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Université de La Rochelle
Avenue Michel Crépeau
17042 La Rochelle Cedex 1
http://www.univ-lr.fr/labo/lmca

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Aldéric Joulin

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On local Poisson-type deviation inequalities for curved continuous time Markov chains, with applications to birth-death processes

Aldéric Joulin*
Laboratoire de Mathématiques et Applications
Université de La Rochelle
Avenue Michel Crépeau
17042 La Rochelle Cedex 1
France

Abstract

In this paper, we present new local Poisson-type deviation inequalities for continuous time Markov chains whose Wasserstein curvature or Γ-curvature is bounded below. Although these two curvatures are equivalent for Brownian motion on Riemannian manifolds, they are not comparable in discrete settings and yield deviation bounds involving different Lipschitz seminorms. In the case of birth-death process, we provide some conditions on the rates of the associated generator for such discrete curvatures to be bounded below, and we extend to this framework the local deviation inequalities of [2] established for continuous time random walks on graphs, seen as models in null curvature. By a limiting argument, deviation bounds are derived for the stationary distribution of birth-death process in the finite state space case and we recover the optimal Gaussian deviation for Ornstein-Uhlenbeck processes constructed as fluid limits of rescaled continuous time Ehrenfest chains. Finally, an extension of these local deviation inequalities to sample vectors of the $M/M/1$ queueing process completes this work.

Key words: continuous time Markov chain, deviation inequality, semigroup, Wasserstein curvature, Γ-curvature, birth-death process, stationary distribution, $M/M/1$ queueing process.
Mathematics Subject Classification. 60E15, 60J27, 47D07, 41A25.

*ajoulin@univ-lr.fr

1 Introduction

In recent years, the area of concentration of measure has been deeply investigated in the context of discrete time Markov chains, using mass transportation and functional inequalities related to the convergence to stationarity. For instance, in the contracting case, Gaussian concentration of measure was put forward by K. Marton [16], via Pinsker-type inequalities derived from information theory. It has been then extended by P.M. Samson [18] to a large class of Markov chains, among them Doeblin recurrent Markov chains, whereas H. Djellout, A. Guillin and L. Wu [9], and lately G. Blower and F. Bolley [4], established similar deviation bounds under assumptions of transportation inequalities. On the other hand, C. Houdré and P. Tetali [13] in the case of reversible Markov chains, and C. Ané and M. Ledoux [2] for continuous time random walks on graphs corresponding to null curvature models, obtained Poisson-type tail estimates using modified logarithmic Sobolev inequalities and the Herbst method.
The purpose of the present paper is to give new local Poisson-type deviation bounds for continuous time Markov chains, which extend and sharpen in the case of curved birth-death processes the tail inequalities of [2] mentioned above. Our approach is based on semigroup analysis and uses the notion of curvature for Markov processes on general metric measure spaces recently investigated in [20], in the context of continuous time Markov chains: the Wasserstein curvature involving the Lipschitz seminorm of the Markov semigroup, and the Γ-curvature related to a commutation relation between the semigroup and the "carré du champ" operator Γ.
In the case of Brownian motion on smooth Riemannian manifolds, Theorem 2 together with Corollary 1 in [20] state that the following assertions are equivalent for any $K \in \mathbb{R}$:

(i) the Brownian Wasserstein curvature is bounded below by K,

(ii) the Brownian Γ-curvature is bounded below by K,

(iii) the Ricci curvature of the manifold is bounded below by K.

Therefore, such an equivalence gives a characterization of uniform lower bounds of the Ricci curvature of the manifold in terms of gradient estimates of heat kernels. However, the equivalence between (i) and (ii) does not hold in the framework of continuous time Markov chains since discrete gradients do not satisfy the chain rule formula. Thus, it is natural to study the role played by each type of discrete curvature in the concentration of measure phenomenon. Actually, the constants in the deviation inequalities we establish in this paper turn out to be different when one or the other discrete curvature above is bounded below. For instance, let $(X_t)_{t\geq 0}$ be a regular continuous time Markov chain on a discrete metric space E, with jumps bounded by some positive b. Let $f : E \to \mathbb{R}$ be a Lipschitz function and denote $g(u) = (1+u)\log(1+u) - u$, $u > 0$. If $(X_t)_{t\geq 0}$ has Wasserstein curvature bounded below by $\rho > 0$ and angle bracket bounded by some positive V^2, we show via Theorem 3.1 the tail probability:

$$\begin{aligned}\sup_{x\in E} \mathbb{P}_x\left(f(X_t) - \mathbb{E}_x\left[f(X_t)\right] \geq y\right) &\leq \exp\left(-\frac{(1-e^{-2\rho t})V^2}{2\rho b^2} g\left(\frac{2\rho b y}{(1-e^{-2\rho t})V^2\|f\|_{\mathrm{Lip}}}\right)\right) \\ &\leq \exp\left(-\frac{y}{2b\|f\|_{\mathrm{Lip}}}\log\left(1+\frac{2\rho b y}{(1-e^{-2\rho t})V^2\|f\|_{\mathrm{Lip}}}\right)\right),\end{aligned}$$

whereas if the Γ-curvature is bounded below by the same ρ and if $\|\Gamma f\|_\infty < +\infty$, we get the estimate:

$$\begin{aligned}\sup_{x\in E} \mathbb{P}_x\left(f(X_t) - \mathbb{E}_x\left[f(X_t)\right] \geq y\right) &\leq \exp\left(-\frac{(1-e^{-2\rho t})\|\Gamma f\|_\infty}{\rho b^2\|f\|_{\mathrm{Lip}}^2} g\left(\frac{\rho b y\|f\|_{\mathrm{Lip}}}{(1-e^{-2\rho t})\|\Gamma f\|_\infty}\right)\right) \\ &\leq \exp\left(-\frac{y}{2b\|f\|_{\mathrm{Lip}}}\log\left(1+\frac{\rho b y\|f\|_{\mathrm{Lip}}}{(1-e^{-2\rho t})\|\Gamma f\|_\infty}\right)\right),\end{aligned}$$

cf. Corollary 4.4. Although the exponential decays above are somewhat similar, we note that a lower bound on the Γ-curvature entails more general inequalities involving the mixed Lipschitz seminorms $\|\cdot\|_{\mathrm{Lip}}$ and $f \mapsto \|\Gamma f\|_\infty^{1/2}$, whereas a lower bound on the Wasserstein curvature leads to deviation results including the sole $\|\cdot\|_{\mathrm{Lip}}$ and enforces the angle bracket of the chain to be bounded.

The paper is organized as follows. In Section 2, some basic material on continuous time Markov chains is recalled and we introduce two notions of curvatures of Markov chains, namely the Wasserstein curvature and the Γ-curvature. In Section 3, Theorem 3.1, a local Poisson-type deviation inequality is established for continuous time Markov chain with Wasserstein curvature bounded below, and we analyze the influence of the sign of such a lower bound in large deviation inequalities. In Section 4, a general estimate is derived in Theorem 4.2 under the hypothesis of a lower bound on the Γ-curvature, and with further assumptions on the chain, these upper bounds are computed to yield local Poisson tail probabilities involving the mixed Lipschitz seminorms $\|\cdot\|_{\mathrm{Lip}}$ and $f \mapsto \|\Gamma f\|_\infty^{1/2}$. The case of irreducible birth-death process on $\mathbb{N}$ or $\{0, 1, \ldots, n\}$ is investigated in Section 5, in which we give some conditions on the rates of the associated generator for such discrete curvatures to be bounded below. As a result, we extend to birth-death processes the deviation inequalities of [2] established for continuous time random walks on graphs, seen as models in null curvature. By a limiting argument, deviation bounds are derived for the stationary distribution of birth-death process on the finite state space $\{0, 1, \ldots, n\}$ and we recover the optimal Gaussian concentration for Ornstein-Uhlenbeck processes constructed as fluid limits of rescaled continuous time Ehrenfest chains. Finally, these local Poisson-type deviation inequalities are extended to sample vectors of the $M/M/1$ queueing process by using a tensorization procedure of the Laplace transform together with an integration by parts formula satisfied by the underlying semigroup.

2 Notation and preliminaries

Throughout the paper, E is a countable set endowed with a non-trivial metric d, $\mathscr{F}(E)$ is the collection of all real-valued functions on E, $\mathscr{B}(E) \subset \mathscr{F}(E)$ is the space of all real-valued bounded functions on E equipped with the supremum norm $\|f\|_\infty = \sup_{x\in E} |f(x)|$, and $\mathrm{Lip}(E)$ is the subspace of $\mathscr{F}(E)$ consisting of Lipschitz functions on E, i.e.

$$\|f\|_{\mathrm{Lip}} := \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x, y)} < +\infty.$$

2.1 Basic material on continuous time Markov chains

On a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, consider an E-valued continuous time Markov chain $(X_t)_{t\geq 0}$ with its natural filtration $(\mathscr{F}_t)_{t\geq 0}$ and homogeneous semigroup $(P_t)_{t\geq 0}$ acting on $\mathrm{Dom}\,(P_t)_{t\geq 0} := \{f \in \mathscr{F}(E) : P_t f \text{ exists for any } t \geq 0\} \supset \mathscr{B}(E)$ as follows:

$$P_t f(x) := \mathbb{E}_x[f(X_t)] = \sum_{y\in E} f(y) P_t(x, y), \quad x \in E.$$

We assume through the paper that the càdlàg chain $(X_t)_{t\geq 0}$ is regular, i.e. the number of its discontinuities is finite on each compact time interval. Define

$$Q_x = \lim_{t\downarrow 0} \frac{1 - P_t(x, x)}{t} \in [0, +\infty], \qquad Q(x, y) = \lim_{t\downarrow 0} \frac{P_t(x, y)}{t} \in [0, +\infty), \quad y \neq x,$$

and denote $Q(x, x) = -Q_x$, $x \in E$. By Theorem 2.2 page 337 in [6], the regularity assumption implies that $(X_t)_{t\geq 0}$ is stable and conservative, i.e. for any $x \in E$, $Q_x < +\infty$, and $\sum_{y\in E} Q(x, y) = 0$, respectively. The generator $\mathscr{L}$ of the chain is given by

$$\mathscr{L} f(x) = \sum_{y\in E} (f(y) - f(x))\, Q(x, y), \quad x \in E,$$

where $f \in \mathrm{Dom}\,\mathscr{L} := \{f \in \mathscr{F}(E) : \mathscr{L}f \text{ exists}\} \supset \mathrm{Dom}\,(P_t)_{t\geq 0}$. In addition, if $\sup_{x\in E} Q_x < +\infty$, then $\mathscr{L}$ is bounded and the chain admits a uniform version, i.e. there exists $\lambda > 0$ and a transition matrix P such that $\mathscr{L} = \lambda(P - I)$, where I denotes the identity on $\mathscr{F}(E)$. The version of the chain with $\lambda = \sup_{x\in E} Q_x$ is called minimal. In the remainder of the paper, the chains we consider are implicitly assumed to be non-explosive. In other words, if $(T_n)_{n\in\mathbb{N}}$ denotes the sequence of jump times of the chain $(X_t)_{t\geq 0}$, i.e. $T_0 = 0$ and $T_{n+1} = \inf\{t > T_n : X_t \neq X_{T_n}\}$, $n \in \mathbb{N}$, then for any initial state $x \in E$, we have $\mathbb{P}_x\left(\lim_{n\to+\infty} T_n = +\infty\right) = 1$.

Given $f \in \mathscr{B}(E)$, the process $M^f = (M_t^f)_{t\geq 0}$ defined by

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathscr{L}f(X_s)ds, \quad t \geq 0,$$

is a $(\mathbb{P}_x, \mathscr{F}_t)$-martingale for any $x \in E$, which has the representation:

$$M_t^f \;=\; \sum_{y,z\in E} \int_0^t (f(y) - f(z))\, 1_{\{X_{s-}=z\}}(N_{z,y} - \sigma_{z,y})(ds),$$

where $(N_{z,y})_{z,y\in E}$ is a family of independent Poisson processes on $\mathbb{R}_+$ with respective intensity $\sigma_{z,y}(dt) = Q(z, y)dt$.

If $(X_t)_{t\geq 0}$ is square-integrable, then the angle bracket process exists and is given by

$$\langle X, X\rangle_t = \sum_{y,z\in E} d(z, y)^2 \int_0^t 1_{\{X_s=z\}} Q(z, y)ds, \quad t \geq 0.$$

If there exists $V > 0$ such that $\left\|\sum_{y\in E} d(\cdot, y)^2 Q(\cdot, y)\right\|_\infty \leq V^2$, then $\langle X, X\rangle_t \leq V^2 t$ and we say that $(X_t)_{t\geq 0}$ has angle bracket bounded by V^2.

Finally, we say that the chain $(X_t)_{t\geq 0}$ has jumps bounded by some positive b if $\sup_{t>0} d(X_{t-}, X_t) \leq b$.

2.2 Curved continuous time Markov chains

Wasserstein curvature of regular Markov chains

Let us introduce the notion of curved Markov chain in the Wasserstein sense.

Definition 2.1 *The Wasserstein curvature at time $t > 0$ of a regular Markov chain with semigroup $(P_t)_{t\geq 0}$ is defined by*

$$K_t := -\frac{1}{t} \sup\left\{\log\left(\frac{\|P_t f\|_{\mathrm{Lip}}}{\|f\|_{\mathrm{Lip}}}\right) : f \in \mathrm{Dom}\,(P_t)_{t\geq 0} \cap \mathrm{Lip}(E),\ f \neq \mathrm{const}\right\} \quad \in [-\infty, +\infty).$$

It is said to be bounded below by $K \in \mathbb{R}$ if $\inf_{t>0} K_t \geq K$.

Remark 2.2 We call this curvature the Wasserstein curvature since it is connected with the so-called Wasserstein distances. Indeed, if $\mathbb{P}(E)$ denotes the space of probability measures on the subsets of E equipped with the weak topology and $\mathbb{P}_1(E)$ is the subset of $\mathbb{P}(E)$ consisting of all μ such that $\sum_{y\in E} d(x,y)\mu(y) < +\infty$ for some (or equivalently for any) $x \in E$, then given $\mu, \nu \in \mathbb{P}_1(E)$, define the Wasserstein distance W between μ and ν by

$$W(\mu,\nu) := \inf_{\pi} \sum_{x,y\in E} d(x,y)\pi(x,y),$$

where the infimum runs over all $\pi \in \mathbb{P}_1(E\times E)$ with marginals μ and ν, making $\mathbb{P}_1(E)$ a Polish space, see for instance [21]. The Kantorovich-Rubinstein duality theorem states that the Wasserstein distance rewrites as

$$W(\mu,\nu) = \sup\left\{ \left| \sum_{x\in E} f(x)(\mu(x) - \nu(x)) \right| : \|f\|_{\mathrm{Lip}} \leq 1 \right\}.$$

Thus, if a Markov kernel $P_t(x,\cdot) \in \mathbb{P}_1(E)$ for some $x \in E$ and any positive t, then the following assertions are equivalent:

(i) $\inf_{t>0} K_t \geq K$;

(ii) $\|P_t f\|_{\mathrm{Lip}} \leq e^{-Kt}\|f\|_{\mathrm{Lip}}$, for any $f \in \mathrm{Dom}\,(P_t)_{t\geq 0} \cap \mathrm{Lip}(E)$ and any $t > 0$;

(iii) $W(P_t(x,\cdot), P_t(y,\cdot)) \leq e^{-Kt} d(x,y)$ for any $x, y \in E$ and any $t > 0$.

Hence, these assertions characterize lower bounds on the Wasserstein curvature in terms of contraction properties of the semigroup in the Wasserstein metric W.

Remark 2.3 By the Kantorovich-Rubinstein duality theorem together with [8, Theorem 5.23], any chain with Wasserstein curvature bounded below by some positive constant K is positive recurrent and thus has a unique stationary distribution $\pi \in \mathbb{P}_1(E)$. Therefore, according to the Kantorovich-Rubinstein duality theorem, we have:

$$\begin{aligned}
W(P_t(x,\cdot),\pi) &= \sup_{\|f\|_{\mathrm{Lip}}\leq 1} \left| \sum_{y\in E} f(y)(P_t(x,y) - \pi(y)) \right| \\
&= \sup_{\|f\|_{\mathrm{Lip}}\leq 1} \left| \sum_{y,z\in E} f(y)(P_t(x,y) - P_t(z,y))\pi(z) \right| \\
&\leq \sup_{\|f\|_{\mathrm{Lip}}\leq 1} \sum_{z\in E} |P_t f(x) - P_t(z)|\pi(z) \\
&\leq e^{-Kt} \sum_{z\in E} d(x,z)\pi(z),
\end{aligned}$$

which goes to 0 as t tends to infinity. Hence, the positive number K describes the speed of convergence of the Markov chain to stationarity with respect to the Wasserstein metric W.

Γ-curvature of regular Markov chains

Recall that the "carré du champ" operator Γ is the symmetric bilinear mapping defined on $\mathrm{Dom}\,\mathscr{L} \times \mathrm{Dom}\,\mathscr{L}$ by

$$\begin{aligned}\Gamma(f,g)(x) &:= \frac{1}{2}\left(\mathscr{L}(fg)(x) - f(x)\mathscr{L}g(x) - g(x)\mathscr{L}f(x)\right) \\ &= \frac{1}{2}\sum_{y\in E}\left(f(y)-f(x)\right)\left(g(y)-g(x)\right)Q(x,y).\end{aligned}$$

We let $\Gamma f = \Gamma(f,f)$ and introduce the notion of curved Markov chains in the Γ-sense:

Definition 2.4 *The Γ-curvature at time $t>0$ of a regular Markov chain with semi-group $(P_t)_{t\geq 0}$ is defined by*

$$\rho_t := -\frac{1}{t}\sup\left\{\log\left(\frac{\left(\Gamma P_t f\right)^{1/2}(x)}{P_t\left(\Gamma f\right)^{1/2}(x)}\right) : f \in \mathrm{Dom}\,(P_t)_{t\geq 0},\ f \neq \mathrm{const},\ x \in E\right\} \quad \in [-\infty, +\infty).$$

It is said to be bounded below by $\rho \in \mathbb{R}$ if $\inf_{t>0}\rho_t \geq \rho$.

Remark 2.5 By definition, the Γ-curvature is bounded below by $\rho \in \mathbb{R}$ if and only if for any $f \in \mathrm{Dom}\,(P_t)_{t\geq 0}$,

$$\left(\Gamma P_t f\right)^{1/2}(x) \leq e^{-\rho t} P_t\left(\Gamma f\right)^{1/2}(x), \quad x \in E, \quad t > 0, \tag{1}$$

which is the analogue in discrete settings of the classical commutation relation between local gradient and heat kernel on Riemannian manifolds with Ricci curvature bounded below, see [3].

Main differences between discrete curvatures

As already mentioned in the introduction, both curvatures are essentially equivalent for Brownian motions on Riemannian manifolds, see [20, Theorem 2]. This is no longer the case in discrete settings since discrete gradients do not satisfy the chain rule formula, and the discrete curvatures defined above are not directly comparable. However, note that the inequality (1) is a pointwise commutation relation between the semigroup and a discrete gradient induced by the operator Γ, whereas a lower bound on the Wasserstein curvature entails via the item (ii) of Remark 2.2 an inequality between Lipschitz seminorms and where the semigroup is dropped in its right-hand-side. Hence, the assumption (ii) is weaker than (1) in some sense and we expect that a lower bound K on the Wasserstein curvature entails weaker deviation results than that established under the assumption of the same lower bound K on the Γ-curvature.

Preliminary comments on tail estimates

Let us make some comments on the deviation inequalities we will establish in the remainder of this paper:

• Our estimates are said to be local since they are given with respect to the probability measures $P_t(x,\cdot)$, $t>0$, uniformly in the initial state $x \in E$. Moreover, we give in general two estimates for each result, to emphasize the good order of magnitude of the exponential decays in the deviation bounds. The second one is easily deduced from the first one by using the elementary inequality $(1+u)\log(1+u) - u \geq u\log(1+u)/2$, $u \geq 0$.

• For the sake of simplicity, our results are concerned with right tail estimates of type $\mathbb{P}_x\left(f(X_t) - \mathbb{E}_x\left[f(X_t)\right] \geq y\right)$, where the level of deviation y is positive. However, replacing in the corresponding inequalities f by $-f$, two-side tail estimates $\mathbb{P}_x\left(\left|f(X_t) - \mathbb{E}_x\left[f(X_t)\right]\right| \geq y\right)$ can be obtained.

• Similarly to the paper [11] for infinitely divisible random vectors with compactly supported Lévy measures, the boundedness assumption on the jumps of the chain allows us to derive explicit Poisson like inequalities, see for instance Theorem 3.1 or Corollary 4.4, whereas the general case yields the formal tail estimate (4) of Theorem 4.2. Moreover, all our results are still available when replacing the upper bound on the jumps $b \geq \sup_{t>0} d(X_{t-}, X_t)$ by the deterministic time-dependent upper bound $b_t \geq \sup_{0<s\leq t} d(X_{s-}, X_s)$, $t>0$.

• We do not investigate in this paper the case of independent product Markov chains, since our results would be sub-optimal with respect to the dimension. Indeed, our proofs are based on the tensorization of the Laplace transform with respect to the ℓ^1-metric, which is not well-adapted to handle dimension-free concentration results, see for instance the discussion in [15, Section 1.6].

• Denote $\log_+(x) = \max(\log(x), 0)$, $x>0$. A classical consequence of our Poisson-type deviation inequalities is the following exponential integrability property:

for any $f \in \mathrm{Lip}(E)$, any positive t and sufficiently small $\lambda > 0$, we have:

$$\sup_{x\in E} \mathbb{E}_x \left[e^{\lambda |f(X_t)-\mathbb{E}_x[f(X_t)]| \log_+ |f(X_t)-\mathbb{E}_x[f(X_t)]|} \right] < +\infty.$$

3 Deviation bounds for curved Markov chains in the Wasserstein sense

In this part, we present Poisson-type deviation results under the assumption of a lower bound on the Wasserstein curvature.

Theorem 3.1 *Let $(X_t)_{t\geq 0}$ be a regular Markov chain on E with jumps and angle bracket bounded respectively by $b > 0$ and $V^2 > 0$. Assume moreover that its Wasserstein curvature is bounded below by $K \in \mathbb{R}$. Let $f \in \mathrm{Lip}(E)$ and define for any $t > 0$ the positive numbers $C_{t,K} = \sup_{0\leq s\leq t} e^{-K(t-s)}$ and $M_{t,K} = (1-e^{-2Kt})/(2K)$ ($M_{t,K} = t$ if $K = 0$). Then for any initial state $x \in E$, any $y > 0$ and any $t > 0$, we have the local Poisson-type deviation inequality:*

$$\begin{aligned} \mathbb{P}_x \left(f(X_t) - \mathbb{E}_x \left[f(X_t) \right] \geq y \right) &\leq \exp\left(-\frac{M_{t,K}V^2}{b^2 C_{t,K}^2} g\left(\frac{bC_{t,K}y}{M_{t,K}V^2 \|f\|_{\mathrm{Lip}}} \right) \right) \qquad (2) \\ &\leq \exp\left(-\frac{y}{2bC_{t,K}\|f\|_{\mathrm{Lip}}} \log\left(1 + \frac{bC_{t,K}y}{M_{t,K}V^2\|f\|_{\mathrm{Lip}}} \right) \right), \end{aligned}$$

where $g(u) = (1+u)\log(1+u) - u$, $u > 0$.

Proof. Fix $x \in E$, $t > 0$, and assume first that f is bounded. The process $\left(Z_s^f\right)_{0\leq s\leq t}$ given by

$$Z_s^f := P_{t-s}f(X_s) - P_t f(X_0)$$

is a real $\mathbb{P}_x$-martingale with respect to the truncated filtration $(\mathscr{F}_s)_{0\leq s\leq t}$ and we have by Itô's formula:

$$Z_s^f = \sum_{y,z\in E} \int_0^s \left(P_{t-\tau}f(y) - P_{t-\tau}f(z) \right) 1_{\{X_{\tau-}=z\}} (N_{z,y} - \sigma_{z,y})(d\tau).$$

Since the Wasserstein curvature is bounded below, the jumps of $\left(Z_s^f\right)_{0\leq s\leq t}$ are bounded for any $s \in [0,t]$:

$$\begin{aligned} \left| Z_s^f - Z_{s-}^f \right| &= \left| P_{t-s}f(X_s) - P_{t-s}f(X_{s-}) \right| \\ &\leq d(X_s, X_{s-}) \|f\|_{\mathrm{Lip}} C_{t,K} \\ &\leq b\|f\|_{\mathrm{Lip}} C_{t,K}, \end{aligned}$$

as its angle bracket:

$$\begin{aligned}\langle Z^f, Z^f\rangle_s &= \sum_{y,z\in E}\int_0^s \left(P_{t-\tau}f(y) - P_{t-\tau}f(z)\right)^2 1_{\{X_{\tau-}=z\}}\,\sigma_{z,y}(d\tau)\\ &\leq \|f\|_{\mathrm{Lip}}^2 \sum_{y,z\in E}\int_0^s e^{-2K(t-\tau)} d(z,y)^2 1_{\{X_{\tau-}=z\}}\,Q(z,y)d\tau\\ &\leq \|f\|_{\mathrm{Lip}}^2 M_{t,K} V^2.\end{aligned}$$

By [14, Lemma 23.19], for any positive λ, the process $(Y_s^{(\lambda)})_{0\leq s\leq t}$ given by

$$Y_s^{(\lambda)} := \exp\left(\lambda Z_s^f - \lambda^2\psi(\lambda b\|f\|_{\mathrm{Lip}}C_{t,K})\langle Z^f, Z^f\rangle_s\right)$$

is a $\mathbb{P}_x$-supermartingale with respect to $(\mathscr{F}_s)_{0\leq s\leq t}$, where $\psi(z) = z^{-2}\left(e^z - z - 1\right)$, $z > 0$. Thus, we get for any $\lambda > 0$:

$$\begin{aligned}\mathbb{E}_x\left[e^{\lambda(f(X_t)-\mathbb{E}_x[f(X_t)])}\right] &= \mathbb{E}_x\left[e^{\lambda Z_t^f}\right]\\ &\leq \exp\left(\lambda^2\|f\|_{\mathrm{Lip}}^2 M_{t,K}V^2\,\psi(\lambda b\|f\|_{\mathrm{Lip}}C_{t,K})\right)\mathbb{E}_x\left[Y_t^{(\lambda)}\right]\\ &\leq \exp\left(\lambda^2\|f\|_{\mathrm{Lip}}^2 M_{t,K}V^2\,\psi(\lambda b\|f\|_{\mathrm{Lip}}C_{t,K})\right)\\ &= \exp\left(\frac{M_{t,K}V^2}{b^2C_{t,K}^2}\left(e^{\lambda b\|f\|_{\mathrm{Lip}}C_{t,K}} - \lambda b\|f\|_{\mathrm{Lip}}C_{t,K} - 1\right)\right).\end{aligned}$$

Finally, using the exponential Chebychev's inequality and optimizing in $\lambda > 0$ in the exponential estimate above, the deviation inequality (2) is established in the bounded case.

To remove the boundedness assumption, let $f \in \mathrm{Lip}(E)$ and consider the bounded function $f_n = \max\{-n, \min\{f, n\}\}$, $n \in \mathbb{N}$. We have the pointwise convergence $f_n \uparrow f$ and by a classical argument, see for instance the proof of Proposition 10 in [5], $(f_n)_{n\in\mathbb{N}}$ is uniformly integrable with respect to the probability measure $P_t(x,\cdot)$, which implies the convergence of $\mathbb{E}_x[f_n(X_t)]$ to $\mathbb{E}_x[f(X_t)]$. Since $\|f_n\|_{\mathrm{Lip}} \leq \|f\|_{\mathrm{Lip}}$ and that g is non-decreasing on $\mathbb{R}_+$, we finally have by Fatou's lemma:

$$\begin{aligned}\mathbb{P}_x\left(f(X_t) - \mathbb{E}_x[f(X_t)] \geq y\right) &\leq \liminf_{n\to+\infty}\mathbb{P}_x\left(f_n(X_t) - \mathbb{E}_x[f_n(X_t)] \geq y\right)\\ &\leq \liminf_{n\to+\infty}\exp\left(-\frac{M_{t,K}V^2}{b^2C_{t,K}^2}g\left(\frac{bC_{t,K}y}{M_{t,K}V^2\|f_n\|_{\mathrm{Lip}}}\right)\right)\\ &\leq \exp\left(-\frac{M_{t,K}V^2}{b^2C_{t,K}^2}g\left(\frac{bC_{t,K}y}{M_{t,K}V^2\|f\|_{\mathrm{Lip}}}\right)\right).\end{aligned}$$

Theorem 3.1 is established in full generality. ∎

Remark 3.2 If $K = 0$, then Theorem 3.1 recovers the tail inequalities of [11, 19] established for Lévy processes, since the independence of the increments implies that the Wasserstein curvature is bounded below by 0. If $K < 0$, then the decay in (2) is slower, due to the exponential factor e^{-Kt}, whereas if $K > 0$, the chain is positive recurrent and such estimates can be extended to the stationary distribution, as illustrated below and in Section 5.2.

Large deviation bounds

Let us now analyze the influence of the sign of the lower bound K of the Wasserstein curvature on some large deviation inequalities which are direct applications of (2). Fix the initial state $x \in E$ and the deviation level $y > 0$. Under the assumptions of Theorem 3.1, we have the following behaviors:

(i) If t tends to 0, then we have the estimate independent of K:

$$\limsup_{t \to 0} -\frac{1}{\log(t)} \log \mathbb{P}_x \left(f(X_t) - \mathbb{E}_x \left[f(X_t) \right] \geq y \right) \leq -\frac{y}{b \| f \|_{\mathrm{Lip}}}.$$

The speed of convergence is $-1/\log(t)$, which is sharp in the case of continuous time Markov chains whose rate functions of the generator are bounded, see [1]. One deduces that the sign of K has no influence in small time in (2).

(ii) If t tends to infinity, then the sign of K is crucial in (2). Indeed, if K is positive, then the existence of a unique stationary distribution π is assured by positive recurrence, as noted in Remark 2.3. The positivity of K achieves the best deviation inequality and as t tends to infinity, (2) entails an inequality for the stationary distribution π:

$$\pi \left(f - \mathbb{E}_\pi[f] \geq y \right) \leq \exp \left(\frac{y}{b \| f \|_{\mathrm{Lip}}} - \left(\frac{y}{b \| f \|_{\mathrm{Lip}}} + \frac{V^2}{2b^2 K} \right) \log \left(1 + \frac{2bKy}{V^2 \| f \|_{\mathrm{Lip}}} \right) \right),$$

where $\mathbb{E}_\pi[f]$ denotes the expectation of f with respect to π. See Section 5.2 for a more careful analysis of deviation estimates for stationary distributions of curved birth-death processes on finite state spaces. On the other hand, if $K \leq 0$, then the worst deviation inequality is realized and the limiting argument above is no longer available in (2), since $C_{t,K}$ and $M_{t,K}$ (which strongly depend on K) are not bounded uniformly in time.

To conclude this section, note that Theorem 3.1 allows us to consider neither Markov chains with unbounded angle bracket nor another Lipschitz seminorms than $\| \cdot \|_{\mathrm{Lip}}$.

To overcome this difficulty, one has to require some assumptions on another curvature of the chain, namely the Γ-curvature.

4 Estimates for curved Markov chains in the Γ-sense

In this section, we adapt to the Markovian case the covariance method of the papers [11, 12] to derive local deviation inequalities for curved Markov chains in the Γ-sense. Although Wasserstein and Γ-curvatures are not comparable in discrete spaces, the results we give in this part are more general than that in Section 3.

4.1 A general bound

Given $(X_t)_{t\geq 0}$ a regular Markov chain on E and two functions $f, g \in \mathscr{B}(E)$, we define the local covariance of $f(X_t)$ and $g(X_t)$ by

$$\mathrm{Cov}_x\left[f(X_t), g(X_t)\right] := \mathbb{E}_x\left[\left(f(X_t) - \mathbb{E}_x\left[f(X_t)\right]\right)\left(g(X_t) - \mathbb{E}_x\left[g(X_t)\right]\right)\right], \quad x \in E.$$

Before turning to Theorem 4.2 below, let us establish the following

Lemma 4.1 *Let $(X_t)_{t\geq 0}$ be a regular Markov chain on E with Γ-curvature bounded below by $\rho \in \mathbb{R}$. Let $g_1, g_2 \in \mathscr{B}(E)$ with $\|\Gamma g_1\|_\infty < +\infty$ and define $L_{t,\rho} = (1 - e^{-2\rho t})/(2\rho)$ if $\rho \neq 0$, and $L_{t,\rho} = t$ otherwise. Then for any initial state $x \in E$ and any time $t > 0$, we have the local covariance inequality:*

$$\mathrm{Cov}_x\left[g_1(X_t), g_2(X_t)\right] \leq 2L_{t,\rho}\|\Gamma g_1\|_\infty^{1/2}\, \mathbb{E}_x\left[(\Gamma g_2)^{1/2}(X_t)\right], \quad t > 0.$$

Proof. Fix $x \in E$ and $t > 0$. As in the proof of Theorem 3.1, we have for $i = 1, 2$:

$$g_i(X_t) - \mathbb{E}_x\left[g_i(X_t)\right] = \sum_{y,z\in E} \int_0^t \left(P_{t-s}g_i(y) - P_{t-s}g_i(z)\right) 1_{\{X_{s-}=z\}}(N_{z,y} - \sigma_{z,y})(ds).$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathrm{Cov}_x\left[g_1(X_t), g_2(X_t)\right] \\
&\quad = \mathbb{E}_x\left[\left(g_1(X_t) - \mathbb{E}_x\left[g_1(X_t)\right]\right)\left(g_2(X_t) - \mathbb{E}_x\left[g_2(X_t)\right]\right)\right] \\
&\quad = 2\,\mathbb{E}_x\left[\int_0^t \Gamma(P_{t-s}g_1, P_{t-s}g_2)(X_s)\,ds\right] \\
&\quad = 2\int_0^t P_s\left(\Gamma(P_{t-s}g_1, P_{t-s}g_2)\right)(x)\,ds
\end{aligned}$$

$$\leq 2\int_0^t P_s\left((\Gamma P_{t-s}g_1)^{1/2}\,(\Gamma P_{t-s}g_2)^{1/2}\right)(x)\,ds$$
$$\leq 2\int_0^t e^{-2\rho(t-s)}P_s\left(P_{t-s}(\Gamma g_1)^{1/2}\,P_{t-s}(\Gamma g_2)^{1/2}\right)(x)\,ds, \qquad (3)$$

where in (3) we used the assumption of a lower bound ρ on the Γ-curvature. Since $(P_t)_{t\geq 0}$ is a contraction operator on $\mathscr{B}(E)$, we have:

$$\begin{aligned}\mathrm{Cov}_x\left[g_1(X_t), g_2(X_t)\right] &\leq 2\,\|\Gamma g_1\|_\infty^{1/2}\int_0^t e^{-2\rho(t-s)}P_s\left(P_{t-s}(\Gamma g_2)^{1/2}\right)(x)\,ds \\ &= 2L_{t,\rho}\|\Gamma g_1\|_\infty^{1/2}\,\mathbb{E}_x\left[(\Gamma g_2)^{1/2}(X_t)\right].\end{aligned}$$

∎

Now, we are able to state Theorem 4.2 which presents a general deviation bound for curved Markov chains in the Γ-sense:

Theorem 4.2 *Let $(X_t)_{t\geq 0}$ be a regular Markov chain on E with Γ-curvature bounded below by $\rho \in \mathbb{R}$. Let $f \in \mathrm{Lip}(E)$ with $\|\Gamma f\|_\infty < +\infty$, and define the function $\psi_{f,t} : (0,+\infty) \to \mathbb{R}_+ \cup \{\infty\}$ by*

$$\psi_{f,t}(\lambda) := \sqrt{2}L_{t,\rho}\|\Gamma f\|_\infty^{1/2}\left\|\sum_{y\in E}(f(y)-f(\cdot))^2\left(\frac{e^{\lambda\|f\|_{\mathrm{Lip}}d(\cdot,y)}-1}{\|f\|_{\mathrm{Lip}}d(\cdot,y)}\right)^2 Q(\cdot,y)\right\|_\infty^{1/2}, \quad t>0,$$

where $L_{t,\rho}$ is defined in Lemma 4.1. Then for any initial state $x \in E$, any deviation level $y > 0$ and any $t > 0$, we get the local tail probability:

$$\mathbb{P}_x\left(f(X_t) - \mathbb{E}_x\left[f(X_t)\right] \geq y\right) \leq \exp\inf_{\lambda\in(0,M_{f,t})}\int_0^\lambda(\psi_{f,t}(\tau)-y)\,d\tau, \qquad (4)$$

where $M_{f,t} = \sup\{\lambda > 0 : \psi_{f,t}(\lambda) < +\infty\}$.

Remark 4.3 Note that $\psi_{f,t}$ is bijective from $(0, M_{f,t})$ to $(0,+\infty)$, so that the term in the exponential is negative and the inequality (4) makes sense.

Proof. Fix $x \in E$ and $t > 0$. Proceeding as in the end of the proof of Theorem 3.1, it is sufficient to establish the result for bounded Lipschitz function f. Applying Lemma 4.1 with $g_1 = f - \mathbb{E}_x[f(X_t)]$ and $g_2 = \exp\left(\lambda(f - \mathbb{E}_x[f(X_t)])\right)$, $\lambda \in (0, M_{f,t})$, we have:

$$\begin{aligned}&\mathbb{E}_x\left[(f(X_t)-\mathbb{E}_x[f(X_t)])\,e^{\lambda(f(X_t)-\mathbb{E}_x[f(X_t)])}\right] \\ &= \mathrm{Cov}_x\left[f(X_t)-\mathbb{E}_x[f(X_t)], e^{\lambda(f(X_t)-\mathbb{E}_x[f(X_t)])}\right]\end{aligned}$$

$$
\begin{aligned}
&\leq 2L_{t,\rho}\|\Gamma f\|_\infty^{1/2}\, e^{-\lambda \mathbb{E}_x[f(X_t)]}\mathbb{E}_x\left[\left(\Gamma e^{\lambda f}\right)^{1/2}(X_t)\right] \\
&= \sqrt{2}\, L_{t,\rho}\|\Gamma f\|_\infty^{1/2}\, \mathbb{E}_x\left[e^{\lambda(f(X_t)-\mathbb{E}_x[f(X_t)])} \left(\sum_{y,z\in E} \left(e^{\lambda(f(y)-f(z))}-1\right)^2 1_{\{X_t=z\}} Q(z,y)\right)^{1/2}\right] \\
&\leq \sqrt{2}\, L_{t,\rho}\|\Gamma f\|_\infty^{1/2} \\
&\times\ \mathbb{E}_x\left[e^{\lambda(f(X_t)-\mathbb{E}_x[f(X_t)])} \left(\sum_{y,z\in E} (f(y)-f(z))^2\, 1_{\{X_t=z\}} \left(\frac{e^{\lambda\|f\|_{\mathrm{Lip}}d(y,z)}-1}{\|f\|_{\mathrm{Lip}}d(y,z)}\right)^2 Q(z,y)\right)^{1/2}\right],
\end{aligned}
$$

where in the last inequality we used the elementary $|e^z-1|\leq e^{|z|}-1$, $z\in\mathbb{R}$, together with the increase of the function $z\mapsto (e^z-1)/z$ on $(0,+\infty)$. Thus, we obtain:

$$
\mathbb{E}_x\left[(f(X_t)-\mathbb{E}_x[f(X_t)])\, e^{\lambda(f(X_t)-\mathbb{E}_x[f(X_t)])}\right] \leq \psi_{f,t}(\lambda)\mathbb{E}_x\left[e^{\lambda(f(X_t)-\mathbb{E}_x[f(X_t)])}\right].
$$

Letting $H_{f,t,x}(\lambda)=\mathbb{E}_x\left[e^{\lambda(f(X_t)-\mathbb{E}_x[f(X_t)])}\right]$, the latter inequality rewrites as

$$
\frac{dH_{f,t,x}(\lambda)}{d\lambda} \leq \psi_{f,t}(\lambda) H_{f,t,x}(\lambda),
$$

and integrating the above differential inequality yields:

$$
\mathbb{E}_x\left[e^{\lambda(f(X_t)-\mathbb{E}_x[f(X_t)])}\right] \leq e^{\int_0^\lambda \psi_{f,t}(\tau)d\tau}, \quad \lambda\in(0,M_{f,t}).
$$

Finally, using the exponential Chebychev's inequality, Theorem 4.2 is established. ■

4.2 Some explicit tail estimates

Since the estimate (4) is very general, let us make further assumptions on the chain to get more explicit inequalities. Denote in the sequel $L_{t,\rho}=(1-e^{-2\rho t})/(2\rho)$ if $\rho\neq 0$, and $L_{t,\rho}=t$ otherwise, and denote the function $g(u)=(1+u)\log(1+u)-u$, $u>0$. Using the notation of Theorem 4.2, we have the

Corollary 4.4 *Under the hypothesis of Theorem 4.2, suppose moreover that $(X_t)_{t\geq 0}$ has jumps bounded by $b>0$. Then for any initial state $x\in E$, any $y>0$ and any $t>0$, we get the local Poisson-type deviation inequality:*

$$
\begin{aligned}
\mathbb{P}_x\left(f(X_t)-\mathbb{E}_x\left[f(X_t)\right]\geq y\right) &\leq \exp\left(-\frac{2L_{t,\rho}\|\Gamma f\|_\infty}{b^2\|f\|_{\mathrm{Lip}}^2}\, g\left(\frac{by\|f\|_{\mathrm{Lip}}}{2L_{t,\rho}\|\Gamma f\|_\infty}\right)\right) \qquad (5)\\
&\leq \exp\left(-\frac{y}{2b\|f\|_{\mathrm{Lip}}}\log\left(1+\frac{by\|f\|_{\mathrm{Lip}}}{2L_{t,\rho}\|\Gamma f\|_\infty}\right)\right).
\end{aligned}
$$

Proof. Under the notation of Theorem 4.2, the boundedness of the jumps implies $M_{f,t} = +\infty$, $t > 0$, and $\psi_{f,t}$ is bounded by

$$\psi_{f,t}(\lambda) \leq 2L_{t,\rho}\|\Gamma f\|_\infty \frac{e^{\lambda b\|f\|_{\mathrm{Lip}}} - 1}{b\|f\|_{\mathrm{Lip}}}, \quad \lambda > 0.$$

Using then Theorem 4.2 and optimizing in $\lambda > 0$, the proof is achieved. ■

Note that (5) is more general than (2), since the finiteness assumption on $\|\Gamma f\|_\infty$ allows us to consider Markov chains with non necessarily bounded angle bracket. Thus, when the angle bracket of the process is bounded, the next corollary exhibits an estimate comparable to that of Theorem 3.1:

Corollary 4.5 *Let $(X_t)_{t\geq 0}$ be a regular Markov chain on E with jumps and angle bracket bounded respectively by $b > 0$ and $V^2 > 0$. Assume moreover that its Γ-curvature is bounded below by $\rho \in \mathbb{R}$, and let $f \in \mathrm{Lip}(E)$. Then for any initial state $x \in E$, any $y > 0$ and any $t > 0$, we get the local Poisson tail probability:*

$$\begin{aligned}\mathbb{P}_x\left(f(X_t) - \mathbb{E}_x\left[f(X_t)\right] \geq y\right) &\leq \exp\left(-\frac{L_{t,\rho}V^2}{b^2} g\left(\frac{by}{L_{t,\rho}V^2\|f\|_{\mathrm{Lip}}}\right)\right) \\ &\leq \exp\left(-\frac{y}{2b\|f\|_{\mathrm{Lip}}}\log\left(1 + \frac{by}{L_{t,\rho}V^2\|f\|_{\mathrm{Lip}}}\right)\right).\end{aligned} \tag{6}$$

Proof. By the boundedness of the jumps and of the angle bracket, the function $\psi_{f,t}$ in Theorem 4.2 is bounded by

$$\psi_{f,t}(\lambda) \leq L_{t,\rho}V^2\|f\|_{\mathrm{Lip}} \frac{e^{\lambda b\|f\|_{\mathrm{Lip}}} - 1}{b}, \quad \lambda > 0.$$

Finally, applying Theorem 4.2 yields the result. ■

Remark 4.6 As in Section 3, a similar discussion about large deviation bounds under the assumption of Γ-curvature bounded below can be derived from the estimates (5) and (6), so we omit it.

5 Deviation probabilities for curved irreducible birth-death processes

Among the main results of the paper [2], some local deviation inequalities are established for continuous time random walks on graphs. Actually, such processes may be seen as models in null curvature, since the rates of the associated generator do not depend on the space-variable.

By using the general results of Sections 3 and 4, the purpose of this section is to extend and sharpen these local tail estimates to birth-death processes whose discrete curvatures are bounded below. Let us introduce now some basic material about birth-death processes. Let $(X_t)_{t\geq 0}$ be a birth-death process on the state space $E = \mathbb{N}$ or $E = \{0, 1, \ldots, n\}$. It is a regular Markov chain with generator defined on $\mathrm{Dom}\,\mathscr{L} = \mathscr{F}(E)$ (recall that $\mathscr{F}(E)$ is the collection of real-valued functions on E) by

$$\mathscr{L} f(x) = \lambda_x \left(f(x+1) - f(x) \right) + \nu_x \left(f(x-1) - f(x) \right), \quad x \in E, \tag{7}$$

where the function rates λ and ν are respectively called the birth and death rates of the chain. The chain $(X_t)_{t\geq 0}$ is irreducible on E if and only if the rates λ and ν are positive with 0 as reflecting state, i.e. $\nu_0 = 0$ (if $E = \{0, 1, \ldots, n\}$, the state n is also reflecting, i.e. $\lambda_n = 0$), and we assume irreducibility in the remainder of the paper. The transition probabilities of the associated semigroup $(P_t)_{t\geq 0}$ are given for any $x \in E$ by

$$P_t(x,y) = \begin{cases} \lambda_x t + o(t) & \text{if } y = x+1, \\ \nu_x t + o(t) & \text{if } y = x-1, \\ 1 - (\lambda_x + \nu_x) t + o(t) & \text{if } y = x, \\ 0 & \text{if } y \in E \setminus \{x-1, x, x+1\}, \end{cases}$$

where the function o is defined in a neighborhood of 0 and is such that $o(t)/t$ converges to 0 as t tends to 0. The chain is positive recurrent if and only if

$$\sum_{x \in E\setminus\{0\}} \prod_{y=1}^{x} \frac{\lambda_{y-1}}{\nu_y} < +\infty,$$

and in this case, the unique stationary distribution π is given for any $x \in E$ by

$$\pi(x) = \pi(0) \prod_{y=1}^{x} \frac{\lambda_{y-1}}{\nu_y}, \quad \text{with} \quad \pi(0) = \left(1 + \sum_{x \in E\setminus\{0\}} \prod_{y=1}^{x} \frac{\lambda_{y-1}}{\nu_y} \right)^{-1}. \tag{8}$$

By Reuter's criterion, any irreducible birth-death process on $E = \mathbb{N}$ is non-explosive in finite time if and only if

$$\sum_{x=1}^{+\infty} \left(\frac{1}{\lambda_x} + \frac{\nu_x}{\lambda_x \lambda_{x-1}} + \cdots + \frac{\nu_x \cdots \nu_1}{\lambda_x \cdots \lambda_1 \lambda_0} \right) = +\infty,$$

see for instance Theorem 4.5 page 352 in [6].

Remark 5.1 If the birth rate λ is bounded, then Reuter's criterion immediately applies.

Before stating the main results of this section, let us give some criteria on the rates of the generator of an irreducible birth-death process on E which ensure that its discrete curvatures are bounded below.

First, we deal with the Wasserstein curvature.

Proposition 5.2 *Let $(X_t)_{t\geq 0}$ be an irreducible birth-death process on E with generator $\mathscr{L}$ given by (7). Assume that there exists a real number K such that*

$$\inf_{x\in E\setminus\{0\}} \lambda_{x-1} - \lambda_x + \nu_x - \nu_{x-1} \geq K. \tag{9}$$

Then the Wasserstein curvature of the chain is bounded below by K.

Remark 5.3 If $E = \mathbb{N}$ and the rates of the generator are bounded and satisfy the assumptions of Proposition 5.2, then necessarily $K \leq 0$.

Proof. Let us establish the result via coupling methods. Consider $(X_t^x)_{t\geq 0}$ and $(X_t^y)_{t\geq 0}$ two independent copies of $(X_t)_{t\geq 0}$, starting respectively from x and y. Then the two-dimensional process $(X_t^x, X_t^y)_{t\geq 0}$, which is an independent coupling of $(X_t)_{t\geq 0}$, see for instance [8, Chapter 5], is an irreducible birth-death process on $E\times E$ starting from (x,y) and with generator given for any $f\in\mathscr{F}(E\times E)$ by $\tilde{\mathscr{L}} f = \mathscr{L}\otimes I + I\otimes \mathscr{L}$, where we recall that I is the identity on $\mathscr{F}(E)$. In other words, $\tilde{\mathscr{L}}$ rewrites as

$$\tilde{\mathscr{L}} f(z,w) = (\mathscr{L} f(\cdot,w))(z) + (\mathscr{L} f(z,\cdot))(w), \quad z,w\in E.$$

Denote by d the classical distance on E, i.e. $d(z,w) = |z-w|$, $z,w\in E$. Since the rates of the generator satisfy the inequality (9), we have immediately the bound $\tilde{\mathscr{L}} d(z,z+1) \leq -K$, $z\in E$, which is equivalent to the inequality $\tilde{\mathscr{L}} d(z,w) \leq -K d(z,w)$ for any $z,w\in E$. Therefore, letting the semigroup act in both sides and integrating the latter inequality yield the estimate $\mathbb{E}_{(x,y)}\left[d(X_t^x, X_t^y)\right] \leq e^{-Kt} d(x,y)$ which in turn implies the following inequality in terms of Wasserstein distance:

$$W(P_t(x,\cdot), P_t(y,\cdot)) \leq e^{-Kt} d(x,y).$$

Finally, by the equivalence between (ii) and (iii) of Remark 2.2, the Wasserstein curvature of $(X_t)_{t\geq 0}$ is bounded below by K. ∎

In order to establish modified logarithmic Sobolev inequalities for continuous time random walks on $\mathbb{Z}$, the authors in [2] used a suitable Γ_2-calculus to give a criterion under which the Γ-curvature is bounded below by 0. Actually, this criterion can be generalized to any real lower bound on the Γ-curvature via Lemma 5.4 below. As in the diffusion case [3], define the Γ_2-operator on $\mathscr{F}(E)$ by

$$\Gamma_2 f(x) = \frac{1}{2}\left(\mathscr{L}\Gamma f(x) - 2\Gamma(f, \mathscr{L} f)(x)\right), \quad x \in E.$$

By adapting the proof in [2] mentioned above, we get the

Lemma 5.4 *Let $(X_t)_{t\geq 0}$ be an irreducible birth-death process on E with generator $\mathscr{L}$ given by (7). Assume that there exists $\rho \in \mathbb{R}$ such that the inequality*

$$\Gamma_2 f(x) - \Gamma\left(\Gamma f\right)^{1/2}(x) \geq \rho \Gamma f(x), \quad x \in E, \tag{10}$$

is satisfied for any $f \in \mathscr{F}(E)$. Then $(X_t)_{t\geq 0}$ has Γ-curvature bounded below by ρ.

Proposition 5.5 *Let $(X_t)_{t\geq 0}$ be an irreducible birth-death process on E with generator $\mathscr{L}$ given by (7). Assume that λ and ν are respectively non-increasing and non-decreasing and that there exists some non-negative number ρ such that*

$$\inf_{x \in E \setminus \{0, \sup E\}} \min\{\lambda_{x-1} - \lambda_x, \nu_{x+1} - \nu_x\} \geq \rho. \tag{11}$$

Then the Γ-curvature is bounded below by ρ.

Remark 5.6 If $E = \mathbb{N}$ and the rates of the generator satisfy the assumptions of Proposition 5.5, then necessarily $\rho = 0$.

Proof. By Lemma 5.4, the result holds true if the Γ_2-inequality (10) above is satisfied, that we prove now. Letting the forward and backward gradients be defined as $\mathrm{d}^+ f = f(\cdot + 1) - f$ and $\mathrm{d}^- f = f(\cdot - 1) - f$, we have for any $x \in E$:

$$\begin{aligned} & 2\Gamma_2 f(x) - 2\Gamma\left(\Gamma f\right)^{1/2}(x) = \\ & \quad \lambda_x(\nu_{x+1} - \nu_x)\left(\mathrm{d}^+ f(x)\right)^2 + \nu_x(\lambda_{x-1} - \lambda_x)\left(\mathrm{d}^- f(x)\right)^2 + I(x) + J(x), \end{aligned}$$

where:

$$\begin{aligned} I(x) &:= \lambda_x \lambda_{x+1} \mathrm{d}^- f(x+1) \mathrm{d}^+ f(x+1) + \lambda_x \nu_x \mathrm{d}^- f(x+1) \mathrm{d}^+ f(x-1) \\ & \quad + \lambda_x \left(\lambda_{x+1}\left(\mathrm{d}^+ f(x+1)\right)^2 + \nu_{x+1}\left(\mathrm{d}^+ f(x)\right)^2\right)^{1/2} \left(\lambda_x\left(\mathrm{d}^+ f(x)\right)^2 + \nu_x\left(\mathrm{d}^- f(x)\right)^2\right)^{1/2}, \end{aligned}$$

and

$$\begin{aligned} J(x) &:= \nu_x \nu_{x-1} \mathrm{d}^+ f(x-1) \mathrm{d}^- f(x-1) + \lambda_x \nu_x \mathrm{d}^- f(x+1) \mathrm{d}^+ f(x-1) \\ & \quad + \nu_x \left(\lambda_{x-1}\left(\mathrm{d}^+ f(x-1)\right)^2 + \nu_{x-1}\left(\mathrm{d}^- f(x-1)\right)^2\right)^{1/2} \left(\lambda_x\left(\mathrm{d}^+ f(x)\right)^2 + \nu_x\left(\mathrm{d}^- f(x)\right)^2\right)^{1/2}. \end{aligned}$$

Since the rates λ and ν are respectively non-increasing and non-decreasing and satisfy furthermore the inequality (11), we get:

$$2\Gamma_2 f(x) - 2\Gamma\left(\Gamma f\right)^{1/2}(x) \geq 2\rho\Gamma f(x) + I(x) + J(x).$$

Proving in the same way that $J \geq 0$, it is sufficient to establish that I is non-negative. Letting $a = \mathrm{d}^- f(x+1)$, $b = \mathrm{d}^+ f(x-1)$ and $c = \mathrm{d}^+ f(x+1)$, we obtain:

$$\begin{aligned} I(x) &= \lambda_x \left(\lambda_{x+1}c^2 + \nu_{x+1}a^2\right)^{1/2} \left(\lambda_x a^2 + \nu_x b^2\right)^{1/2} + \lambda_x\lambda_{x+1}ac + \lambda_x\nu_x ab \\ &\geq \lambda_x \left(\lambda_{x+1}c^2 + \nu_{x+1}a^2\right)^{1/2} \left(\lambda_x a^2 + \nu_x b^2\right)^{1/2} - \lambda_x\lambda_{x+1}|ac| - \lambda_x\nu_x|ab| \\ &= \lambda_x\left(I_1(x) - I_2(x)\right), \end{aligned}$$

where

$$I_1(x) := \left(\lambda_{x+1}c^2 + \nu_{x+1}a^2\right)^{1/2} \left(\lambda_x a^2 + \nu_x b^2\right)^{1/2} \quad \text{and} \quad I_2(x) := \lambda_{x+1}|ac| + \nu_x|ab|.$$

Using again the monotonic assumptions on the rates of the generator, we have:

$$\begin{aligned} &\left(I_1(x)\right)^2 - \left(I_2(x)\right)^2 \\ &\quad = \lambda_{x+1}(\lambda_x - \lambda_{x+1})a^2c^2 + \nu_x(\nu_{x+1} - \nu_x)a^2b^2 + \lambda_x\nu_{x+1}a^4 + \lambda_{x+1}\nu_x b^2c^2 - 2\nu_x\lambda_{x+1}a^2bc \\ &\quad \geq \nu_x\lambda_{x+1}(a^2 - bc)^2 \geq 0. \end{aligned}$$

The proof is complete. ■

5.1 The case $E = \mathbb{N}$

An estimate for bounded generators

In order to apply the deviation inequalities of Theorem 3.1, one has to require that regular Markov chain has Wasserstein curvature bounded below and bounded angle bracket. In the case of an irreducible birth-death process on $\mathbb{N}$, the latter assumption is equivalent to assume that the rates of the generator are bounded.

Theorem 5.7 *Let $(X_t)_{t\geq 0}$ be an irreducible birth-death process on $\mathbb{N}$ with generator $\mathscr{L}$ given by (7), where $\lambda, \nu \in \mathscr{B}(\mathbb{N})$. Assume that there exists $K \leq 0$ such that $\inf_{x\in\mathbb{N}\setminus\{0\}} \lambda_{x-1} - \lambda_x + \nu_x - \nu_{x-1} \geq K$, and let $f \in \mathrm{Lip}(\mathbb{N})$. Then for any initial state $x \in \mathbb{N}$, any deviation level $y > 0$ and any $t > 0$, we have the local Poisson-type tail estimate:*

$$\mathbb{P}_x\left(f(X_t) - \mathbb{E}_x\left[f(X_t)\right] \geq y\right) \tag{12}$$

$$\leq \exp\left(-\frac{\sinh(tK)\|\lambda+\nu\|_\infty}{Ke^{tK}} g\left(\frac{yK}{\sinh(tK)\|\lambda+\nu\|_\infty\|f\|_{\mathrm{Lip}}}\right)\right)$$
$$\leq \exp\left(-\frac{ye^{-tK}}{2\|f\|_{\mathrm{Lip}}}\log\left(1+\frac{yK}{\sinh(tK)\|\lambda+\nu\|_\infty\|f\|_{\mathrm{Lip}}}\right)\right),$$

where $g(u) = (1+u)\log(1+u) - u$, $u > 0$. *If* $K = 0$, *then replace (12) by its limit as* $K \to 0$.

Proof. By Proposition 5.2, the Wasserstein curvature is bounded below by K. Since λ is bounded, the chain is non-explosive in finite time, and the use of Theorem 3.1 achieves the proof. ■

An inequality for non necessarily bounded generators

In this part, no particular boundedness assumption is made on the generator of birth-death process.

Theorem 5.8 *Let* $(X_t)_{t\geq 0}$ *be an irreducible birth-death process on* $\mathbb{N}$ *with generator* $\mathscr{L}$ *given by (7). Assume that* λ *and* ν *are respectively non-increasing and non-decreasing. Let* $f \in \mathrm{Lip}(\mathbb{N})$ *with furthermore* $\|\Gamma f\|_\infty < +\infty$. *Then for any initial state* $x \in \mathbb{N}$, *any* $y > 0$ *and any* $t > 0$, *we have the local deviation estimate:*

$$\begin{aligned}\mathbb{P}_x\left(f(X_t) - \mathbb{E}_x\left[f(X_t)\right] \geq y\right) \quad &\leq \quad \exp\left(-\frac{2t\|\Gamma f\|_\infty}{\|f\|_{\mathrm{Lip}}^2} g\left(\frac{y\|f\|_{\mathrm{Lip}}}{2t\|\Gamma f\|_\infty}\right)\right) \\ &\leq \quad \exp\left(-\frac{y}{2\|f\|_{\mathrm{Lip}}}\log\left(1+\frac{y\|f\|_{\mathrm{Lip}}}{2t\|\Gamma f\|_\infty}\right)\right),\end{aligned}$$

where $g(u) = (1+u)\log(1+u) - u$, $u > 0$.

Proof. By Proposition 5.5, the Γ-curvature is bounded below by 0. Since the birth rate λ is bounded above by λ_0, the chain is non-explosive, and applying Corollary 4.4 with the lower bound $\rho = 0$ yields the result. ■

Remark 5.9 As claimed above, Theorem 5.8 is available for birth-death processes with non necessarily bounded generator, in contrast to Theorem 5.7. However, the price to pay in the unbounded case is to require that f is bounded, since the unboundedness of the death rate ν together with the finiteness assumption of $\|\Gamma f\|_\infty$ imply that f is a convergent sequence, hence bounded.

5.2 The case $E = \{0, 1, \ldots, n\}$

If π denotes the stationary distribution of an irreducible Markov chain on a finite state space, then it satisfies a logarithmic Sobolev inequality, see [17], which in turn implies via the Herbst method that Lipschitz functions have Gaussian tails under π. However, it is sometimes interesting to weaken the upper bound in terms of the deviation level to have a better control of the tail with respect to some parameters, see for instance the discussion in [5] about concentration for Bernoulli distributions and penalties.
In this way, the purpose of this part is to refine Theorem 5.7 and Theorem 5.8 when the state space is finite, in order to establish by a limiting argument Poisson-type deviation estimates for stationary distributions of birth-death processes. To do so, the crucial point is to obtain positive lower bounds on discrete curvatures.
Our estimates below may be compared to that of [13, Proposition 4] established under reversibility assumptions and without notion of discrete curvatures.

Theorem 5.10 *Let $(X_t)_{t\geq 0}$ be an irreducible birth-death process on $\{0, 1, \ldots, n\}$ with generator $\mathscr{L}$ given by (7) and stationary distribution π. Assume that there exists $K > 0$ such that $\min_{x\in\{1,\ldots,n\}} \lambda_{x-1} - \lambda_x + \nu_x - \nu_{x-1} \geq K$, and let $f \in \mathrm{Lip}(\{0, 1, \ldots, n\})$. Then for any initial state $x \in \{0, 1, \ldots, n\}$, any deviation level $y > 0$ and any $t > 0$, we have:*

$$\begin{aligned} &\mathbb{P}_x\left(f(X_t) - \mathbb{E}_x\left[f(X_t)\right] \geq y\right) \\ &\quad \leq \exp\left(-\frac{(1 - e^{-2Kt})\|\lambda + \nu\|_\infty}{2K} g\left(\frac{2Ky}{(1 - e^{-2Kt})\|\lambda + \nu\|_\infty \|f\|_{\mathrm{Lip}}}\right)\right), \end{aligned}$$

where $g(u) = (1 + u)\log(1 + u) - u$, $u > 0$.
In particular, letting t going to infinity in the above local inequality yields the deviation estimate under π:

$$\pi\left(f - \mathbb{E}_\pi[f] \geq y\right) \leq \exp\left(\frac{y}{\|f\|_{\mathrm{Lip}}} - \left(\frac{y}{\|f\|_{\mathrm{Lip}}} + \frac{\|\lambda + \nu\|_\infty}{2K}\right)\log\left(1 + \frac{2Ky}{\|\lambda + \nu\|_\infty \|f\|_{\mathrm{Lip}}}\right)\right).$$

Proof. By Proposition 5.2, the Wasserstein curvature is bounded below by K. Therefore, it remains to apply Theorem 3.1 to get the result. ■

Under different assumptions on the rates of the generator, we get a somewhat similar estimate:

Theorem 5.11 *Let $(X_t)_{t\geq 0}$ be an irreducible birth-death process on $\{0, 1, \ldots, n\}$ with generator $\mathscr{L}$ given by (7) and stationary distribution π. Assume that the rates λ and*

ν are respectively non-increasing and non-decreasing and that there exists $\rho > 0$ such that $\min_{x\in\{1,\ldots,n-1\}} \min\{\lambda_{x-1} - \lambda_x, \nu_{x+1} - \nu_x\} \geq \rho$. *Let* $f \in \mathrm{Lip}(\{0, 1, \ldots, n\})$. *Then for any initial state $x \in \{0, 1, \ldots, n\}$, any deviation level $y > 0$ and any $t > 0$, we have:*

$$\begin{aligned} &\mathbb{P}_x \left(f(X_t) - \mathbb{E}_x \left[f(X_t) \right] \geq y \right) \\ &\qquad \leq \exp\left(-\frac{(1-e^{-2\rho t})(\lambda_0 + \nu_n)}{2\rho} g\left(\frac{2\rho y}{(1-e^{-2\rho t})(\lambda_0+\nu_n)\|f\|_{\mathrm{Lip}}} \right) \right), \end{aligned}$$

where $g(u) = (1+u)\log(1+u) - u$, $u > 0$.

In particular, letting t going to infinity in the above local inequality entails the following tail probability under the stationary distribution π:

$$\pi\left(f - \mathbb{E}_\pi[f] \geq y \right) \leq \exp\left(\frac{y}{\|f\|_{\mathrm{Lip}}} - \left(\frac{y}{\|f\|_{\mathrm{Lip}}} + \frac{\lambda_0 + \nu_n}{2\rho} \right) \log\left(1 + \frac{2\rho y}{(\lambda_0 + \nu_n)\|f\|_{\mathrm{Lip}}} \right) \right).$$

Proof. By Proposition 5.5, the Γ-curvature is bounded below by ρ. Hence, applying Corollary 4.5 achieves the proof. ■

Remark 5.12 In order to obtain deviation bounds for stationary distributions, the positivity of lower bounds of discrete curvatures is crucial and thus does not allow us to extend such estimates to birth-death processes on the infinite state space $E = \mathbb{N}$, see the Remark 5.3 and Remark 5.6.

In particular, it excludes the $M/M/1$ and $M/M/\infty$ queueing processes recently investigated by D. Chafai [7]. The stationary distributions for these queues are respectively the Poisson and geometric distributions on $\mathbb{N}$, which deviation is of Poisson-type, see [11]. Therefore, we expect to recover such estimates by taking the limit as $t \to +\infty$ in some appropriate local deviation inequalities satisfied by the queueing processes above, and such an interesting problem will be addressed in a forthcoming research.

Note also that while both Theorem 5.7 and Theorem 5.8 apply for the $M/M/1$ queue, cf. Section 5.4 below, the sole Theorem 5.8 with $\rho = 0$ is available for the $M/M/\infty$ queueing process, and such a result does not reflect the positive exact curvature of this queue emphasized in [7].

5.3 Ornstein-Uhlenbeck processes as fluid limits of rescaled Ehrenfest chains

In this part, we recover via Theorem 5.10 the optimal Gaussian concentration for an Ornstein-Uhlenbeck process constructed as a fluid limit of a rescaled continuous time

Ehrenfest chain.

Given $n \in \mathbb{N}$, let $(X_t^n)_{t\geq 0}$ be the continuous time Ehrenfest chain on $\{0,1,\ldots,n\}$ starting from some $x_n \in \{0,1,\ldots,n\}$ and with generator given by:

$$\mathscr{L}_n f(x) = \lambda(n-x)\left(f(x+1)-f(x)\right) + \nu x\left(f(x-1)-f(x)\right), \quad x \in \{0,1,\ldots,n\},$$

where $0 < \lambda \leq \nu < 1$ are such that $\lambda + \nu = 1$.

Let $y(t) = \lambda + (y_0 - \lambda)e^{-t}$, $t > 0$, where $y_0 = \lim_{n\to+\infty} X_0^n/n$, and define for any $n \in \mathbb{N}\backslash\{0\}$ the process $(Z_t^n)_{t\geq 0}$ by $Z_t^n = (X_t^n - ny(t))/\sqrt{n}$, $t > 0$. Assume furthermore that the sequence of initial states $(Z_0^n)_{n\in\mathbb{N}}$ converges to z_0 (say).

By the central limit theorem in [10, Chapter 11], the sequence of processes $(Z_t^n)_{t\geq 0}$ converges as n goes to infinity to the process $(Z_t)_{t\geq 0}$ which is the unique solution of the equation

$$Z_t = z_0 + \int_0^t \sqrt{\lambda + (\nu-\lambda)y(s)}\,dB_s - \int_0^t Z_s ds, \quad t > 0,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion.

In particular, if $y_0 = \lambda$, then $y(t) = \lambda$ for any $t > 0$ and $(Z_t)_{t\geq 0}$ rewrites as the Ornstein-Uhlenbeck process $(U_t)_{t\geq 0}$:

$$U_t = z_0 e^{-t} + \sqrt{2\lambda\nu}\int_0^t e^{-(t-s)}dB_s, \quad t > 0.$$

Now, fix $n \in \mathbb{N}\backslash\{0\}$ and time $t > 0$, and let $f \in \mathrm{Lip}(\mathbb{R})$. If h_n denotes the function $h_n = f \circ \phi_n$, where ϕ_n is defined on $\{0,1,\ldots,n\}$ by $\phi_n(x) = (x-n\lambda)/\sqrt{n}$, then $h_n \in \mathrm{Lip}(\{0,1,\ldots,n\})$ with constant at most $n^{-1/2}\|f\|_{\mathrm{Lip}}$. Therefore we can apply Theorem 5.10 to $(X_t^n)_{t\geq 0}$ and h_n, with $K = 1$, to get for any fixed $n \in \mathbb{N}\backslash\{0\}$, any deviation level $y > 0$ and any $t > 0$, the local deviation estimate:

$$\begin{aligned}&\mathbb{P}_{x_n}\left(h_n(X_t^n) - \mathbb{E}_{x_n}\left[h_n(X_t^n)\right] \geq y\right)\\ &\quad\leq \exp\left(-\frac{(1-e^{-2t})n\nu}{2}g\left(\frac{2\sqrt{n}y}{(1-e^{-2t})n\nu\|f\|_{\mathrm{Lip}}}\right)\right),\end{aligned}$$

where $g(u) = (1+u)\log(1+u) - u$, $u > 0$. Finally, letting n going to infinity in the above inequality yields for any $y > 0$ and any $t > 0$ the optimal Gaussian deviation:

$$\mathbb{P}_{z_0}\left(f(U_t) - \mathbb{E}_{z_0}\left[f(U_t)\right] \geq y\right) \leq \exp\left(-\frac{2y^2}{(1-e^{-2t})\nu\|f\|_{\mathrm{Lip}}^2}\right),$$

see for instance Theorems 5.1 and 5.3 in [15].

5.4 A local deviation inequality for sample vectors of the $M/M/1$ queue

In this part, we give a local deviation estimate for sample vectors of the $M/M/1$ queueing process. Recall it is an irreducible birth-death process whose generator is given by

$$\mathscr{L} f(x) = \lambda \left(f(x+1) - f(x) \right) + \nu 1_{\{x \neq 0\}} \left(f(x-1) - f(x) \right), \quad x \in \mathbb{N},$$

where the positive numbers λ and ν correspond respectively to the input rate and service rate of the queue: the independent and identically distributed interarrival times and independent and identically distributed service times of the customers follow an exponential law with respective parameters λ and ν. As noticed in Remark 5.12, we are unable to recover via Theorem 5.7 or Theorem 5.8 the Poisson-type deviation inequality satisfied by the geometric distribution, cf. [11], which is by (8) the unique stationary distribution of the $M/M/1$ queueing process in the positive recurrent case $\lambda < \nu$. However, the existence of an integration by parts formula for the associated semigroup together with a tensorization procedure of the Laplace transform allow us to provide with Theorem 5.13 below a local inequality for sample vectors of the $M/M/1$ queue.

We say in the sequel that a function $f : \mathbb{N}^d \to \mathbb{R}$ is ℓ^1-Lipschitz if

$$\|f\|_{\mathrm{Lip}(d)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|_1} < +\infty,$$

where $\| \cdot \|_1$ denotes the ℓ^1-norm $\|z\|_1 = \sum_{i=1}^d |z_i|$, $z \in \mathbb{N}^d$.

Now, we can state the following

Theorem 5.13 *Let $(X_t)_{t \geq 0}$ be the $M/M/1$ queue with input and service rates λ and ν. Let f be ℓ^1-Lipschitz on $\mathbb{N}^d$ and consider the sample vector $X_d = (X_{t_1}, \ldots, X_{t_d})$, $t_1 < \cdots < t_d = T$. Then for any initial state $x \in \mathbb{N}$ and any deviation level $y > 0$, we have the mulitdimensional local Poisson like deviation inequality:*

$$\begin{aligned} \mathbb{P}_x \left(f(X_d) - \mathbb{E}_x[f(X_d)] \geq y \right) &\leq \exp\left(-T(\lambda + \nu) g\left(\frac{y}{Td(\lambda + \nu)\|f\|_{\mathrm{Lip}(d)}} \right) \right) \qquad (13) \\ &\leq \exp\left(-\frac{y}{2d\|f\|_{\mathrm{Lip}(d)}} \log\left(1 + \frac{y}{Td(\lambda + \nu)\|f\|_{\mathrm{Lip}(d)}} \right) \right), \end{aligned}$$

where $g(u) = (1+u)\log(1+u) - u$, $u > 0$.

Proof. Fix the initial state $x \in \mathbb{N}$. If u is a one dimensional Lipschitz function on $\mathbb{N}$ and $t > 0$, then rewriting the proof of Theorem 4.2 for the $M/M/1$ queue yields for any $\tau > 0$:

$$\mathbb{E}_x \left[e^{\tau u(X_t)} \right] \quad \leq \quad \exp \left(\tau \mathbb{E}_x[u(X_t)] + h(\tau, t, \|u\|_{\mathrm{Lip}}) \right), \tag{14}$$

where h is the function defined on $(\mathbb{R}_+)^3$ by $h(\tau, t, z) = t(\lambda + \nu)\left(e^{\tau z} - \tau z - 1\right)$ and $\| \cdot \|_{\mathrm{Lip}}$ remains for the classical Lipschitz seminorm on $\mathbb{N}$.

To obtain a multidimensional version of (14), the idea is to tensorize it via an integration by parts formula satisfied by the semigroup $(P_t)_{t \geq 0}$ of the $M/M/1$ queueing process. We sketch now the argument for $d = 2$. Let $0 < s < t$ and denote f_y the function $f_y(z) = f(y, z)$ and $f_1(y) = \sum_{z \in \mathbb{N}} f(y, z) P_{t-s}(y, z)$. By the Markov property together with (14), we have:

$$\begin{aligned}
&\mathbb{E}_x \left[\exp \left(\tau f(X_s, X_t) \right) \right] \\
&\quad = \sum_{y, z \in \mathbb{N}} \exp \left(\tau f_y(z) \right) P_{t-s}(y, z) P_s(x, y) \\
&\quad \leq \sum_{y \in \mathbb{N}} \exp \left(\tau \sum_{z \in \mathbb{N}} f_y(z) P_{t-s}(y, z) + h(\tau, t-s, \|f_y\|_{\mathrm{Lip}}) \right) P_s(x, y) \\
&\quad \leq \exp \big(h(\tau, t-s, \|f\|_{\mathrm{Lip}(2)}) \big) \sum_{y \in \mathbb{N}} \exp \left(\tau f_1(y) \right) P_s(x, y) \\
&\quad \leq \exp \left\{ h(\tau, t-s, 2\|f\|_{\mathrm{Lip}(2)}) + h(\tau, s, \|f_1\|_{\mathrm{Lip}}) + \tau \mathbb{E}_x[f(X_s, X_t)] \right\},
\end{aligned} \tag{15}$$

since the function $z \mapsto h(\cdot, \cdot, z)$ is non-decreasing on $[0, +\infty)$. Now, let us bound $\|f_1\|_{\mathrm{Lip}}$ by $2\|f\|_{\mathrm{Lip}(2)}$. To do so, observe that we have the commutation relation $\mathscr{L} \mathrm{d}^+ = \mathrm{d}^+ \mathscr{L}$, where d^+ is the forward gradient $\mathrm{d}^+ f(x) = f(x+1) - f(x)$, $x \in \mathbb{N}$. It implies $P_t \mathrm{d}^+ = \mathrm{d}^+ P_t$ for any non-negative t, which in turn entails for any $u \in \mathscr{F}(\mathbb{N})$ the integration by parts formula:

$$\sum_{y \in \mathbb{N}} u(y) P_t(x+1, y) = \sum_{y \in \mathbb{N}} u(y+1) P_t(x, y), \quad x \in \mathbb{N}.$$

Thus, we have:

$$\begin{aligned}
\|f_1\|_{\mathrm{Lip}} &= \sup_{y \in \mathbb{N}} |f_1(y+1) - f_1(y)| \\
&= \sup_{y \in \mathbb{N}} \left| \sum_{z \in \mathbb{N}} f(y+1, z) P_{t-s}(y+1, z) - \sum_{z \in \mathbb{N}} f(y, z) P_{t-s}(y, z) \right| \\
&= \sup_{y \in \mathbb{N}} \left| \sum_{z \in \mathbb{N}} \left(f(y+1, z+1) - f(y, z) \right) P_{t-s}(y, z) \right| \\
&\leq 2\|f\|_{\mathrm{Lip}(2)}.
\end{aligned}$$

Therefore, plugging this into (15) entails:

$$\mathbb{E}_x\left[\exp\left(\tau f(X_s,X_t)\right)\right] \;\le\; \exp\left\{t(\lambda+\nu)\left(e^{2\tau\|f\|_{\mathrm{Lip}(2)}}-2\tau\|f\|_{\mathrm{Lip}(2)}-1\right)+\tau\mathbb{E}_x[f(X_s,X_t)]\right\}.$$

In the general case, we show similarly that for any $i=1,\ldots,d$, the function f_i defined on $\mathbb{N}^i$ by

$$f_i(x_1,\ldots,x_i) := \sum_{x_{i+1},\ldots,x_d\in\mathbb{N}} f(x_1,\ldots,x_i,\ldots,x_d)P_{t_{i+1}-t_i}(x_i,x_{i+1})\cdots P_{t_d-t_{d-1}}(x_{d-1},x_d),$$

has Lipschitz seminorm (with respect to the i^{th} variable) smaller than $(d-i+1)\|f\|_{\mathrm{Lip}(d)}$, and thus than $d\|f\|_{\mathrm{Lip}(d)}$. Therefore, since h is non-decreasing in its third variable, we obtain by using recursively (15) ($t_0=0$ by convention):

$$\begin{aligned}
&\mathbb{E}_x\left[e^{\tau f(X_d)}\right]\\
&\quad\le\; \exp\left(\tau\mathbb{E}_x[f(X_d)]+\sum_{i=1}^d h(\tau,t_i-t_{i-1},d\|f\|_{\mathrm{Lip}(d)})\right)\\
&\quad=\; \exp\left\{\tau\mathbb{E}_x[f(X_d)]+T(\lambda+\nu)\left(e^{\tau d\|f\|_{\mathrm{Lip}(d)}}-\tau d\|f\|_{\mathrm{Lip}(d)}-1\right)\right\}.
\end{aligned}$$

Finally, dividing in both sides by $e^{\tau\mathbb{E}_x[f(X_d)]}$ and using the exponential Chebychev's inequality achieve the proof. ■

Remark 5.14 To conclude this work, note that Theorem 5.13 does not allow us to extend such inequality to functionals on path spaces. Thus, it would be an interesting project to refine suitably (13) in terms of the increments $\Delta_i=t_i-t_{i-1}$, as $\Delta_i\to 0$.

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06-02 Aldéric Joulin. On local Poisson-type deviation inequalities for curved continuous time Markov chains, with applications to birth-death processes