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Some remarks on the asymptotic behavior of the Caginalp system with singular potentials

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Abstract

This article is devoted to the study of the asymptotic behavior of a Caginalp phase-field system with Neumann boundary conditions and singular potentials. We first prove the existence and uniqueness of solutions, and then the existence of exponentials attractors (and thus of finite dimensional global attractors). We finally study the convergence of solutions to steady states as time goes to infinity ; in particular, we are able to prove that, in some cases, the trajectories converge to spatially homogeneous steady states exponentially fast.

Key words and phrases. Caginalp system, singular potentials, exponential attractors, Lojasiewicz inequality.
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1 Introduction

We consider in this article the following system of partial differential equations in a bounded smooth domain Ω of $\mathbb{R}^3$:

$$\begin{cases} \delta\,\partial_t\phi - \Delta\phi + f(\phi) - u = g, \\ \varepsilon\,\partial_t u + \partial_t\phi - \Delta u = 0, \\ \frac{\partial u}{\partial n}{}_{/\partial\Omega} = 0, \quad \frac{\partial \phi}{\partial n}{}_{/\partial\Omega} = 0, \\ \phi_{/t=0} = \phi_0, \quad u_{/t=0} = u_0, \end{cases}$$

$0 < \varepsilon < 1$, $\delta > 0$. This system of equations was proposed by G. Caginalp in [4] in order to model melting-solidification phenomena in certain classes of materials. Here, u corresponds to the relative temperature and ϕ is the order parameter, or phase field, which describes the proportion of either of the phases ; $\phi = \pm 1$ correspond to the pure states.

This system, with various types of boundary conditions and for a regular potential f, has been much studied, see, e.g., [1], [2], [3], [4], [8], [13], [17] and the references therein. In particular, one has satisfactory results on the existence and uniqueness of solutions, the existence of finite dimensional attractors and the convergence of solutions to steady states. We note however that, for regular potentials, we are not able to prove that the order parameter remains in the physically relevant interval $[-1, 1]$.

In this article, we consider the case of singular potentials f ; in particular, we have in mind the following thermodynamically relevant logarithmic potential:

$$f(r) = -\kappa_0 r + \kappa_1 \ln \frac{1+r}{1-r}, \quad r \in (-1,1), \quad 0 < \kappa_0 < \kappa_1.$$

This problem, with Dirichlet boundary conditions, was considered in [9] ; in particular, the existence and uniqueness of solutions and the existence of exponential attractors (see [5]) was proven in [9]. The convergence of solutions to steady states, based on the Lojasiewicz inequality and the analyticity of f (see [11], [12] and [16]), was proven in [10] for mixed Dirichlet (for the temperature) and Neumann (for the order parameter) boundary conditions. We can note that such singular potentials allow to prove that the order parameter remains strictly between -1 and 1, as it is expected from the physical point of view, contrary to regular potentials.

In this article, we endow both equations with Neumann boundary conditions. In particular, this yields that the quantity

$$\int_\Omega (\varepsilon u + \phi)\, dx$$

is conserved. Then, by adapting the techniques of [9], [13] and [17], we study the existence and uniqueness of solutions, the existence of exponential attractors (and thus of finite dimensional global attractors) and the convergence of solutions to steady states. Furthermore, when the above conserved quantity is, in absolute value, large enough, we can prove that the solutions converge to spatially homogeneous steady states (which are given explicitly) exponentially fast.

2 Setting of the problem

In this article, we are interested in the study of the long time behavior of the following problem:

$$\begin{cases} \delta\, \partial_t \phi - \Delta\phi + f(\phi) - u = g, \\ \varepsilon\, \partial_t u + \partial_t \phi - \Delta u = 0, \\ \frac{\partial u}{\partial n}{}_{/\partial\Omega} = 0, \quad \frac{\partial \phi}{\partial n}{}_{/\partial\Omega} = 0, \\ \phi_{/t=0} = \phi_0, \quad u_{/t=0} = u_0. \end{cases} \tag{2.1}$$

We assume that Ω is a bounded smooth domain of $\mathbb{R}^3$, that $0 < \varepsilon < 1$ and $\delta > 0$, and that the function f satisfies the following conditions:

$$f \in \mathcal{C}^3(-1,1), \; \lim_{r\to\pm1} f(r) = \pm\infty, \; \lim_{r\to\pm1} f'(r) = +\infty. \tag{H_1}$$

Such a function f satisfies the following properties (see [9]):

$$\begin{aligned} & f'(r) \geq -K \quad \forall r \in (-1,1), \\ & -c \leq F(r) \leq f(r)r + C \quad \forall r \in (-1,1), \quad F(r) = \int_0^r f(s)ds, \end{aligned} \tag{2.2}$$

where K, c and C are strictly positive constants.

Throughout this article, we denote by $\|.\|$ and $((.,.))$ the norm and the scalar product in $L^2(\Omega)$, and we set $< u >= \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u(x)\,dx$ for $u \in L^1(\Omega)$. We also set $A\phi = -\Delta\phi + \phi$, $D(A) = H^2_N(\Omega)$ $(= \{\phi \in H^2(\Omega), \dfrac{\partial u}{\partial n}_{/\partial\Omega} = 0\})$. Furthermore, we set $\|\phi\|_1 = ((A^{\frac{1}{2}}\phi, A^{\frac{1}{2}}\phi))^{\frac{1}{2}}$, and this norm is equivalent to the usual one in $H^1(\Omega)$. Finally, the singularities of the potential f lead us to define the quantity $D[u(t)] = (1 - \|u(t)\|_{L^\infty})^{-1}$. Hereafter, C will denote a positive constant which may vary from line to line, and $Q,\ Q_\varepsilon$ will denote increasing functions, the latter depending on ε.

3 Existence and uniqueness of the solution

We start with the following theorem, which is the analogue of Theorem 1.1 in [9]. Nevertheless, contrary to [9], our estimates are not independent of ε.

Theorem 3.1 *Let $M > 0$, the nonlinearity f satisfy assumption (H1), and g belong to $L^\infty(\Omega)$. Then, for any initial datum (ϕ_0, u_0) satisfying*

$$D[\phi_0] + \|\phi_0\|^2_{H^2} + \|u_0\|^2_{H^2} < \infty, \qquad |< \varepsilon\, u_0 + \phi_0 >| \leq M, \tag{3.3}$$

equation (2.1) possesses a unique solution $(\phi(t), u(t))$ which satisfies, for $t \geq 0$, the estimate

$$D[\phi(t)] + \|\phi(t)\|^2_{H^2} + \|u(t)\|^2_{H^2} \leq Q_\varepsilon(D[\phi_0] + \|\phi_0\|^2_{H^2} + \|u_0\|^2_{H^2})\, e^{-\alpha t} + Q_\varepsilon(\|g\|_{L^\infty}), \tag{3.4}$$

where the positive constant α and the increasing function Q_ε are independent of (ϕ_0, u_0) but depend on M.

Proof : We rewrite (2.1) in the form

$$\begin{cases} \delta\, \partial_t\phi + A\phi + \tilde{f}(\phi) - u = g, \\ \varepsilon\, \partial_t u + \partial_t\phi - \Delta u = 0, \\ \frac{\partial u}{\partial n}_{/\partial\Omega} = 0, \quad \frac{\partial \phi}{\partial n}_{/\partial\Omega} = 0, \\ \phi_{/t=0} = \phi_0, \quad u_{/t=0} = u_0, \end{cases} \tag{3.5}$$

with $\tilde{f}(\phi) = f(\phi) - \phi$. The function $\tilde{f}$ still satisfies (2.2). We set $\tilde{F}(\phi) = \displaystyle\int_0^r \tilde{f}(s)\,ds$. Clearly, the first equation of (2.1) (or (3.5)) has a sense provided that ϕ is separated from the singular points of f, namely $-1 < \phi(x,t) < 1$ for almost $(x,t) \in \Omega \times \mathbb{R}^+$. Hence we assume that, a priori,

$$\|\phi\|_{L^\infty(\Omega\times\mathbb{R}^+)} < 1.$$

Integrating the second equation of (3.5) over Ω, we obtain the following conservation law:

$$\varepsilon < u(t) > + < \phi(t) > = \varepsilon < u_0 > + < \phi_0 > =: I_0 \quad \forall\, t \geq 0.$$

We multiply the first equation of (3.5) by $\phi(t) + \partial_t \phi(t)$, the second one by $u(t)$, we sum and integrate over Ω. We obtain, after straightforward simplifications,

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\{\delta\|\phi(t)\|^2 + \|\phi(t)\|_1^2 + 2((\tilde{F}(\phi(t)),1)) + \varepsilon\|u(t)\|^2\} \;+\; \tfrac{\delta}{2}\|\partial_t\phi(t)\|^2 + \|\phi(t)\|_1^2$$

$$+((\tilde{F}(\phi(t)),1)) + \|\nabla u(t)\|^2 \;\leq\; ((u(t),\phi(t))) + c\|g\|^2 + c'.$$

This, together with the analogue of Friedrich's inequality

$$\|\nabla u(t)\|^2 \geq C_1 \, \|u(t) - < u(t) > \|^2 \;=\; C_1(\|u(t)\|^2 - |\Omega| < u(t) >^2),$$

lead to

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\{\delta\|\phi(t)\|^2 + \|\phi(t)\|_1^2 + 2((\tilde{F}(\phi(t)),1)) + \varepsilon\|u(t)\|^2\} \;+\; \tfrac{\delta}{2}\|\partial_t\phi(t)\|^2 + \|\phi(t)\|_1^2$$

$$+((\tilde{F}(\phi(t)),1)) + \tfrac{C_1}{2}\|u(t)\|^2 \leq C\|g\|^2 + C' + C_1|\Omega|\tfrac{(I_0+1)^2}{\varepsilon^2}$$

$$\leq C\|g\|^2 + C' + \frac{C''}{\varepsilon^2}.$$

Thus, for some appropriate positive constant $\alpha > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\{\delta\|\phi(t)\|^2 + \|\phi(t)\|_1^2 + 2((\tilde{F}(\phi(t)),1)) + \varepsilon\|u(t)\|^2\} \;+ \delta\|\partial_t\phi(t)\|^2$$

$$+\alpha\,\{\delta\|\phi(t)\|^2 + \|\phi(t)\|_1^2 + 2((\tilde{F}(\phi(t)),1)) + \varepsilon\|u(t)\|^2\} \;\leq\; C\|g\|^2 + C' + \frac{C''}{\varepsilon^2}$$

and Gronwall's Lemma implies

$$\begin{aligned} &\|\phi(t)\|_1^2 + \varepsilon\|u(t)\|^2 + \frac{\delta}{2}\int_0^t \|\partial_t\phi(s)\|^2\, e^{-\alpha(t-s)}\, ds \\ &\quad\leq\; Q(D[\phi_0] + \|\phi_0\|_1^2 + \|u_0\|^2)\, e^{-\alpha t} + Q_\varepsilon'(\|g\|). \end{aligned} \tag{3.6}$$

Next, we rewrite the second equation of (3.5) as

$$\varepsilon\partial_t u + Au = u - \partial_t\phi,$$

multiply this equation by $Au(t) + \partial_t u(t)$ and integrate over Ω. Again, standard transformations, as well as (3.6) and Gronwall's Lemma, yield, for $\alpha > 0$ small enough,

$$\|u(t)\|_1^2 + \int_0^t (\|\partial_t u(s)\|^2 + \|Au(s)\|^2)\, e^{-\alpha(t-s)}\, ds$$

$$\leq\; Q_\varepsilon(D[\phi_0] + \|\phi_0\|_1^2 + \|u_0\|_1^2)\, e^{-\alpha t} + Q_\varepsilon(\|g\|).$$

Thus, combining (3.6) and the latter inequality, we find

$$
\|u(t)\|_1^2 + \|\phi(t)\|_1^2 + \int_0^t (\|\partial_t u(s)\|^2 + \|Au(s)\|^2 + \|\partial_t \phi(s)\|^2)\, e^{-\alpha(t-s)}\, ds
$$
$$
\leq\; Q_\varepsilon(D[\phi_0] + \|\phi_0\|_1^2 + \|u_0\|_1^2)\, e^{-\alpha t} + Q_\varepsilon(\|g\|). \tag{3.7}
$$

We differentiate the first equation of (3.5) with respect to t:

$$
\delta\, \partial_{tt}^2\phi(t) + A\partial_t\phi(t) + \tilde{f}'(\phi(t))\partial_t\phi(t) - \partial_t u(t) = 0,
$$

multiply this equation by $\partial_t\phi(t)$ and integrate over Ω. Using the fact that $f'(r) \geq -K$, we obtain

$$
\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\partial_t\phi(t)\|^2 + \|\partial_t\phi(t)\|_1^2 \;\leq\; (K + \frac{1}{2})\, \|\partial_t\phi(t)\|^2 + \frac{1}{2}\|\partial_t u(t)\|^2.
$$

Since $\partial_t\phi(0) = \dfrac{1}{\delta}\,\{\Delta\phi(0) - f(\phi(0)) + u(0) + g\,\}$, Gronwall's Lemma and (3.7) imply

$$
\|\partial_t\phi(t)\|^2 + \int_0^t \|\partial_t\phi(s)\|_1^2\, e^{-\alpha(t-s)}\, ds \;\leq\; Q_\varepsilon(D[\phi_0] + \|\phi_0\|_{H^2}^2 + \|u_0\|_1^2)\, e^{-\alpha t} + Q_\varepsilon(\|g\|). \tag{3.8}
$$

Next, we multiply the first equation of (2.1) by $-\Delta\phi(t)$ and integrate again over Ω. This yields, after straightforward simplifications,

$$
\frac{1}{2}\,\|\Delta\phi(t)\|^2 \;\leq\; K\|\nabla\phi(t)\|^2 + C(\|\partial_t\phi(t)\|^2 + \|u(t)\|^2 + \|g\|^2).
$$

Therefore, combining (3.7) and (3.8), we find

$$
\|\phi(t)\|_{H^2}^2 \leq\; Q_\varepsilon(D[\phi_0] + \|\phi_0\|_{H^2}^2 + \|u_0\|_1^2)\, e^{-\alpha t} \;+\; Q_\varepsilon(\|g\|). \tag{3.9}
$$

In order to obtain the u-part of the H^2-estimate, we now multiply the equation

$$
\varepsilon\partial_t u - \Delta u =\; -\partial_t\phi
$$

by $-\Delta u(t) - \partial_t\Delta u(t)$, and integrate over Ω. This yields

$$
\frac{\mathrm{d}}{\mathrm{dt}}\{\|\Delta u(t)\|^2 + \varepsilon\|\nabla u(t)\|^2\} \;+ \|\Delta u(t)\|^2 + \varepsilon\|\nabla\partial_t u(t)\|^2 \;\leq\; \frac{1}{\varepsilon}\|\nabla\partial_t\phi(t)\|^2 + \|\partial_t\phi(t)\|^2.
$$

Thus, applying Gronwall's Lemma and estimates (3.7), (3.8), (3.9), we have

$$
\|u(t)\|_{H^2}^2 + \|\phi(t)\|_{H^2}^2 + \int_0^t \|\nabla\partial_t u(s)\|^2\, e^{-\alpha(t-s)} ds \;\leq\; Q_\varepsilon(D[\phi_0] + \|u_0\|_{H^2}^2 + \|\phi_0\|_{H^2}^2) e^{-\alpha t} + Q_\varepsilon(\|g\|). \tag{3.10}
$$

Finally, we write

$$
\delta\, \partial_t\phi(t) - \Delta\phi(t) + f(\phi(t)) = h_u(t) = u(t) + g,
$$

with

$$\|h_u(t)\|_{L^\infty} \le c\,\|u(t)\|_{H^2} + \|g\|_{L^\infty} \le Q(D[\phi_0] + \|u_0\|^2_{H^2} + \|\phi_0\|^2_{H^2})\, e^{-\alpha t} + Q(\|g\|_{L^\infty}).$$

Arguing as in [9] (the main argument being the comparison principle for second-order parabolic PDEs), we conclude that

$$D[\phi(t)] \le Q(D[\phi_0] + \|\phi_0\|^2_{H^2} + \|u_0\|^2_{H^2})e^{-\alpha t} + Q(\|g\|_\infty) \quad \forall t \ge 0.$$

In particular, we have proven that

$$\|\phi(t)\|_{L^\infty} \le 1 - \delta \quad \forall t \ge 0,$$

for some $\delta > 0$ depending on $D[\phi_0]$, $\|\phi_0\|_{H^2}$ and $\|u_0\|_{H^2}$. Hence every solution $(\phi(t), u(t))$ of (3.5) is a priori strictly separated from the singularities $r = \pm 1$ of the nonlinearity f. Thus the existence of a solution of problem (3.5) (or (2.1)) can be studied exactly as in the case of regular nonlinearities (see, e.g., [13]). The uniqueness of the solution will follow from Lemma 3.3 below. Furthermore, mimicking ([9], Theorem 1.2), we can also prove the

Theorem 3.2 *Under the assumptions of Theorem 3.1, every solution $(\phi(t), u(t))$ of problem (2.1) satisfies*

$$D[\phi(t)] + \|\phi(t)\|^2_{H^2} + \|u(t)\|^2_{H^2} \le Q_\varepsilon(t^{-1} + \|u_0\|^2)\, e^{-\alpha t} + Q_\varepsilon(\|g\|_{L^\infty}), \quad t > 0.$$

Remark 3.1 *We have established that the solutions of problem (2.1) are a priori strictly separated from the singularities -1 and 1 (which implies the existence and uniqueness of the solution) for the Caginalp system endowed with Neumann boundary conditions. Our arguments still hold for Dirichlet boundary conditions (see [9]), or for the mixed conditions*

$$\phi_{/\partial\Omega} = 0, \quad \frac{\partial u}{\partial n}_{/\partial\Omega} = 0.$$

However, our results are not valid in the case

$$u_{/\partial\Omega} = 0, \quad \frac{\partial \phi}{\partial n}_{/\partial\Omega} = 0.$$

Indeed, estimate (3.10) does not hold anymore, since we cannot apply Green's formula to the integral $\int_\Omega \partial_t\phi\, \Delta(\partial_t u)\, dx$.

Next, we give several estimates on the difference of two solutions which are useful for proving the uniqueness ; they will also be essential in the following section in order to define the solving semigroup and to establish the existence of an exponential attractor.

Lemma 3.1 *Let f, g satisfy the assumptions of Theorem 3.1, and let (ϕ_1, u_1), (ϕ_2, u_2) be two solutions of problem (2.1) with initial data $(\phi_i(0), u_i(0))$, $i = 1, 2$, satisfying (3.3). Then the following estimate holds for $t \ge 0$:*

$$\|\phi_1(t) - \phi_2(t)\|^2 + \|u_1(t) - u_2(t)\|^2 \le K_2\, e^{K_1 t}\, (\|\phi_1(0) - \phi_2(0)\|^2 + \|u_1(0) - u_2(0)\|^2\,), \tag{3.11}$$

where the positive constants K_1, K_2 depend on ε, but are independent of the initial data.

Proof: We set $\psi = \phi_1 - \phi_2$ and $v = u_1 - u_2$. Thus (ψ, v) is solution of

$$\begin{cases} \delta\, \partial_t \psi - \Delta\psi + l(t)\psi - v = 0, \\ \varepsilon\, \partial_t v + \partial_t \psi - \Delta v = 0, \\ \frac{\partial v}{\partial n}_{/\partial\Omega} = 0, \quad \frac{\partial \psi}{\partial n}_{/\partial\Omega} = 0, \\ \psi_{/t=0} = \phi_1(0) - \phi_2(0), \quad v_{/t=0} = u_1(0) - u_2(0), \end{cases} \tag{3.12}$$

where $l(t) = \int_0^1 f'(s\phi_1(t) + (1-s)\phi_2(t))\, ds$. Integrating the second equation of (3.12) over Ω, we have

$$\partial_t(\varepsilon < v(t) > + < \psi(t) >) = 0. \tag{3.13}$$

Hence, we can rewrite the second equation of (3.12) as

$$\partial_t\{(\varepsilon v(t) + \psi(t)) - < \varepsilon v(t) + \psi(t) >\} - \Delta(v(t) - < v(t) >) = 0.$$

Multiplying this equation by $(-\Delta)^{-1}(\varepsilon v(t) + \psi(t) - < \varepsilon v(t) + \psi(t) >)$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\| \varepsilon v(t) + \psi(t) - < \varepsilon v(t) + \psi(t) > \|^2_{H^{-1}} + ((v(t) - < v(t) >, \varepsilon v(t) + \psi(t) - < \varepsilon v(t) + \psi(t) >)) = 0.$$

Thus we have

$$\begin{aligned} \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\| \varepsilon v(t) + \psi(t) - < \varepsilon v(t) + \psi(t) > \|^2_{H^{-1}} + \varepsilon \| v(t) - < v(t) > \|^2 + ((v(t), \psi)) \\ - < v(t) >< \psi(t) > |\Omega| = 0. \end{aligned}$$

Furthermore, noting that

$$< \varepsilon v(t) + \psi(t) >^2 = \varepsilon^2 < v(t) >^2 + < \psi(t) >^2 + 2\varepsilon < v(t) >< \psi(t) >,$$

we find

$$< v(t) >< \psi(t) > = \frac{1}{2\varepsilon} < \varepsilon v(t) + \psi(t) >^2 - \frac{\varepsilon}{2} < v(t) >^2 - \frac{1}{2\varepsilon} < \psi(t) >^2 \leq \frac{1}{2\varepsilon} < \varepsilon v(t) + \psi(t) >^2 .$$

Then it follows from (3.13) that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\{\| \varepsilon v(t) + \psi(t) - < \varepsilon v(t) + \psi(t) > \|^2_{H^{-1}} + < \varepsilon v(t) + \psi(t) >^2\} + \varepsilon \| v(t) - < v(t) > \|^2 \\ + ((v(t), \psi(t))) \ \leq \ \frac{|\Omega|}{2\varepsilon} < \varepsilon v(t) + \psi(t) >^2 . \end{aligned}$$

Next, we multiply the first equation of (3.12) by ψ, the second one by $\varepsilon(\varepsilon v + \psi)$, we sum these equations and integrate over Ω. Standard transformations, together with (2.2), yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\{\delta \|\psi(t)\|^2 + \varepsilon \|\varepsilon v(t) + \psi(t)\|^2\} + \frac{1}{2}|\nabla \psi(t)|^2 - ((v(t), \psi(t))) + \frac{\varepsilon^2}{2}\|\nabla v(t)\|^2 \ \leq \ K\|\psi(t)\|^2.$$

Combining the last two inequalities, we deduce that

$$\frac{\mathrm{d}}{\mathrm{dt}}\{\|\varepsilon v(t)+\psi(t)- < \varepsilon v(t)+\psi(t) > \|^2_{H^{-1}} + < \varepsilon v(t)+\psi(t) >^2 + \delta\|\psi(t)\|^2 + \varepsilon\|\varepsilon v(t)+\psi(t)\|^2\}$$

$$\leq \tfrac{|\Omega|}{\varepsilon} < \varepsilon v(t)+\psi(t) >^2 + 2K\|\psi(t)\|^2.$$

Thus, setting

$$E(t) = \|\varepsilon v(t)+\psi(t)- < \varepsilon v(t)+\psi(t) > \|^2_{H^{-1}} + < \varepsilon v(t)+\psi(t) >^2 + \delta\|\psi(t)\|^2 + \varepsilon\|\varepsilon v(t)+\psi(t)\|^2,$$

we have

$$\frac{\mathrm{d}}{\mathrm{dt}}E(t) \leq K_1\, E(t)$$

and it follows from Gronwall's Lemma that

$$\|\psi(t)\|^2 + \varepsilon\|\varepsilon v(t)+\psi(t)\|^2 \leq C\, e^{K_1 t}\,(\|\psi(0)\|^2 + \|v(0)\|^2).$$

Finally we conclude that

$$\|\psi(t)\|^2 + \|v(t)\|^2 \leq K_2 e^{K_1 t}\,(\|\psi(0)\|^2 + \|v(0)\|^2),$$

where K_1, K_2 depend on ε. This finishes the proof of Lemma 3.1.

Lemma 3.2 *Let f, g satisfy the assumptions of Theorem 3.1, and let (ϕ_1, u_1), (ϕ_2, u_2) be two solutions of (2.1) with initial data satisfying (3.3). Then the following estimate holds for $t \geq 0$:*

$$\begin{aligned}&\|\phi_1(t)-\phi_2(t)\|^2_{H^2} + \|u_1(t)-u_2(t)\|^2_{H^1} + \|\partial_t\phi_1(t)-\partial_t\phi_2(t)\|^2 \\ &\quad \leq L_2\, e^{L_1 t}\left(\|\phi_1(0)-\phi_2(0)\|^2_{H^2} + \|u_1(0)-u_2(0)\|^2_{H^1}\right),\end{aligned} \tag{3.14}$$

where L_1, L_2 depend on $\|\phi_i(0)\|_{H^2}$, $\|u_i(0)\|_{H^2}$, $D[\phi_i(0)]$, $i = 1,2$, and ε.

Proof: We again set $\psi = \phi_1 - \phi_2$, $v = u_1 - u_2$, where (ψ, v) is solution of problem (3.12). We multiply the first equation of (3.12) by $\psi + \partial_t\psi$, the second one by v, sum and integrate over Ω to find

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\{\|\nabla\psi(t)\|^2 + \delta\|\psi(t)\|^2 + \varepsilon\|v(t)\|^2\} + \delta\|\partial_t\psi(t)\|^2 + \|\nabla v(t)\|^2 + \|\nabla\psi(t)\|^2$$

$$\leq (K + \tfrac{1}{2})\|\psi(t)\|^2 + \tfrac{1}{2}\|v(t)\|^2 + |\,((l(t)\psi(t),\, \partial_t\psi(t)))\,|.$$

Moreover, according to (3.4), we have, $\forall t \geq 0$,

$$\|\phi_i(t)\|_{L^\infty} \leq 1-\delta_i, \quad \delta_i = \delta_i(\|\phi_i(0)\|_{H^2}, \|u_i(0)\|_{H^2}, D[\phi_i(0)], \varepsilon), \quad i = 1,2.$$

Then, setting $\delta_0 = \min(\delta_1, \delta_2)$, we deduce that

$$\|s\phi_1(t) + (1-s)\phi_2(t)\|_{L^\infty} \leq 1-\delta_0 \quad \forall\, 0 \leq s \leq 1,$$

and, consequently,

$$\|l(t)\|_{L^\infty} \leq C\,(= C\,(\delta_0)). \tag{3.15}$$

Hence we have

$$\begin{aligned} |\left((l(t)\psi,\, \partial_t\psi)\right)| &\leq C\|\psi\|\,\|\partial_t\psi\|, \\ &\leq \tfrac{\delta}{2}\|\partial_t\psi\|^2 + C'\|\psi\|^2. \end{aligned}$$

Combining the above inequalities, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}\{\|\nabla\psi(t)\|^2 + \delta\|\psi(t)\|^2 + \varepsilon\|v(t)\|^2\} \leq C'\left(\|\nabla\psi(t)\|^2 + \delta\|\psi(t)\|^2 + \varepsilon\|v(t)\|^2\right). \quad (3.16)$$

Next, we differentiate the first equation of (3.12) with respect to t to find

$$\delta\frac{\partial^2\psi}{\partial t^2} - \Delta\partial_t\psi + \partial_t l(t)\,\psi + l(t)\partial_t\psi - \partial_t v\, = 0.$$

Then we multiply this equation by $\partial_t\psi$, multiply the second equation of (3.12) by $\partial_t v$, sum and integrate over Ω. We know from (3.8) that

$$\|\partial_t l(t)\|^2 \,\leq\, C\,(\|\partial_t\phi_1(t)\|^2 + \|\partial_t\phi_2(t)\|^2) \,\leq\, \tilde{C}\,(= \tilde{C}(D[\phi_i(0)], \|\phi_{i0}\|_{H^2}, \|u_{i0}\|_{H^2}, \varepsilon)\,),\ i = 1, 2.$$

Therefore, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\{\delta\|\partial_t\psi\|^2 + \|\nabla v\|^2\} + \|\nabla\partial_t\psi\|^2 + \varepsilon\|\partial_t v\|^2 &\leq C\|\partial_t\psi\|^2 + \tilde{C}\|\psi\partial_t\psi\| \\ &\leq C\|\partial_t\psi\|^2 + \tilde{C}\|\psi\|_{L^4}\,\|\partial_t\psi\|_{L^4} \\ &\leq C\|\partial_t\psi\|^2 + C'\|\psi\|_1\,\|\partial_t\psi\|_1 \\ &\leq (C + \tfrac{1}{2})\|\partial_t\psi\|^2 + \tfrac{1}{2}\|\nabla\partial_t\psi\|^2 + C''\|\psi\|_1^2. \end{aligned}$$

We thus deduce from (3.16) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}\{\|\nabla\psi(t)\|^2 + \delta\|\psi(t)\|^2 + \varepsilon\|v(t)\|^2 + \|\nabla v(t)\|^2 + \delta\|\partial_t\psi(t)\|^2\,\} \\ &\qquad \leq\, L_1\,(\|\nabla\psi(t)\|^2 + \delta\|\psi(t)\|^2 + \varepsilon\|v(t)\|^2 + \|\nabla v(t)\|^2 + \delta\|\partial_t\psi(t)\|^2\,) \end{aligned}$$

and, applying Gronwall's Lemma, we infer

$$\|\psi(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\partial_t\psi(t)\|^2 \,\leq L\,e^{L_1 t}\,\left(\|\psi(0)\|_{H^1}^2 + \|v(0)\|_{H^1}^2 + \|\partial_t\psi(0)\|^2\right), \quad (3.17)$$

where the constants L_1, L depend on $\|\phi_i(0)\|_{H^2}$, $\|u_i(0)\|_{H^2}$, $D[\phi_i(0)]$, $i = 1, 2$, and ε. Moreover, since $\delta\partial_t\psi(0) = \Delta\psi(0) - l(0)\psi(0) + v(0)$, we deduce that

$$\|\delta\partial_t\psi(0)\|_{L^2} \,\leq\, C\,\|\psi(0)\|_{H^2} + \|v(0)\|_{L^2}. \quad (3.18)$$

In the same way, we can write

$$\|\Delta\psi(t)\| \,\leq\, \delta\|\partial_t\psi(t)\|_{L^2} + \|l(t)\psi(t)\| + \|v(t)\|,$$

which, together with (3.15), (3.17) and (3.18), allow to conclude that

$$\|\psi(t)\|_{H^2}^2 + \|v(t)\|_{H^1}^2 + \delta\|\partial_t\psi(t)\|^2 \,\leq L_2\,e^{L_1 t}\,\left(\|\psi(0)\|_{H^2}^2 + \|v(0)\|_{H^1}^2\right),$$

which finishes the proof of Lemma 3.2.

Since we have shown that the solutions of problem (2.1) are strictly separated from the singularities, everything holds as for a regular potential. Thus, mimicking ([13], Lemmata 2.6 and 2.7), we can derive the following smoothing estimates for the difference of two solutions of (2.1). These will be necessary in order to construct an exponential attractor.

Lemma 3.3 *Let f, g satisfy the assumptions of Theorem 3.1, and let (ϕ_1, u_1), (ϕ_2, u_2) be two solutions of (2.1) with initial data satisfying (3.3). Then, there holds for $t \geq 0$*

$$\begin{aligned} &\|\phi_1(t) - \phi_2(t)\|_{H^2}^2 + \|u_1(t) - u_2(t)\|_{H^2}^2 + \|\partial_t\phi_1(t) - \partial_t\phi_2(t)\|^2 \\ &\qquad \leq K\ e^{Lt} \left(\|\phi_1(0) - \phi_2(0)\|_{H^2}^2 + \|u_1(0) - u_2(0)\|_{H^2}^2\right), \end{aligned} \tag{3.19}$$

where K, L depend on $\|\phi_i(0)\|_{H^2}$, $\|u_i(0)\|_{H^2}$, $D[\phi_i(0)]$, $i = 1, 2$, and ε. Furthermore, we also have the smoothing estimate

$$\begin{aligned} &\|\phi_1(t) - \phi_2(t)\|_{H^3}^2 + \|u_1(t) - u_2(t)\|_{H^3}^2 \\ &\qquad \leq K\ e^{Lt}\,\frac{t+1}{t} \left(\|\phi_1(0) - \phi_2(0)\|_{H^2}^2 + \|u_1(0) - u_2(0)\|_{H^2}^2\right), \quad t > 0. \end{aligned} \tag{3.20}$$

4 Existence of an exponential attractor

4.1 The solving semigroup

Theorem 3.1 allows to define the solving semigroup S_t^M associated with problem (2.1) by the following expression:

$$S_t^M : \Phi_M \to \Phi_M, \quad S_t^M(\phi_0, u_0) := (\phi(t), u(t)),$$

where $(\phi(t), u(t))$ is the unique solution of problem (2.1) with initial datum (ϕ_0, u_0), and

$$\Phi_M = \{(\phi, u) \in H_N^2(\Omega) \times H_N^2(\Omega),\ \|\phi\|_{L^\infty} < 1,\ | < \varepsilon u + \phi > | \leq M\},$$

$$\|(\phi, u)\|_{\Phi_M} = (\|\phi\|_{H^2}^2 + \|u\|_{H^2}^2)^{\frac{1}{2}}.$$

Moreover, applying Lemma 3.1, we can extend S_t^M to a semigroup (still denoted by S_t^M) acting on the closure L_M of Φ_M in $[L^2(\Omega)]^2$, i.e.

$$S_t^M : L_M \to L_M\,; \quad L_M = \{(\phi, u) \in L^\infty(\Omega) \times L^2(\Omega),\ \|\phi\|_{L^\infty} \leq 1,\ | < \varepsilon u + \phi > | \leq M\}.$$

When $(\phi_0, u_0) \notin \Phi_M$, we have, as usual,

$$S_t^M(\phi_0, u_0) = [L^2(\Omega)]^2 - \lim_{n\to+\infty} S_t^M(\phi_0^n, u_0^n),$$

where $(\phi_0^n, u_0^n) \in \Phi_M$ is such that $\|u_0^n - u_0\| + \|\phi_0^n - \phi_0\| \underset{n\to+\infty}{\longrightarrow} 0$.

Finally, we have, thanks to Theorem 3.2, the smoothing property

$$S_t^M : L_M \to \Phi_M \quad \forall t > 0.$$

We conclude this section by an auxiliary lemma which will be needed in order to prove the finite dimensionality of exponential attractors.

Lemma 4.1 *Let T be a strictly positive constant. Under the assumptions of Theorem 3.1, every solution $(\phi(t), u(t))$ of equation (2.1) is Hölder continuous with respect to t, i.e $\forall\, t \in [0, T]$ and $0 \le s \le 1$, we have*

$$\|\phi(t+s) - \phi(t)\|_{H^2} + \|u(t+s) - u(t)\|_{H^2} \;\le\; Q(\|(\phi_0, u_0)\|_{\Phi_M}) s^{\frac{1}{3}}.$$

Proof: We infer from (3.8), (2.1) and (3.4) that

$$\|\phi(t+s) - \phi(t)\| + \|u(t+s) - u(t)\| \;\le\; s\, C(\|(\phi_0, u_0)\|_{\Phi_M}).$$

Thus, applying (3.20) and the following interpolation inequality:

$$\|v\|_{H^2} \;\le\; \|v\|_{L^2}^{\frac{1}{3}} \, \|v\|_{H^3}^{\frac{2}{3}} \quad \forall\, v \,\in\, H^3(\Omega),$$

we finish the proof of Lemma 4.1.

4.2 Exponential attractors

We now state the main result of this section, namely the existence of an exponential attractor associated with the semigroup S_t^M.

Theorem 4.1 *Let the nonlinearity f satisfy assumption (H1), and g belong to $L^\infty(\Omega)$. Then there exists a compact set $\mathcal{M}^M \subset \Phi_M$, called exponential attractor, which satisfies the following properties:*

(i) $\mathcal{M}^M$ is semi-invariant with respect to the flow S_t^M associated with problem (2.1), i.e.

$$S_t^M \, \mathcal{M}^M \;\subset\; \mathcal{M}^M \quad \forall t \ge 0.$$

(ii) The fractal dimension of the set $\mathcal{M}^M$ is finite, i.e.

$$dim_F\,(\mathcal{M}^M,\, \Phi_M) \;\le\; C < +\infty.$$

(iii) $\mathcal{M}^M$ attracts exponentially fast the bounded subsets of Φ_M, i.e. there exists $\alpha > 0$ such that

$$dist_{\Phi_M}\,(S_t^M\, B, \mathcal{M}^M) \;\le\; Q(\|B\|_{\phi_M})\, e^{-\alpha t} \quad \forall\, B \text{ bounded in } \Phi_M,$$

where $dist_{\Phi_M}$ denotes the nonsymmetric Hausdorff distance between sets in Φ_M.

For the proof of Theorem 4.1, we proceed as in [6], [7], [13], [9] and first construct an exponential attractor for a discrete semigroup. For the sake of completeness, we briefly outline the proof, recalling the main argument, namely the following existence result for exponential attractors of discrete maps.

Proposition 4.1 *[6] Let Φ_1, Φ be two Banach spaces such that Φ_1 is compactly embedded into Φ. Let B be a closed and bounded subset of Φ, and $S_* : B \to B$ be such that, for every $b^1,\ b^2 \in B$,*

$$\|S_* b^1 - S_* b^2\|_{\Phi_1} \leq K \|b^1 - b^2\|_{\Phi}. \tag{4.21}$$

Then, the map S_* possesses an exponential attractor $\mathcal{M}^* \subset \Phi$, i.e. a compact set with finite fractal dimension which satisfies

$$S_* \mathcal{M}_* \subset \mathcal{M}_*, \tag{4.22}$$

$$dist_{\Phi}\,(S_*^{(k)} B,\, \mathcal{M}_*) \leq c\, e^{-\gamma k} \text{ with } c > 0, \gamma > 0 \text{ and } k \in \mathbb{N}. \tag{4.23}$$

Proof of Theorem 4.1: We set $\Phi = \Phi_M$, $B = \{(\phi, u) \in \Phi_M\, ;\, \|(\phi, u)\|_{\Phi_M} \leq 2\, Q(\|g\|_{L^\infty})\}$. It follows from Theorem 3.1 that there exists t^* large enough such that $S_{t^*}^M\, B \subset B$. We set $S_* = S_{t^*}^M$. Estimate (4.21) is then a direct consequence of (3.20). Thus Proposition 4.1 holds and we infer the existence of a discrete exponential attractor $\mathcal{M}_*$ which satisfies (4.22) and (4.23).

As usual, we now set

$$\mathcal{M}^M = \cup_{t \in [0, t_*]}\, S_t^M \mathcal{M}_*. \tag{4.24}$$

It is rather standard to verify that the set $\mathcal{M}^M$ is the desired continuous exponential attractor. Indeed, (i) follows from (4.22). Furthermore, as a consequence of Lemma 4.1, the semigroup S_t^M is Hölder continuous on $[0, t_*] \times \mathcal{M}_*$, hence

$$\dim_F (\mathcal{M}^M,\, \Phi_M) \leq \dim_F (\mathcal{M}_*,\, \Phi_M) + 3.$$

Thus, applying Proposition 4.1, we have (ii). Finally, (iii) follows from (4.23) and (3.19).

Remark 4.1 *A direct consequence of Theorem 3.2 is the existence of the global attractor $\mathcal{A}^M$, contained in $\mathcal{M}^M$. Moreover, according to (iii), we also have*

$$dist_{L^2 \times L^2}\, (S_t^M B,\, \mathcal{M}^M) \leq C\, e^{-\gamma t} \quad \forall\, B \text{ bounded in } L_M,\ \forall\, t > 0.$$

5 Convergence to a spatially homogeneous equilibrium when $|I_0|$ is large enough

We assume in this section that $g = 0$. Our aim is to prove that every solution of (2.1) converges exponentially fast to the spatially homogeneous equilibrium defined by

$$\hat{u} = f(\hat{\phi}), \quad \varepsilon \hat{u} + \hat{\phi} = I_0\, (=< \varepsilon u_0 + \phi_0 >), \tag{5.25}$$

provided that its initial data are such that $|I_0|$ is large enough. To this aim, we give a preliminary Lemma which will be useful in what follows.

Lemma 5.1 *Let the nonlinearity f satisfy assumption (H1), and let $M_0 < 1$ be defined by*

$$M_0 = \max\{|w|;\, \exists z \in]-1, 1[,\ f(z) = f(w) \text{ and } f'(z) = 0\}.$$

Then,

$$|I_0| \geq M_1 \text{ implies } |\hat{\phi}| \geq M_0,$$

where $\hat{\phi}$ is defined by (5.25), and $M_1 = M_0 + \sup\{|f(M_0)|, |f(-M_0)|\}$.

Proof: We assume that $|\hat{\phi}| < M_0$, with M_0 defined in Lemma 5.1. Then the following obvious inequality holds:

$$f(-M_0) < f(\hat{\phi}) < f(M_0).$$

Thus $|f(\hat{\phi})| < \sup\{|f(M_0)|, |f(-M_0)|\}$, and, consequently,

$$|I_0| = |\varepsilon f(\hat{\phi}) + \hat{\phi}| < |\hat{\phi}| + |f(\hat{\phi})| < \sup\{|f(M_0)|, |f(-M_0)|\} + M_0 = M_1,$$

which finishes the proof of Lemma 5.1.

Next we recall a result established in [14].

Lemma 5.2 *Let f satisfy assumption (H1), and m be a real number such that $|m| \geq M_0$ (M_0 being defined in Lemma 5.1). Then there holds*

$$(f(m+v) - f(m)).v \geq 0 \quad \forall v \in (-1-m,\, 1-m). \tag{5.26}$$

Theorem 5.1 *We assume that $g = 0$, and that f satisfies assumption (H1). Then every solution of (2.1) with initial data such that $|I_0| \geq M_1$ (M_1 being given in Lemma 5.1) satisfies*

$$\varepsilon^3 \|u(t) - \hat{u}\|^2 + \|\phi(t) - \hat{\phi}\|^2 \leq C\, e^{-\alpha t} \quad \forall t > 0,$$

where $(\hat{u}, \hat{\phi})$ is given by (5.25), and the constant C depends on $\|u_0\|, \|\phi_0\|, \hat{\phi}, \hat{u}$.

Proof: Let $(\phi(t), u(t)) = S_t^M(\phi_0, u_0)$. We set $w(t) = u(t) - \hat{u},\ \varphi(t) = \phi(t) - \hat{\phi}$. Then (w, φ) is solution of the problem

$$\begin{cases} \delta\, \partial_t \varphi - \Delta\varphi + f(\phi) - f(\hat{\phi}) - w = 0, \\ \varepsilon\, \partial_t w + \partial_t \varphi - \Delta w = 0, \\ \frac{\partial w}{\partial n}_{/\partial\Omega} = 0, \quad \frac{\partial \varphi}{\partial n}_{/\partial\Omega} = 0, \\ \varphi_{/t=0} = \phi_0 - \hat{\phi}, \quad w_{/t=0} = u_0 - \hat{u}. \end{cases} \tag{5.27}$$

From the second equation of (5.27), we infer the conservation law

$$\varepsilon < w(t) > + < \varphi(t) > = \varepsilon < w(0) > + < \varphi(0) > = I_0 - \varepsilon\, \hat{u} - \hat{\phi} = 0.$$

We multiply the second equation of (5.27) by $(-\Delta)^{-1}(\varepsilon w(t) + \varphi(t))$, the first one by $\varphi(t)$, sum and integrate over Ω to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\{\|\varepsilon w(t) + \varphi(t)\|^2_{H^{-1}} + \delta\|\varphi(t)\|^2\} + \varepsilon\|w(t)\|^2 + \|\nabla\varphi(t)\|^2 + ((f(\phi(t)) - f(\hat{\phi}), \varphi(t))) = 0. \tag{5.28}$$

Since $|I_0| \geq M_1$, we infer from Lemma 5.1 that $1 > |\hat{\phi}| \geq M_0$. This allows us to apply (5.26) with $v(t) = \varphi(t) \in (-1 - \hat{\phi}, 1 - \hat{\phi})$ and $m = \hat{\phi}$ and we have

$$((f(\phi(t)) - f(\hat{\phi}), \varphi(t))) \geq 0.$$

Next, we multiply the second equation of (5.27) by $\varepsilon(\varepsilon w(t)+\varphi(t))$, and integrate again over Ω. This leads to

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\varepsilon w(t)+\varphi(t)\|^2+\frac{\varepsilon^2}{2}\,\|\nabla w(t)\|^2\ \leq\ \frac{1}{2}\,\|\nabla\varphi(t)\|^2.$$

Combining (5.28) with the last two estimates yields

$$\frac{\mathrm{d}}{\mathrm{dt}}\{\varepsilon\|\varepsilon w(t)+\varphi(t)\|^2+\delta\|\varphi(t)\|^2+\|\varepsilon w(t)+\varphi(t)\|^2_{-1}\}+\|\nabla\varphi(t)\|^2+\varepsilon^2\,\|\nabla w(t)\|^2+2\varepsilon\|w(t)\|^2\ \leq 0. \tag{5.29}$$

On the other hand, taking into account the fact that $0<\varepsilon<1$, we have

$$\varepsilon\|\varepsilon w(t)+\varphi(t)\|^2+\delta\|\varphi(t)\|^2+\|\varepsilon w(t)+\varphi(t)\|^2_{H^{-1}}\ \leq\ C\,(\varepsilon^2\|w(t)\|^2+\|\varphi(t)\|^2). \tag{5.30}$$

Furthermore, Friedrich's inequality yields

$$\|w(t)\|^2\ \leq\ C'\,(\|\nabla w(t)\|^2+<w(t)>^2),$$
$$\|\varphi(t)\|^2\ \leq\ C'\,(\|\nabla\varphi(t)\|^2+<\varphi(t)>^2),$$

so that, applying the conservation law $<\varphi(t)>^2=\ \varepsilon^2\ <w(t)>^2$, we obtain

$$\varepsilon^2\|w(t)\|^2+\|\varphi(t)\|^2\leq\ C'\,(\|\nabla\varphi(t)\|^2+\varepsilon^2\|\nabla w(t)\|^2+2\varepsilon^2<w(t)>^2\,).$$

Thus, according to (5.30), we find

$$\varepsilon\|\varepsilon w(t)+\varphi(t)\|^2+\delta\|\varphi(t)\|^2+\|\varepsilon w(t)+\varphi(t)\|^2_{H^{-1}}\ \leq\ C''(\|\nabla\varphi(t)\|^2+\varepsilon^2\|\nabla w(t)\|^2+2\varepsilon\|w(t)\|^2).$$

Consequently, we infer from (5.29) the existence of a constant $\alpha>0$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\{\varepsilon\|\varepsilon w(t)+\varphi(t)\|^2+\delta\|\varphi(t)\|^2+\|\varepsilon w(t)+\varphi(t)\|^2_{H^{-1}}\}$$
$$+\alpha\left(\varepsilon\|\varepsilon w(t)+\varphi(t)\|^2+\delta\|\varphi(t)\|^2+\|\varepsilon w(t)+\varphi(t)\|^2_{H^{-1}}\right)\ \leq\ 0$$

and we finish the proof by employing Gronwall's Lemma.

6 Convergence to an equilibrium

We assume in this section that $g=0$. We again consider problem (2.1) and the corresponding equilibrium problem

$$\begin{cases} -\Delta\bar{\phi}+f(\bar{\phi})-\bar{u}=0, \\ \varepsilon\bar{u}+<\bar{\phi}>=I_0\ (=\varepsilon<u_0>+<\phi_0>), \\ \frac{\partial\bar{\phi}}{\partial n}{}_{/\partial\Omega}=0. \end{cases} \tag{6.31}$$

Hereafter, the function f will be assumed to satisfy $(H1)$ and

$$f \text{ is analytic in } (-1,1). \tag{$H2$}$$

The main result of this section is given in the following Theorem.

Theorem 6.1 *Let f satisfy assumptions (H1) and (H2), and let $(\phi, u) \in \Phi_M$ be a solution of (2.1) with $g = 0$. Then, there exists a solution $(\bar{\phi}, \bar{u})$ of problem (6.31) such that*

$$\begin{aligned} \phi(x,t) &\rightarrow \bar{\phi}(x), \\ u(x,t) &\rightarrow \bar{u}, \end{aligned}$$

strongly in $H^2(\Omega)$ as t tends to $+\infty$.

Remark 6.1 *Here, contrary to the previous section, we are not able to say to which equilibrium the solution converges in general ; furthermore, the convergence is not exponential. Of course, when $|I_0|$ is large enough, then the solution converges to the spatially homogeneous equilibrium exhibited in the previous section.*

Our proof follows [17], where a similar result is established, but for the regular potential $f(v) = \frac{1}{2}(v^3 - v)$. Thus our task consists in verifying that the result remains valid for a singular potential satisfying (H1) and (H2). To this aim, we introduce the function

$$V(\phi(t), u(t)) = \int_\Omega \left(\frac{1}{2} \|\nabla \phi(t)\|^2 + F(\phi(t)) + \frac{\varepsilon}{2} \|u(t)\|^2 \right) dx.$$

Our first step consists in proving that $(\Phi_M,\ S_t^M,\ V)$ is a gradient system (see the Appendix ; see also [17]), from which it follows that the ω-limit set $\omega(\phi_0, u_0)$ consists of equilibria.

6.1 (Φ_M, S_t^M, V) is a gradient system

The function V is a Lyapounov function, since it satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} V(\phi(t), u(t)) &= ((\nabla \phi(t),\ \nabla \partial_t \phi(t))) + \int_\Omega f(\phi(t))\, \partial_t \phi(t)\, dx + \varepsilon((u(t),\ \partial_t u(t))) \\ &= ((-\Delta \phi(t) + f(\phi(t)),\ \partial_t \phi(t))) + \varepsilon\, ((u(t), \partial_t u(t))) \\ &= ((-\delta \partial_t \phi(t) + u(t), \partial_t \phi(t))) + \varepsilon((u(t), \partial_t u(t))) \\ &= -\delta \|\partial_t \phi(t)\|^2 + ((u(t), \partial_t \phi(t) + \varepsilon \partial_t u(t))) \\ &= -\delta \|\partial_t \phi(t)\|^2 + ((u(t), \Delta u(t))) \\ &= -\delta \|\partial_t \phi(t)\|^2 - \|\nabla u(t)\|^2 \ \leq 0. \end{aligned} \tag{6.32}$$

Next, we verify the first point of the definition of a gradient system (see the Appendix). Let $t_1 > 0$ be such that $V(S(t_1)(\phi_0, u_0)) = V(\phi_0, u_0)$. Then, (6.32) implies

$$\partial_t \phi(t) = 0,\ \nabla u(t) = 0 \quad \forall t \in\ [0, t_1]$$

and it follows from the second equation of (2.1) that $\partial_t u(t) = 0 \quad \forall t \in [0, t_1]$. As a consequence, (ϕ_0, u_0) is a stationnary solution.

We then prove that, for every initial datum in Φ_M, there exists a time $t_0 > 0$ such that the orbit actually lies in $H^3(\Omega) \times H^3(\Omega)$. Since $H^3(\Omega)$ is compactly embedded into $H^2(\Omega)$, this yields the second point of the definition of a gradient system.

We have already stated in Theorem 3.1 that the solutions are strictly separated from the singularities. Hence we have

$$\|f'(\phi(t))\|_{L^\infty} \ \leq C, \tag{6.33}$$

where C depends on $D[\phi_0]$, $\|\phi_0\|_{H^2}$, $\|u_0\|_{H^2}$, ε. We first differentiate the first equation of (2.1) with respect to t, multiply the resulting equation by $\partial^2_{tt}\phi$ and integrate over Ω to find

$$\delta\|\partial^2_{tt}\phi(t)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla\partial_t\phi(t)\|^2 \le \frac{\delta}{2}\|\partial^2_{tt}\phi(t)\|^2 + C\|f'(\phi(t))\partial_t\phi\|^2 + C\|\partial_t u(t)\|^2.$$

Thus

$$\delta\,\|\partial^2_{tt}\phi(t)\|^2 + \frac{\mathrm{d}}{\mathrm{dt}}\|\nabla\partial_t\phi(t)\|^2 \le C'\,(\|\partial_t\phi(t)\|^2 + \|\partial_t u(t)\|^2).$$

Moreover, we have

$$\int_0^t s\,\frac{d}{ds}\|\nabla\partial_t\phi(s)\|^2 ds\; t\,\|\nabla\partial_t\phi(t)\|^2 - \int_0^t \|\nabla\partial_t\phi(s)\|^2\,ds.$$

Combining the last two estimates, we obtain

$$\delta\int_0^t s\,\|\partial^2_{tt}\phi(s)\|^2\,ds + t\,\|\nabla\partial_t\phi(t)\|^2 \le C\,t\int_0^t (\|\partial_t\phi(s)\|^2 + \|\partial_t u(s)\|^2)\,ds + \int_0^t \|\nabla\partial_t\phi(s)\|^2 ds,$$

which yields, applying (3.7) and (3.8) with $\alpha = 0$,

$$\delta\int_0^t s\,\|\partial^2_{tt}\phi(s)\|^2\,ds + t\,\|\nabla\partial_t\phi(t)\|^2 \le C't + C'', \tag{6.34}$$

where the constants C', C'' depend on $\|\phi_0\|_{H^2}$, $\|u_0\|_1$, $D[\phi_0]$ and ε. Hence we conclude that

$$\|\nabla\partial_t\phi(t)\|^2 \le C' + \frac{C''}{t} \le C' + \frac{C''}{t_0} = C_0 \quad \forall t \ge t_0 > 0.$$

Next, we differentiate the second equation of (2.1) with respect to t, multiply the resulting equation by $t\,\partial^2_{tt}u$ and integrate over Ω. It follows that

$$\varepsilon\, t\,\|\partial^2_{tt}u(t)\|^2 + \frac{t}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla\partial_t u(t)\|^2 \le \frac{\varepsilon t}{2}\|\partial^2_{tt}u(t)\|^2 + \frac{t}{2\varepsilon}\|\partial^2_{tt}\phi(t)\|^2.$$

Thus, integrating over $[0,t]$, we obtain

$$\int_0^t s\,\frac{\mathrm{d}}{\mathrm{ds}}\|\nabla\partial_t u(s)\|^2 ds + \varepsilon\int_0^t s\,\|\partial^2_{tt}u(s)\|^2\,ds \le \int_0^t \frac{s}{\varepsilon}\,\|\partial^2_{tt}\phi(s)\|^2\,ds$$

and, therefore,

$$t\,\|\nabla\partial_t u(t)\|^2 + \varepsilon\int_0^t s\,\|\partial^2_{tt}u(s)\|^2 ds \le \int_0^t \frac{s}{\varepsilon}\,\|\partial^2_{tt}\phi(s)\|^2\,ds + \int_0^t \|\nabla\partial_t u(s)\|^2\,ds.$$

Estimates (3.10) and (6.34) then lead to

$$t\,\|\nabla\partial_t u(t)\|^2 + \varepsilon\int_0^t s\,\|\partial^2_{tt}u(s)\|^2 ds \le C + C'\,t.$$

Hence, we have

$$\|\nabla\partial_t u(t)\|^2 \le C_0' \quad \forall t \ge t_0 > 0. \tag{6.35}$$

Rewriting the first equation of (2.1) as

$$\Delta\phi(t) \;=\; \delta\partial_t\phi(t) + f(\phi(t)) - u,$$

and applying (3.4) and (6.34), we infer

$$\begin{aligned}\|\nabla(\Delta\phi(t))\| \;&\leq\; \delta\|\nabla\partial_t\phi(t)\| + \|\nabla f(\phi(t))\| + \|\nabla u(t)\| \\ &\leq\; C_0^{''} \quad \forall t > t_0.\end{aligned}$$

We thus deduce that

$$\|\phi(t)\|_{H^3} \;\leq\; C_0^{'''} \quad \forall t \geq t_0,$$

where $C_0^{'''}$ depends on $D[\phi_0]$, $\|u_0\|_{H^2}$, $\|\phi_0\|_{H^2}, \varepsilon$, and t_0. In the same way, (6.34) and (6.35) allow to conclude that

$$\begin{aligned}\|\nabla(\Delta u(t))\| \;&\leq\; \varepsilon\|\nabla\partial_t u(t)\| + \|\nabla\partial_t\phi(t)\| \\ &\leq\; C_0^{'''} \quad \forall t \geq t_0 > 0\end{aligned}$$

and, finally, we deduce that $\|u(t)\|_{H^3}$ is bounded for all $t > t_0$, hence

Lemma 6.1 *The orbit* $\cup_{t\geq t_0} S(t)(\phi_0, u_0)$ *is relatively compact in* Φ_M*. Therefore,* $(\Phi_M, S(t), V)$ *is a gradient system.*

6.2 Proof of theorem 6.1

We proceed as in [17]. However, for the sake of completeness, we give the details of the proof. Thanks to Lemma 6.1, we know that, for any given $(\phi_0, u_0) \in \Phi_M$, the $\omega-$limit set $\omega(\phi_0, u_0)$ consists of equilibria. Thus, by definition of the $\omega-$limit set, there exist $(\bar\phi, \bar u) \in \omega(\phi_0, u_0)$ and a sequence $t_n \to +\infty$ such that

$$\phi(t_n) \to \bar\phi,\; u(t_n) \to \bar u \quad \textit{in } H^2(\Omega).$$

Using (6.32), we necessarily have

$$V(\phi(t), u(t)) - V(\bar\phi, \bar u) \geq 0 \quad \forall t \geq 0.$$

We first assume that there exists $t_0 \in \mathbb{R}^+$ such that $V(\phi(t), u(t)) - V(\bar\phi, \bar u) \;=0 \quad \forall t \geq t_0$. Then, it follows from (6.32) that

$$u(t) = \bar u,\; \phi(t) = \bar\phi \quad \forall t \geq t_0,$$

hence the result. We thus now assume that $V(\phi(t), u(t)) \;>\; V(\bar\phi, \bar u)$ $\forall t \geq 0$. It follows from (6.32) that

$$\frac{\mathrm{d}}{\mathrm{dt}}(V(\phi(t), u(t)) - V(\bar\phi, \bar u)) + \delta\|\partial_t\phi(t)\|^2 + \|\nabla u(t)\|^2 = 0.$$

Therefore,

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{dt}}\{(V(\phi(t), u(t)) - V(\bar\phi, \bar u))^\theta\} \;&=\; \theta\,(V(\phi(t), u(t)) - V(\bar\phi, \bar u))^{\theta-1}\;(\|\nabla u(t)\|^2 + \delta\|\partial_t\phi(t)\|^2) \\ &\geq\; \theta\,(V(\phi(t), u(t)) - V(\bar\phi, \bar u))^{\theta-1}\;(\sqrt{\tfrac{\delta}{2}}\,\|\partial_t\phi(t)\| + \tfrac{1}{\sqrt{2}}\|\nabla u(t)\|)^2.\end{aligned} \tag{6.36}$$

According to Lemma 7.2, there exists $T > 0$ such that

$$\forall t \geq T, \quad \|\phi(t) - \bar{\phi}\|_{H^2} \leq \sigma, \quad \|u(t) - \bar{u}\|_{H^2} \leq \sigma,$$

with $\sigma > 0$ given in Lemma 7.1. Furthermore, we infer from Lemma 7.1 that

$$-\frac{\mathrm{d}}{\mathrm{dt}} \left\{ (V(\phi(t), u(t)) - V(\bar{\phi}, \bar{u}))^{\theta} \right\} \geq \theta \left(\sqrt{\frac{\delta}{2}} \|\partial_t \phi(t)\| + \frac{1}{\sqrt{2}} \|\nabla u(t)\| \right).$$

Thus, integrating over $[T, +\infty[$, we obtain

$$\sqrt{\frac{\delta}{2}} \int_T^{+\infty} \|\partial_t \phi(s)\| ds + \frac{1}{\sqrt{2}} \int_T^{+\infty} \|\nabla u(s)\| ds < +\infty.$$

Since we have

$$\begin{aligned} \|\phi(t) - \bar{\phi}\| &\leq \|\phi(t) - \phi(t_n)\| + \|\phi(t_n) - \bar{\phi}\| \\ &\leq \int_{t_n}^{t} \|\partial_t \phi(s)\| ds + \|\phi(t_n) - \bar{\phi}\|_{H^2}, \end{aligned}$$

and

$$\begin{aligned} \|u(t) - \bar{u}\|_{H^{-1}} &\leq \int_{t_n}^{t} \|\partial_t u(s)\|_{H^{-1}} ds + \|u(t_n) - \bar{u}\|_{H^{-1}} \\ &\leq C_{\varepsilon} \left(\int_{t_n}^{t} \{ \|\nabla u(s)\| + \|\partial_t \phi(s)\| \} ds + \|u(t_n) - \bar{u}\|_{H^2} \right), \end{aligned}$$

we deduce that

$$(\phi(t), u(t)) \underset{t \to +\infty}{\longrightarrow} (\bar{\phi}, \bar{u}) \text{ in } L^2(\Omega) \times H^{-1}(\Omega).$$

By the relative compactness of $\cup_{t \geq t_0} S(t)(\phi_0, u_0)$, we finally conclude that

$$(\phi(t), u(t)) \underset{t \to +\infty}{\longrightarrow} (\bar{\phi}, \bar{u}) \text{ in } H^2(\Omega) \times H^2(\Omega),$$

which finishes the proof of Theorem 6.1.

7 Appendix

Definition 7.1 *Assume that Φ is a complete metric space, $S(t)$ is a nonlinear semigroup defined on Φ and $V(\phi, u)$ is a Lyapounov function. Then the system $(\Phi, S(t), V)$ is called a gradient system if the following conditions are satisfied:*
(i) Let $(u_0, \phi_0) \in \Phi$. If for all $t > 0$, $V(S(t)(\phi_0, u_0)) = V(\phi_0, u_0)$, then (ϕ_0, u_0) is a fixed point of the semigroup $S(t)$.
(ii) For any $(\phi_0, u_0) \in \Phi$, there exists $t_0 > 0$ such that $\cup_{t \geq t_0} S(t)(\phi_0, u_0)$ is relatively compact in Φ.

Next we state two auxiliary Lemmas. The first one corresponds to a Lojasiewicz-Simon type inequality. We omit its proof and refer the reader to [17], [15], [10]. Indeed, even though our potential is not regular as in [17], we established in Theorem 3.1 that every solution $(\phi(t), u(t))$ satisfies

$$\|\phi(t)\|_{L^\infty} \le 1-\delta,$$

and our function f is analytic over $[-1+\delta, 1-\delta]$. Thus, as in [15], [10], we can introduce a proper extension $\tilde{f}$ of f outside $[-1+\delta, 1-\delta]$, where $\tilde{f} \in \mathcal{C}^1(\mathbb{R})$ is such that

$$\begin{array}{ll} \tilde{f}(y) = f(y) & \forall\, y \in [-1+\delta, 1-\delta], \\ |\tilde{f}(y)| \le c & \forall y \in \mathbb{R}, \quad c > 0. \end{array}$$

Lemma 7.1 *Let $(\bar{\phi}, \bar{u})$ be a solution of (6.31). Then, there exist $\sigma > 0$, $\theta \in (0, \frac{1}{2})$ depending on $(\bar{\phi}, \bar{u})$ such that, $\forall\, (\phi(t), u(t)) \in \Phi_M$ which satisfies the following two conditions:*

(i) $\varepsilon < u(t) > + < \phi(t) > = I_0$,

(ii) $\|\phi(t) - \bar{\phi}\|_{H^2} \le \sigma, \quad \|u(t) - \bar{u}\|_{H^2} \le \sigma,$

the following estimate holds:

$$|V(\phi(t), u(t)) - V(\bar{\phi}, \bar{u})|^{1-\theta} \le \frac{1}{\sqrt{2\delta}}\, \|\Delta\phi(t) - f(\phi(t)) + u(t)\| + \frac{1}{\sqrt{2}}\, \|\nabla u(t)\|.$$

Lemma 7.2 *Let $(\phi, u) \in \Phi_M$ be a solution of (2.1) with $g = 0$ and assume that there exists a sequence $t_n \to +\infty$ such that*

$$\phi(t_n) \longrightarrow \bar{\phi} \text{ and } u(t_n) \longrightarrow \bar{u} \text{ strongly in } H^2(\Omega).$$

Then there exists $T > 0$ such that

$$\|\phi(t) - \bar{\phi}\|_{H^2} \le \sigma \quad \text{and} \quad \|u(t) - \bar{u}\|_{H^2} \le \sigma \quad \forall t \ge T,$$

with σ given in Lemma 7.1.

Proof of Lemma 7.2: By assumption, for every $0 < \nu < \sigma$, there exists N such that

$$\begin{array}{l} \forall\, n \ge N, \quad \|u(t_n) - \bar{u}\|_{H^2} \le \frac{\nu}{2}, \quad \|\phi(t_n) - \bar{\phi}\|_{H^2} \le \frac{\nu}{2} \\ \quad \text{and} \quad C_0(V(\phi(t_n), u(t_n)) - V(\bar{\phi}, \bar{u}))^\theta \le \frac{\nu}{2}, \quad C_0 = \frac{\sqrt{2}}{\theta\varepsilon}\,(\frac{1}{\sqrt{\delta}} + 1). \end{array} \tag{7.37}$$

We set

$$\tilde{t}_n = \sup\,\{t > t_n \,/\, \|u(s) - \bar{u}\|_{H^2} < \sigma,\ \|\phi(s) - \bar{\phi}\|_{H^2} < \sigma\,;\, \forall s \in [t_n, t]\}.$$

Let us assume that $\tilde{t}_n < +\infty$ $\forall n \ge N$. Then, $\forall\, t \in [t_n, \tilde{t}_n]$, (6.36) and Lemma 7.1 imply

$$-\frac{\mathrm{d}}{\mathrm{dt}}\,(V(\phi(t), u(t)) - V(\bar{\phi}, \bar{u}))^\theta \ge \theta \left(\sqrt{\frac{\delta}{2}}\, \|\partial_t \phi(t)\| + \frac{1}{\sqrt{2}} \|\nabla u(t)\| \right).$$

Thus, integrating over $[t_n, \tilde{t}_n]$, we have

$$\int_{t_n}^{\tilde{t}_n} \|\partial_t \phi(s)\|\, ds \le \frac{1}{\theta} \sqrt{\frac{2}{\delta}}\, (V(\phi(t_n), u(t_n)) - V(\bar{\phi}, \bar{u}))^\theta \le \frac{\nu}{2}$$

and, using (7.37), we infer

$$\begin{array}{rcl}\|\phi(\tilde{t}_n)-\bar{\phi}\| & \leq & \|\phi(\tilde{t}_n)-\phi(t_n)\|+\|\phi(t_n)-\bar{\phi}\| \\ & \leq & \int_{t_n}^{\tilde{t}_n}\|\partial_t\phi(s)\|ds\,+\,\|\phi(t_n)-\bar{\phi}\|_{H^2} \\ & \leq & \nu.\end{array}$$

We thus deduce that

$$\phi(\tilde{t}_n)\underset{n\to+\infty}{\longrightarrow}\bar{\phi}\ \text{in}\ \ L^2(\Omega).$$

In the same way, we can prove that $u(\tilde{t}_n)\underset{n\to+\infty}{\longrightarrow}\bar{u}$ in $H^{-1}(\Omega)$. By the relative compactness of the orbit, there exists a subsequence $(\phi(\tilde{t}_n),u(\tilde{t}_n))$, still denoted by $(\phi(\tilde{t}_n),u(\tilde{t}_n))$, such that $\phi(\tilde{t}_n)\to\bar{\phi}$ in $H^2(\Omega)$. So, if n is big enough, we have

$$\|\phi(\tilde{t}_n)-\bar{\phi}\|_{H^2}<\frac{\sigma}{2}\quad\text{and}\quad\|u(\tilde{t}_n)-\bar{u}\|_{H^2}<\frac{\sigma}{2},$$

which contradicts the definition of $\tilde{t}_n$. Thus, there exists $n_0>0$ such that $\tilde{t}_{n_0}=+\infty$ and Lemma 7.2 is proven, with $T=t_{n_0}$.

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