any LF polynomial automorphism of $\mathbb{A}^N$ the exponential of a LF derivation of $\mathbb{C}[X]$?

We were even not able to answer the following subquestion.

Question 2.2. Is any semisimple polynomial automorphism of $\mathbb{A}^N$ the exponential of a semisimple derivation of $\mathbb{C}[X]$?

Remark. Of course, if l is a LF linear endomorphism, then l is semisimple if and only if $\exp l$ is semisimple (this is just the generalization of the corresponding fact in the finite dimensional case).

At this point, let us recall that a famous linearization conjecture asserts that if F is a finite order automorphism of $\mathbb{A}^N$ (i.e. $F^k = I$ for some non negative integer k), then F should be conjugate to some linear automorphism (i.e. there should exist an automorphism φ such that $\varphi \circ F \circ \varphi^{-1}$ is linear). This conjecture is still open for $N \geq 3$. Since the polynomial $T^k - 1$ has single roots, F is necessarily semisimple. One can generalize the linearization conjecture in the following manner.

Question 2.3. Is any semisimple polynomial automorphism of $\mathbb{A}^N$ linearizable ?

It had also been conjectured by Kambayashi in 1979 (see [18] or section 9.4 in [10]) that any (algebraic) action of a reductive algebraic group G on $\mathbb{A}^N$ is linearizable. However, Schwarz gave a counterexample in 1989 (see [25]) for $G = SL_2$ (and some other groups) and Knop gave counterexamples in 1991 (see [19]) when G is any non commutative connected reductive (algebraic) group. What happens if G is a commutative connected reductive group, i.e. $G = (\mathbb{C}^*)^p$ is a torus ? The next question (which seems very difficult) is still open.

Question 2.4. Is any action of a torus $(\mathbb{C}^*)^p$ on the affine space $\mathbb{A}^N$ linearizable ?

It has been pointed to us by Mathieu that a positive answer to question 2.3 would imply a positive answer to question 2.4. Indeed, if we are given an action of $G = (\mathbb{C}^*)^p$ on $\mathbb{A}^N$ and if we choose an element $g \in G$ such that the subgroup generated by g in G is Zariski dense, then the automorphism of $\mathbb{A}^N$ induced by g is semisimple. Therefore, it should be linearizable and the G-action also.

Finally, we can ask a question similar to question 2.3 at the level of derivations.

Question 2.5. Is any semisimple derivation of $\mathbb{C}[X]$ "linearizable" ?
In other words, is it conjugate to some $D = \sum_{1 \leq L \leq N} \lambda_L x_L \frac{\partial}{\partial x_L}$, $\lambda_L \in \mathbb{C}$?

We can express question 2.5 in the following way: does there exist an automorphism $F = (F_1, \ldots, F_N)$ of $\mathbb{A}^N$ such that $F_1, \ldots, F_N$ are eigenvectors of D ? A positive answer to questions 2.2 and 2.5 would imply a positive answer to question 2.3.

III. NILPOTENT ENDOMORPHISMS.

In the linear case, it is well known that if F is a nilpotent linear endomorphism of $\mathbb{C}^N$, then $F^N = 0$. It turns out that this result is still true for polynomial endomorphisms.

Theorem 3.1. Let $F \in End(\mathbb{A}^N)$ be nilpotent, then $F^N = 0$.

Proof. Let F be any polynomial endomorphism of $\mathbb{A}^N$ and let us endow $\mathbb{A}^N$ with the Zariski topology. If k is a non negative integer, we set $V_k := \overline{F^k(\mathbb{A}^N)}$. This is an irreducible closed variety of $\mathbb{A}^N$. Indeed, $F^k(\mathbb{A}^N)$ is irreducible since it is the image of the irreducible variety $\mathbb{A}^N$ and we know that the closure of an irreducible subset remains irreducible. We have $V_{k+1} = \overline{F^k(F(\mathbb{A}^N))} \subset \overline{F^k(\mathbb{A}^N)} = V_k$, so that $\mathbb{A}^N = V_0 \supset V_1 \supset \ldots \supset V_k \supset V_{k+1} \supset \ldots$. Let us show that $V_{k+1} = \overline{F(V_k)}$. We have $F(V_k) = F\left(\overline{F^k(\mathbb{A}^N)}\right) \subset \overline{F(F^k(\mathbb{A}^N))} = V_{k+1}$, whence $\overline{F(V_k)} \subset V_{k+1}$. We have used the fact that if F is a continuous map, then for any set A, we have $F(\overline{A}) \subset \overline{F(A)}$. Indeed, A is a subset of the closed set $F^{-1}\left(\overline{F(A)}\right)$, so that $\overline{A} \subset F^{-1}\left(\overline{F(A)}\right)$, which proves that $F(\overline{A}) \subset \overline{F(A)}$. On the converse $F(V_k) = F\left(\overline{F^k(\mathbb{A}^N)}\right) \supset F\left(F^k(\mathbb{A}^N)\right)$ so that $\overline{F(V_k)} \supset \overline{F^{k+1}(\mathbb{A}^N)} = V_{k+1}$. If we assume that $\dim V_k = \dim V_{k+1}$ for some k, since V_{k+1} is a closed subvariety of the irreducible variety V_k, this implies that $V_{k+1} = V_k$. Hence, we also have $\overline{F(V_{k+1})} = \overline{F(V_k)}$, i.e. $V_{k+2} = V_{k+1}$. Finally, we will have $V_k = V_{k+1} = \ldots = V_n$ for each $n \geq k$. Let us now assume that F is nilpotent and let m be the smallest integer such that $F^m = \{0\}$. If $k < m$, we cannot have $\dim V_k = \dim V_{k+1}$, because otherwise we would have $V_k = V_{k+1} = \ldots = V_m = \{0\}$. Therefore, $N = \dim V_0 > \dim V_1 > \ldots > \dim V_m = 0$ and $m \leq N$. □

Remark. If F is a nilpotent linear endomorphism, it is well known that the sequence $u_n := \dim \operatorname{Im} F^n - \dim \operatorname{Im} F^{n+1}$ is decreasing. In the polynomial case, it is no longer true. If we take the endomorphism $F := (xz, yz, 0)$ of $\mathbb{A}^3$, we have $\dim \operatorname{Im} F^0 = 3$, $\dim \operatorname{Im} F^1 = 2$ and $\dim \operatorname{Im} F^2 = 0$.

IV. DIMENSION TWO.

From now on, we set $N = 2$. In subsection 1 (resp. 2), we analyse LF polynomial endomorphisms of $\mathbb{A}^2$ which are (resp. which are not) invertible. In subsection 3 (resp. 4), we apply these results to characteristic (resp. minimal) polynomials.

1. THE INVERTIBLE CASE.

One of the direct consequences of the amalgamated structure of the group of polynomial automorphisms of $\mathbb{A}^2$ (see [17], [20], [26], [11]) is the well known fact that an automorphism of $\mathbb{A}^2$ is dynamically trivial if and only if it is conjugate to a triangular automorphism. One could show easily that for an automorphism F the following assertions are equivalent (see [12]):

(i) F is dynamically trivial ; (ii) F is triangularizable ; (iii) F is LF ;
(iv) $\deg F^2 \leq \deg F$; (v) $\forall\, n \in \mathbb{N}$, $\deg F^n \leq \deg F$.

In fact, any triangularizable automorphism F can be triangularized in a "good" way with respect to the degree:

Lemma 4.1. If F is a triangularizable automorphism of $\mathbb{A}^2$, then there exist a triangular automorphism G and an automorphism φ such that

$$F = \varphi \circ G \circ \varphi^{-1} \text{ and } \deg F \ = \ \deg G \ (\deg \varphi)^2.$$

Proof. Let Aut be the group of polynomial automorphisms of $\mathbb{A}^2$ and let $\mathcal{A}$ (resp. $\mathcal{T}$) be the subgroup of affine (resp. upper triangular) automorphisms. We have $\mathcal{A} = \{K \in Aut,\ \deg K \ = \ 1\}$ and $\mathcal{T} = \{K = (K_1, K_2) \in Aut,\ \dfrac{\partial K_2}{\partial x_1} \ = \ 0\}$.
Let $F = A^{[1]} \circ T^{[1]} \circ A^{[2]} \circ T^{[2]} \circ \ldots \circ A^{[l]} \circ T^{[l]} \circ A^{[l+1]}$ be a reduced expression of F where the $A^{[k]}$ (resp. $T^{[k]}$) belong to $\mathcal{A}$ (resp. $\mathcal{T}$): this means that $\forall\ k,\ T^{[k]} \notin \mathcal{A}$ and that $\forall\ k \in \{2, \ldots, l\},\ A^{[k]} \notin \mathcal{T}$ (see [26]).
Let B (resp. E) be the composition (in the same order) of the first (resp. last) l terms of the sequence $A^{[1]},\ T^{[1]},\ A^{[2]},\ \ldots,\ A^{[l]},\ T^{[l]},\ A^{[l+1]}$ and let M be the middle term (i.e. $M = A^{[k+1]}$ if $l = 2k$ and $M = T^{[k+1]}$ if $l = 2k+1$), so that we have $F = B \circ M \circ E$.
The triangularizability of F is equivalent to saying that $E \circ B \ \in \ \mathcal{A} \cap \mathcal{T}$ (see prop. 4 of [12]). Thus we have $F = B \circ H \circ B^{-1}$ where $H := M \circ E \circ B \in \mathcal{A} \cup \mathcal{T}$. The first expression of F being reduced, we get $\deg F \ = \ \prod_k \deg T^{[k]} \ = \ \deg B \deg M \deg E$.
But $\deg E \ = \ \deg B^{-1} \ = \ \deg B$ and $\deg M \ = \ \deg M \circ E \circ B \ = \ \deg H$, so that $\deg F \ = \ \deg H (\deg B)^2$. If $H \in \mathcal{T}$, we can just set $\varphi := B$ and $G := H$.
If $H \in \mathcal{A}$, let $A \in \mathcal{A}$ be such that $G := A^{-1} \circ H \circ A \ \in \ \mathcal{A} \cap \mathcal{T}$. We can now just set $\varphi := B \circ A$ and we are done since $\deg G \ = \ \deg H$ $(= \ 1)$ and $\deg \varphi \ = \ \deg B$. $\square$

Remark. If $F(0) = 0$, we can assume that $\varphi(0) = 0$ and $G(0) = 0$ by using the groups $Aut_0 \ := \ \{F \in Aut,\ F(0) = 0\}$, $\mathcal{A}_0 := \mathcal{A} \cap Aut_0$ and $\mathcal{T}_0 := \mathcal{T} \cap Aut_0$.

Before computing a vanishing polynomial for triangularizable automorphisms (see lemma 4.3 below), we deal with the triangular case:

Lemma 4.2. Let $G = (ax + r(y), by)$ be a triangular endomorphism of degree d with $a, b \in \mathbb{C}$ and $r(y) \in \mathbb{C}[y]$ satisfying $r(0) = 0$. Then $p(T) := (T-a)(T-b)(T-b^2)\ldots(T-b^d)$ is a vanishing polynomial of G.

Proof. We may assume that $r = \sum_{l=1}^{d} r_l y^l$ is a fixed polynomial.

<u>First case.</u> We assume that $\forall\ l \in \{1, \ldots, d\},\ a \neq b^l$. By induction, we get (for any $n \geq 0$) $G^n \ = \ \left(a^n x + \sum_{k=0}^{n-1} a^k r(b^{n-1-k} y), b^n y \right)$. But we have

$$\begin{aligned} \sum_{k=0}^{n-1} a^k \ r(b^{n-1-k} y) \ &= \ \sum_{k=0}^{n-1} a^k \sum_{l=1}^{d} r_l y^l \ (b^{n-1-k})^l \ = \ \sum_{l=1}^{d} r_l y^l \sum_{k=0}^{n-1} a^k (b^l)^{n-1-k} \\ &= \ \sum_{l=1}^{d} r_l y^l \ \frac{a^n - (b^l)^n}{a - b^l}. \end{aligned}$$

Therefore there exist endomorphisms $K_0, \ldots, K_l$ such that
$$\forall\ n \in \mathbb{N},\ G^n \ = \ a^n K_0 + b^n K_1 + (b^2)^n K_2 + \ldots + (b^d)^n K_d.$$
If we set $\Omega := \{a, b, \ldots, b^d\}$, this means that the sequence $(G^n)_{n \in \mathbb{N}}$ is of Ω-exponential type (see rem. 3 following prop. 1.2) and this proves our result in this case.

<u>Second case.</u> The general case.

If we set $G_{a,b} := (ax+r(y), by)$ and $p_{a,b} := (T-a)(T-b)(T-b^2)\dots(T-b^d)$, we have shown above that $p_{a,b}(G_{a,b}) = 0$ for all $(a,b) \in \mathbb{C}^2$ outside the curve $(a-b)(a-b^2)\dots(a-b^l) = 0$. Therefore, by density, this equality remains true for any $(a,b) \in \mathbb{C}^2$. $\square$

Lemma 4.3. Let $F = \varphi \circ G \circ \varphi^{-1}$ be an endomorphism of $\mathbb{A}^2$ where φ is an automorphism of degree e with $\varphi(0) = 0$ and where $G = (ax + r(y), by)$ is a triangular endomorphism of degree d with $a, b \in \mathbb{C}$ and $r(y) \in \mathbb{C}[y]$ satisfying $r(0) = 0$. Then F is a zero of

$$p(T) := \prod_{\substack{(k,l)\in\mathbb{N}^2 \\ 0 \,<\, dk+l \,\leq\, de}} (T - a^k b^l).$$

Proof. <u>First case.</u> We assume that $\forall\ l \in \{1,\dots,d\},\ a \neq b^l$.
We have seen in the proof of lemma 4.2 that in this case the sequence $(G^n)_{n\in\mathbb{N}}$ is of Ω-exponential type where $\Omega := \{a, b, b^2, \dots, b^d\}$.
The sequence $(G^n \circ \varphi^{-1})_{n\in\mathbb{N}}$ will still be of Ω-exponential type.
If we write $G^n \circ \varphi^{-1} = (u_1(n), u_2(n))$ and $\varphi = (\varphi_1, \varphi_2)$, we have $F^n = \varphi \circ G^n \circ \varphi^{-1} = (\varphi_1(u_1(n), u_2(n)), \varphi_2(u_1(n), u_2(n)))$. Since the sequences u_1 and u_2 are of Ω-exponential type, the sequences $\varphi_1(u_1, u_2)$ and $\varphi_2(u_1, u_2)$ are of Ω'-exponential type with $\Omega' = \bigcup_{k=1}^{e} \Omega^k$ (see rem. 3 following prop. 1.2).
But $\Omega' = \{a^{j_0} b^{j_1+2j_2+\ldots+dj_d},\ j = (j_0,\dots,j_d) \in \mathbb{N}^{d+1},\ 0 < |j| \leq e\}$ is included into $\Omega'' = \{a^k b^l,\ (k,l) \in \mathbb{N}^2,\ 0 < dk + l \leq de\}$ because the inequality $j_0 + \dots j_d \leq e$ implies the inequality $dj_0 + (j_1 + 2j_2 + \ldots + dj_d) \ \leq\ de$.
So, the sequence $(F^n)_{n\in\mathbb{N}}$ is of Ω''-exponential type and this implies that $p(F) = 0$.

<u>Second case.</u> The general case. As in lemma 4.2, we conclude by a density argument. $\square$

2. THE NON INVERTIBLE CASE.

In the following lines, we will identify a polynomial map $u\ :\ \mathbb{A}^2 \to \mathbb{A}^1$ to a polynomial $u(x,y) \in \mathbb{C}[x,y]$ and we will identify a polynomial map $v\ :\ \mathbb{A}^1 \to \mathbb{A}^2$ to a pair $v = (v_1, v_2)$ where $v_1, v_2 \in \mathbb{C}[x]$.

Lemma 4.4. Let F be a LF endomorphism of $\mathbb{A}^2$ which is not invertible and such that $F(0) = 0$. Then, there exist polynomial maps $u\ :\ \mathbb{A}^2 \to \mathbb{A}^1$ and $v\ :\ \mathbb{A}^1 \to \mathbb{A}^2$ such that
(i) $F = v \circ u$; (ii) $u(0,0) = 0$ and $v(0) = (0,0)$;
(iii) the map $L := u \circ v\ :\ \mathbb{A}^1 \to \mathbb{A}^1$ is linear, i.e. $L(x) = ax$ for some $a \in \mathbb{C}$.

Proof. We may assume that $F \neq 0$. We have already seen that $\mathrm{Jac}(F_1, F_2) = 0$. This condition is equivalent to saying that F_1 and F_2 are algebraically dependant over $\mathbb{C}$ or to saying that there exist $u(x,y) \in \mathbb{C}[x,y]$ and $v_1(x), v_2(x) \in \mathbb{C}[x]$ such that $F_1 = v_1(u)$ and $F_2 = v_2(u)$ (see [14], [23] and [22]). We may assume that $u(0,0) = 0$ and since $F(0) = 0$, we obtain $v_1(0) = v_2(0) = 0$.
If we set $L(x) := u \circ v(x) \in \mathbb{C}[x]$, we have $\forall\ k \in \mathbb{N},\ (F^{\#})^k(u) \ =\ u \circ F^k \ =\ L^k \circ u$. Since the degree of $(F^{\#})^k(u)$ must be upper bounded and since $\deg(L^k \circ u) \ =\ (\deg L)^k \deg u$, this implies $\deg L \ \leq\ 1$ (since $\deg u \neq 0$). $\square$

Lemma 4.5. Let F be a LF endomorphism of $\mathbb{A}^2$ which is not invertible and such that $F(0) = 0$. Let us write $F = v \circ u$ as in lemma 4.4 and let a be such that $u \circ v(x) = ax$. If $d := \deg F$, then $p = T(T-a)(T-a^2)\dots(T-a^d)$ is a vanishing polynomial of F.

Proof. If $u = 0$, we obtain at once $F = 0$, $p = T$ and $p(F) = 0$. If $u \neq 0$ let us note that $\deg v_1$ and $\deg v_2 \leq d$ and that $\forall\, n \in \mathbb{N}$, $F^{n+1} = (v_1(a^n u), v_2(a^n u))$. Let us set $\Omega := \{a, a^2, \ldots, a^d\}$. The sequences $n \mapsto p_1(a^n u)$ and $n \mapsto p_2(a^n u)$ are of Ω-exponential type, so that the sequence $n \mapsto F^{n+1}$ is also of Ω-exponential type. This means that $q := (T-a)(T-a^2)\ldots(T-a^d)$ is a vanishing polynomial of this sequence. This is equivalent to saying that $p(T) = Tq(T)$ is a vanishing polynomial of the sequence $n \mapsto F^n$. By prop. 1.3, this is still equivalent to $p(F) = 0$. $\square$

Remark. If $\operatorname{supp} r := \{k,\ r_k \neq 0\}$ for $r = \sum_k r_k x^k$ and if $\Omega' := \{a^k,\ k \in \operatorname{supp} v_1 \cup \operatorname{supp} v_2\}$, we could show that $\mu_F = T \prod_{\omega \in \Omega'} (T - \omega)$ when $u \neq 0$.

We will now explain how to build any LF polynomial endomorphism F of $\mathbb{A}^2$ which is not invertible and such that $F(0) = 0$. We will distinguish two cases:

<u>First case.</u> F is nilpotent.
1. Choose any nonzero polynomial map $v : \mathbb{A}^1 \to \mathbb{A}^2$ such that $v(0) = (0,0)$;
2. Since v is proper, its image is a closed curve of $\mathbb{A}^2$. Therefore, $\mathcal{I}_v := \{r \in \mathbb{C}[x,y],\ r \circ v = 0\}$ is a nonzero principal ideal of $\mathbb{C}[x,y]$, i.e. $\mathcal{I}_v = (r)$ for some (nonzero) element $r \in \mathbb{C}[x,y]$;
3. If $q \in \mathbb{C}[x,y]$, then $u := qr \in \mathcal{I}_v$ defines a map $u : \mathbb{A}^2 \to \mathbb{A}^1$ such that $u \circ v = 0$;
4. If we set $F := v \circ u$, then F is a nilpotent endomorphism of $\mathbb{A}^2$ such that $F(0) = 0$.

<u>Second case.</u> F is not nilpotent.
We will now show that F is conjugate to a polynomial endomorphism of the shape $G = (\lambda x + yq(x,y), 0)$ where $\lambda \in \mathbb{C}^*$ and $q(x,y) \in \mathbb{C}[x,y]$. This will imply that $\operatorname{Im} F$ is a closed curve of $\mathbb{A}^2$ isomorphic to $\mathbb{A}^1$ and that $\operatorname{Im} F^n = \operatorname{Im} F$ (for $n \geq 1$), since $G^n = (\lambda^n x + \lambda^{n-1} yq(x,y), 0)$ (for $n \geq 1$).
Let us write $F = v \circ u$ as in lemma 4.4. We have $u \circ v(x) = ax$ with $a \neq 0$. By the Abhyankar-Moh theorem (see [1]), there exists an automorphism φ of $\mathbb{A}^2$ such that $\varphi \circ v(x) = (x, 0)$. Therefore, if we set $G := \varphi \circ F \circ \varphi^{-1}$, then the second coordinate of G is zero. We can write the first coordinate under the shape $G_1 = r(x) + yq(x,y)$. Since the sequence $n \mapsto G^n$ is of bounded degree, the sequence $n \mapsto G^n \circ (x,0)$ also. But $G^n \circ (x, 0) = (r^n(x), 0)$, where r^n stands for the composition $\underbrace{r \circ r \circ \ldots \circ r}_{n}$. We must have $\deg r \leq 1$ and finally we obtain $r(x) = \lambda x$ for some nonzero complex number λ.

3. THE CHARACTERISTIC POLYNOMIAL.

Theorem 4.1. Let $F \in End(\mathbb{A}^2)$ be LF and such that $F(0) = 0$. If $d := \deg F$ and if λ_1, λ_2 are the eigenvalues of $\mathcal{L}(F)$, then $\prod_{\substack{\alpha \in \mathbb{N}^2 \\ 0 < |\alpha| \leq d}} (T - \lambda^\alpha)$ is a vanishing polynomial of F.

Proof. This comes from th. 1.2. since $\deg F^n \leq d$ for $n \geq 0$ (if F is invertible, it has already been said and if F is not, it is a consequence of lemma 4.4). $\square$

Remark. This characteristic polynomial is of degree $\dfrac{d(d+3)}{2}$. If $d = 1$, we find the classical characteristic polynomial of a linear endomorphism (in dimension two).

4. THE MINIMAL POLYNOMIAL.

Theorem 4.2. Let F be a LF endomorphism of $\mathbb{A}^2$ such that $F(0)=0$ and let μ_F be the minimal polynomial of F, then $\deg\,\mu_F \;\leq\; \deg\,F\;+\;1$.

Proof. If F is not invertible, this comes from lemma 4.5. If F is invertible, we can write $F=\varphi\circ G\circ\varphi^{-1}$ with $\varphi(0)=G(0)=0$ and $\deg\,F \;=\; de^2$, where $d=\deg\,G$ and $e=\deg\,\varphi$ (see lemma 4.1 and the remark following it). By lemma 4.3, $\deg\,\mu_F$ is less than or equal to the cardinal of the set $A:=\{(k,l)\in\mathbb{N}^2,\ 0<dk+l\leq de\}$. But

$$\begin{aligned}|A|+1 \;&=\; |\{(k,l)\in\mathbb{N}^2,\ 0\leq dk+l\leq de\}| \;=\; \sum_{k=0}^{e}(de-dk+1) \;=\; e+1+d\sum_{k=0}^{e}(e-k)\\ &=\; e+1+d\,\frac{e(e+1)}{2} \;=\; (e+1)\left(\frac{de}{2}+1\right) \text{ so that } |A|=e\left(\frac{de}{2}+\frac{d}{2}+1\right) \text{ and}\end{aligned}$$

we want to prove that $|A| \;\leq\; de^2+1$. If $\dfrac{de}{2}+\dfrac{d}{2}+1 \;\leq\; de$, i.e. $2\;\leq\; d(e-1)$, we are done. Otherwise, we get $e=1$ or $(e,d)=(2,1)$ so that $|A|=de^2+1$. $\square$

Acknowledgements. The second author is grateful to the University of La Rochelle for financing two pleasant stays at this place.

References

[1] S.S. Abhyankar, T.T. Moh, Embeddings of the line in the plane, J. Reine Angew. Math., 276 (1975), 148-166.

[2] J. Ax, The elementary theory of finite fields, Ann. of Math. (2) 88 (1968), 239-271.

[3] J. Ax, Injective endomorphisms of varieties and schemes, Pac. J. Math. 31 (1969), 1-7.

[4] M. de Bondt, Quasi-translations and counterexamples to the homogeneous dependence problem, to appear in the Proc. of the A.M.S.

[5] A. Borel, Injective endomorphisms of algebraic varieties, Arch. Math. (Basel) 20 (1969), 531-537.

[6] N. Bourbaki, Éléments de Mathématiques, Algèbre, chap. 1-3, Hermann, Paris, 1970.

[7] L. Cerlienco, M. Mignotte, F. Piras, Suites récurrentes linéaires, propriétés algébriques et arithmétiques, L'Enseignement Mathématique 33 (1987), 67-108.

[8] S. Cynk, R. Rusek, Injective endomorphisms of algebraic and analytic sets, Ann. Pol. Math. 56, n°1 (1991), 29-35.

[9] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150, Springer-Verlag, New York, 1995.

[10] A. van den Essen, Polynomial Automorphisms and the Jacobian Conjecture, Progress in Math, vol 190, Birkhäuser Verlag, Basel, Boston, Berlin, 2000.

[11] S. Friedland, J. Milnor, Dynamical properties of plane polynomial automorphisms, Ergod. Th. & Dynam. Syst 9 (1989), 67-99.

[12] J.-P. Furter, On the degree of iterates of automorphisms of the affine plane, Manuscripta Mathematica, 98 (1999), 183-193.

[13] J.-P. Furter, Quasi-locally finite polynomial endomorphisms, submitted, available at http://www.univ-lr.fr/labo/lmca/publications/06-07/06-07.pdf.

[14] P. Gordan, Über biquadratische Gleichungen, Math. Annalen 29 (1987), 318-326.

[15] A. Grothendieck, EGA 4, Inst. Hautes Études Sci. Publ. Math. n°32, 1967.

[16] J.E. Humphreys, Linear Algebraic Groups, Springer-Verlag, New York, 1975, 1981.

[17] H.W.E. Jung, Über ganze birationale Transformationen der Ebene, J. Reine Angew. Math., 184 (1942), 161-174.

[18] T. Kambayashi, Automorphism group of a polynomial ring and algebraic group actions on affine space, J. Algebra 60, n°2 (1979), 439-451.

[19] F. Knop, Nichtlinearisierbare Operationen halbeinfacher Gruppen aud affinen Raümen, Invent. Math. 105, n°1 (1991), 217-220.

[20] W. van der Kulk, On polynomial rings in two variables, Nieuw Archief voor Wiskunde, 3, n° 1 (1953), 33-41.

[21] M. Nagata, On the automorphism group of k[X,Y], Kyoto Univ. Lectures in Math. 5, Kyoto University, Kinokuniya-Tokio, 1972.

[22] A. Nowicki, On the jacobian equation J(f,g)=0 for polynomials in k[x,y], Nagoya Math. J., 109 (1988), 151-157.

[23] O. Perron, Algebra I, Die Grundlagen, Walter de Gruyter & Co., Berlin, 1951.

[24] W. Rudin, Injective polynomial maps are automorphisms, Amer. Math. Monthly 102, n°6 (1995), 540-543.

[25] G. Schwarz, Exotic algebraic group actions, C.R. Acac. Sci. Paris Sér. I Math., 309, n°2 (1989), 89-94.

[26] J.-P. Serre, Trees, Springer Verlag, Berlin, Heidelberg, New York, 1980.

[27] I.P. Shestakov, U.U. Umirbaev, Poisson brackets and two-generated subalgebras of rings of polynomials, J. Amer. Math. Soc 17 (2004), 181-196.

[28] I.P. Shestakov, U.U. Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables, J. Amer. Math. Soc 17 (2004), 197-227.

[29] M.K. Smith, Stably Tame Automorphisms, J. of Pure and Applied Algebra, 58 (1989), 209-212.

[30] B.L. van der Waerden, Die Alternative bei nichtlinearen Gleichungen. Nachr. Gesells. Wiss. Göttingen, Math. Phys. Klasse (1928), 77-87.

Liste des prépublications

99-1 Monique Jeanblanc et Nicolas Privault. A complete market model with Poisson and Brownian components. A paraître dans *Proceedings of the Seminar on Stochastic Analysis, Random Fields and Applications*, Ascona, 1999.

99-2 Laurence Cherfils et Alain Miranville. Generalized Cahn-Hilliard equations with a logarithmic free energy. A paraître dans *Revista de la Real Academia de Ciencias*.

99-3 Jean-Jacques Prat et Nicolas Privault. Explicit stochastic analysis of Brownian motion and point measures on Riemannian manifolds. *Journal of Functional Analysis* **167** (1999) 201-242.

99-4 Changgui Zhang. Sur la fonction q-Gamma de Jackson. A paraître dans *Aequationes Math.*

99-5 Nicolas Privault. A characterization of grand canonical Gibbs measures by duality. A paraître dans *Potential Analysis*.

99-6 Guy Wallet. La variété des équations surstables. A paraître dans *Bulletin de la Société Mathématique de France*.

99-7 Nicolas Privault et Jiang-Lun Wu. Poisson stochastic integration in Hilbert spaces. *Annales Mathématiques Blaise Pascal*, **6** (1999) 41-61.

99-8 Augustin Fruchard et Reinhard Schäfke. Sursabilité et résonance.

99-9 Nicolas Privault. Connections and curvature in the Riemannian geometry of configuration spaces. *C. R. Acad. Sci. Paris, Série I* **330** (2000) 899-904.

99-10 Fabienne Marotte et Changgui Zhang. Multisommabilité des séries entières solutions formelles d'une équation aux q-différences linéaire analytique. A paraître dans *Annales de l'Institut Fourier*, 2000.

99-11 Knut Aase, Bernt Øksendal, Nicolas Privault et Jan Ubøe. White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance. *Finance and Stochastics*, **4** (2000) 465-496.

00-01 Eric Benoît. Canards en un point pseudo-singulier nœud. A paraître dans *Bulletin de la Société Mathématique de France*.

00-02 Nicolas Privault. Hypothesis testing and Skorokhod stochastic integration. *Journal of Applied Probability*, **37** (2000) 560-574.

00-03 Changgui Zhang. La fonction théta de Jacobi et la sommabilité des séries entières q-Gevrey, I. *C. R. Acad. Sci. Paris, Série I* **331** (2000) 31-34.

00-04 Guy Wallet. Déformation topologique par changement d'échelle.

00-05 Nicolas Privault. Quantum stochastic calculus for the uniform measure and Boolean convolution. A paraître dans *Séminaire de Probabilités* **XXXV**.

00-06 Changgui Zhang. Sur les fonctions q-Bessel de Jackson.

00-07 Laure Coutin, David Nualart et Ciprian A. Tudor. Tanaka formula for the fractional Brownian motion. A paraître dans *Stochastic Processes and their Applications*.

00-08 Nicolas Privault. On logarithmic Sobolev inequalities for normal martingales. *Annales de la Faculté des Sciences de Toulouse* **9** (2000) 509-518.

01-01 Emanuelle Augeraud-Veron et Laurent Augier. Stabilizing endogenous fluctuations by fiscal policies; Global analysis on piecewise continuous dynamical systems. A paraître dans *Studies in Nonlinear Dynamics and Econometrics*

01-02 Delphine Boucher. About the polynomial solutions of homogeneous linear differential equations depending on parameters. A paraître dans *Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation: ISSAC 99, Sam Dooley Ed., ACM, New York 1999.*

01-03 Nicolas Privault. Quasi-invariance for Lévy processes under anticipating shifts.

01-04 Nicolas Privault. Distribution-valued iterated gradient and chaotic decompositions of Poisson jump times functionals.

01-05 Christian Houdré et Nicolas Privault. Deviation inequalities: an approach via covariance representations.

01-06 Abdallah El Hamidi. Remarques sur les sentinelles pour les systèmes distribués

02-01 Eric Benoît, Abdallah El Hamidi et Augustin Fruchard. On combined asymptotic expansions in singular perturbation.

02-02 Rachid Bebbouchi et Eric Benoît. Equations différentielles et familles bien nées de courbes planes.

02-03 Abdallah El Hamidi et Gennady G. Laptev. Nonexistence of solutions to systems of higher-order semilinear inequalities in cone-like domains.

02-04 Hassan Lakhel, Youssef Ouknine, et Ciprian A. Tudor. Besov regularity for the indefinite Skorohod integral with respect to the fractional Brownian motion: the singular case.

02-05 Nicolas Privault et Jean-Claude Zambrini. Markovian bridges and reversible diffusions with jumps.

02-06 Abdallah El Hamidi et Gennady G. Laptev. Existence and Nonexistence Results for Reaction-Diffusion Equations in Product of Cones.

02-07 Guy Wallet. Nonstandard generic points.

02-08 Gilles Bailly-Maitre. On the monodromy representation of polynomials.

02-09 Abdallah El Hamidi. Necessary conditions for local and global solvability of nondiagonal degenerate systems.

02-10 Abdallah El Hamidi et Amira Obeid. Systems of Semilinear higher order evolution inequalities on the Heisenberg group.

03-01 Abdallah El Hamidi et Gennady G. Laptev. Non existence de solutions d'inéquations semilinéaires dans des domaines coniques.

03-02 Eric Benoît et Marie-Joëlle Rochet. A continuous model of biomass size spectra governed by predation and the effects of fishing on them.

03-03 Catherine Stenger: On a conjecture of Wolfgang Wasow concerning the nature of turning points.

03-04 Christian Houdré et Nicolas Privault. Surface measures and related functional inequalities on configuration spaces.

03-05 Abdallah El Hamidi et Mokhtar Kirane. Nonexistence results of solutions to systems of semilinear differential inequalities on the Heisenberg group.

03-06 Uwe Franz, Nicolas Privault et René Schott. Non-Gaussian Malliavin calculus on real Lie algebras.

04-01 Abdallah El Hamidi. Multiple solutions to a nonlinear elliptic equation involving Paneitz type operators.

04-02 Mohamed Amara, Amira Obeid et Guy Vallet. Relaxed formulation and existence result of the degenerated elliptic small disturbance model.

04-03 Hippolyte d'Albis et Emmanuelle Augeraud-Veron. Competitive Growth in a Life-cycle Model: Existence and Dynamics

04-04 Sadjia Aït-Mokhtar: Third order differential equations with fixed critical points.

04-05 Mokhtar Kirane et Nasser-eddine Tatar. Asymptotic Behavior for a Reaction Diffusion System with Unbounded Coefficients.

04-06 Mokhtar Kirane, Eric Nabana et Stanislav I. Pohozaev. Nonexistence of Global Solutions to an Elliptic Equation with a Dynamical Boundary Condition.

04-07 Khaled M. Furati, Nasser-eddine Tatar and Mokhtar Kirane. Existence and asymptotic behavior for a convection Problem.

04-08 José Alfredo López-Mimbela et Nicolas Privault. Blow-up and stability of semilinear PDE's with gamma generator.

04-09 Abdallah El Hamidi. Multiple solutions with changing sign energy to a nonlinear elliptic equation.

04-10 Sadjia Aït-Mokhtar: A singularly perturbed Riccati equation.

04-11 Mohamed Amara, Amira Obeid et Guy Vallet. Weighted Sobolev spaces for a degenerated nonlinear elliptic equation.

04-12 Abdallah El Hamidi. Existence results to elliptic systems with nonstandard growth conditions.

04-13 Eric Edo et Jean-Philippe Furter: Some families of polynomial automorphisms.

04-14 Laurence Cherfils et Yavdat Il'yasov. On the stationary solutions of generalized reaction diffusion equations with p & q- Laplacian.

04-15 Jean-Christophe Breton et Youri Davydov. Local limit theorem for supremum of an empirical processes for i.i.d. random variables.

04-16 Jean-Christophe Breton, Christian Houdré et Nicolas Privault. Dimension free and infinite variance tail estimates on Poisson space.

04-17 Abdallah El Hamidi et Gennady G. Laptev. Existence and nonexistence results for higher-order semilinear evolution inequalities with critical potential.

05-01 Mokhtar Kirane et Nasser-eddine Tatar. Nonexistence of Solutions to a Hyperbolic Equation with a Time Fractional Damping.

05-02 Mokhtar Kirane et Yamina Laskri. Nonexistence of Global Solutions to a Hyperbolic Equation with a Time Fractional Damping.

05-03 Mokhtar Kirane, Yamina Laskri et Nasser-eddine Tatar. Critical Exponents of Fujita Type for Certain Evolution Equations and Systems with Spatio-Temporal Fractional Derivatives.

05-04 Abdallah El Hamidi et Jean-Michel Rakotoson. Compactness and quasilinear problems with critical exponents

05-05 Claudianor O. Alves et Abdallah El Hamidi. Nehari manifold and existence of positive solutions to a class of quasilinear problems.

05-06 Khalid Adriouch et Abdallah El Hamidi. The Nehari manifold for systems of nonlinear elliptic equations.

05-07 Eric Benoît. Equation fonctionnelle: Transport et convolution.

05-08 Jean-Philippe Furter et Stefan Maubach. Locally Finite Polynomial Endomorphisms.