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# Jet Groups

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**Abstract.**

We describe some classes of subgroups of the $n$-jet group of analytic automorphisms of $\mathbb{C}^N$ fixing the origin. We apply the results to show that any subgroup of the algebraic automorphism group strictly containing the affine one is dense for the Krull topology. We also show that any algebraic or analytic automorphism can be interpolated at any order and at any finite set of points, by a tame one.

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## INTRODUCTION.

Let $N \geq 2$. Let $\mathbb{A}^N$ be the vector space $V = \mathbb{C}^N$ when it is seen as an affine space, let $(e_1, \ldots, e_N)$ be the canonical basis of $V$ and let $(x_1, \ldots, x_N)$ be the dual basis of $V^*$. Let $E$ (resp. $\widetilde{E}$) be the space of algebraic (resp. analytic) endomorphisms of $\mathbb{C}^N$. We will identify any element $f$ of $E$ (resp. $\widetilde{E}$) to the $N$-uple of its coordinate functions $f = (f_1, \ldots, f_N)$ where each $f_L$ belongs to the ring $R = \mathbb{C}[x_1, \ldots, x_N]$ of regular functions on $\mathbb{A}^N$ (resp. the ring $\widetilde{R}$ of analytic functions on $\mathbb{A}^N$). We will denote by $\mathrm{id} := (x_1, \ldots, x_N)$ the identity element of $E$. If $f \in \widetilde{E}$, we will denote by $f_{(k)}$ its homogeneous part of degree $k$. Let $A$ (resp. $\widetilde{A}$) be the group of algebraic (resp. analytic) automorphisms of $\mathbb{C}^N$ and let $SL$ (resp. $GL$, resp. $GA$) denote the special linear (resp. linear, resp. affine) group of $\mathbb{C}^N$. Let $T$ be the group of algebraic tame automorphisms, i.e. the subgroup of $A$ generated by $GA$ and by the elementary automorphisms $\mathrm{id} + p(x)e_L$, where $p$ is a polynomial independant of $x_L$. Let $\widetilde{T}$ be the group of analytic tame automorphims, i.e. the subgroup of $\widetilde{A}$ generated by $GA$ and by the overshears $(x', p(x') + q(x')x_N)$, where $x' = (x_1, \ldots, x_{N-1})$, $p, q : \mathbb{C}^{N-1} \to \mathbb{C}$ are analytic and $q$ does not vanish (or similar ones obtained by permuting the variables). The relation $G_1 \leq G_2$ means that $G_1$ is a subgroup of $G_2$. For $k \geq 0$, we set $E_k := \{f \in E, \forall \lambda \in \mathbb{C}, \forall a \in \mathbb{C}^N,\ f(\lambda a) = \lambda^k f(a)\}$ and for $K \subset \mathbb{N}$, we set $E_K := \bigoplus_{k \in K} E_k \subset E$ and $A_K := A \cap E_K$. If $S$ is a sub-semigroup of $\mathbb{N}$, it is shown in § $III.2$ below (see prop. 3.1) that $A_{1+S} \leq A$ (we call monoidal such a subgroup). If $G \leq \widetilde{A}$, let us agree, that $J_n(G)$ denotes the group of $n$-jets at the origin of the elements of $G$ fixing the origin. We have $J_n(A_{1+S}) \leq J_n(A)$. Our first result is the following.

**Theorem A.** Any group $G$ such that $GL \leq G \leq J_n(A)$ is equal to some $J_n(A_{1+S})$ where $S$ is a sub-semigroup of $\mathbb{N}$.

For a more general description of the groups $G$ such that $SL \leq G \leq J_n(A)$, see th. 4.2. If $1 \leq k \leq n$, let $J_{n,k} : J_n(\widetilde{A}) \to J_k(\widetilde{A})$ be the natural group-morphism associating to a $n$-jet its restricted $k$-jet. Since $J_k(A) \leq J_k(\widetilde{A})$, we get $J_{n,k}^{-1}(J_k(A)) \leq J_n(\widetilde{A})$. Actually:

**Theorem B.** Any group $G$ such that $J_n(A) \leq G \leq J_n(\widetilde{A})$ is equal to some $J_{n,k}^{-1}(J_k(A))$ where $1 \leq k \leq n$.

Let us agree that the Krull topology on $E$ is the one making it a topological vector space and such that a basis of open neighbourhoods of the origin is composed by the sets $U_n$ ($n \geq 1$) of endomorphisms whose $n$-jet at the origin is zero. With the induced topology, $A$ is a topological group. In [3], the author shows that $T$ is dense in $A$ for the Krull topology and concludes with these words: "very little of the full strength of $T$ is needed in order to approximate any element of $A$". The next result completes this statement.

**Theorem C.** Any subgroup of $A$ strictly larger than $GA$ is dense for the Krull topology.

The last assertion (to be used in [13]) gives the conditions under which it is possible to interpolate finitely many analytic (resp. algebraic) automorphisms by an analytic (resp. algebraic) tame automorphism.

**Theorem D.** Let $n \geq 1$, let $u^{[1]}, \ldots, u^{[m]}$ be distinct points of $\mathbb{C}^N$ and let $f^{[1]}, \ldots, f^{[m]}$ be analytic (resp. algebraic) automorphisms of $\mathbb{C}^N$. There exists an analytic (resp. algebraic) tame automorphism $f$ such that the $n$-jets of $f$ and $f^{[k]}$ coincide at each $u^{[k]}$ if and only if the $f^{[k]}(u^{[k]})$ are distinct (resp. the $f^{[k]}(u^{[k]})$ are distinct and the Jacobians of the $f^{[k]}$ are equal).

**Corollary.** For any analytic (resp. algebraic) automorphism $f$, any finite set of points and any $n \geq 1$, there exists a tame analytic (resp. algebraic) automorphism $g$ such that the $n$-jets of $f$ and $g$ coincide at these points.

Our paper is divided into six sections. Sections I and II are devoted to establish preliminary results on the vector space $E$ considered either as a $GL$-module or as a Lie algebra. In section III, we introduce the notations and tools that we use in section IV (resp. section V) to prove th. A,B,C (resp. th. D). Finally, in section VI, we apply some of the previous notions to variables (or coordinates) and recover a result of [9].

## I. THE SPACE E AS A GL-MODULE.

The main aim of this paragraph is to give the decomposition into irreducible submodules of the $GL$-module $E_m$, essentially showing that $E_m$ splits into two irreducible submodules $E_m = E_m^0 \oplus E_m^1$, where $E_m^0 := \{f \in E_m, \nabla f = 0\}$ and $E_m^1 := \{\Delta\, r, r \in R_{m-1}\}$, with $\nabla f = \sum_{1\leq L\leq N} \dfrac{\partial f_L}{\partial x_L}$ and $\Delta\, r = r\,\mathrm{id} = (r\,x_1, \ldots, r\,x_N)$. However, we will study this subject in detail. Let $SV^* \simeq \mathbb{C}[x_1, \ldots, x_N]$ be the symmetric algebra of $V^*$. The isomorphism $\varphi\,:\; SV^* \otimes V \to E,\; p \otimes v \mapsto pv$ will be the main thread, allowing to identify $f = (f_1, \ldots, f_N) = \sum_{1\leq L\leq N} f_L e_L \in E$ with $\varphi^{-1}(f) = \sum_{1\leq L\leq N} f_L \otimes e_L \in SV^* \otimes V$. Since $V$ is naturally a $GL$ module, so is $E \simeq SV^* \otimes V$, the action being the following: $GL \times E \to E$, $(g, f) \mapsto g \circ f \circ g^{-1}$.

We begin to describe two direct sum decompositions of the $GL$-module $E$. The first one is the obvious $E = \bigoplus_{m\geq 0} E_m$ which corresponds via $\varphi$ to $SV^* \otimes V = \bigoplus_{m\geq 0} S^m V^* \otimes V$. The second is closely linked with the natural maps of contraction $c\,:\; SV^* \otimes V \to SV^*$ and multiplication $m\,:\; SV^* \to SV^* \otimes V$ which we now introduce. The contraction map $c\,:\; SV^* \otimes V \to SV^*$ is defined by its restriction

$$\begin{array}{ccccc} S^{m+1}V^* & \otimes & V & \to & S^m V^* \\ v_1^* \ldots v_{m+1}^* & \otimes & v & \mapsto & \displaystyle\sum_i < v, v_i^* > v_1^* \ldots \widehat{v_i^*} \ldots v_{m+1}^* \end{array}$$

and the induced map (using $\varphi$) on $E$ is nothing else than the operator $\nabla\,:\; E \to R$. Indeed, if $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ and $L \in \{1, \ldots, N\}$, we have

$$c(x^\alpha \otimes e_L) = c(x_1^{\alpha_1} \ldots x_N^{\alpha_N} \otimes e_L) = \sum_{M=1}^{N} \sum_{s=1}^{\alpha_M} < e_L, x_M > \frac{x^\alpha}{x_M} = \alpha_L \frac{x^\alpha}{x_L} = \nabla(x^\alpha e_L).$$

The multiplication map $m\,:\; SV^* \to SV^* \otimes V$ is the multiplication by the identity element $\mathrm{id} \in V^* \otimes V = Hom(V, V)$. Its restriction $S^m V^* \to S^{m+1} V^* \otimes V$ is the composition of the two maps:

$$\begin{array}{ccccc} S^m V^* & \to & S^m V^* & \otimes & V^* \otimes V \\ t & \mapsto & t & \otimes & \mathrm{id} \end{array}$$

and

$$\begin{array}{cccccccccc} S^m V^* & \otimes & V^* & \otimes & V & \to & S^{m+1} V^* & \otimes & V \\ t & \otimes & u & \otimes & v & \mapsto & tu & \otimes & v \end{array}$$

The induced map (using $\varphi$) on $R = SV^*$ is the operator $\Delta\,:\; R \to E$.

**Lemma 1.1.** The maps $\nabla\,:\; E \to R$ and $\Delta\,:\; R \to E$ are $GL$-morphisms.

**Proof.** Since $\nabla$ and $\Delta$ correspond to the natural maps $c\,:\; SV^* \otimes V \to SV^*$ and $m\,:\; SV^* \to SV^* \otimes V$, the checking is straightforward. Let us for example show that $c$ is a $GL$-morphism. It is enough to prove that if $g \in GL$, $f = v_1^* \ldots v_{m+1}^* \otimes v \in S^{m+1} V^* \otimes V$, then $c(g.f) = g.c(f)$. We obtain:

$$\begin{aligned} c(g.f) &= c\left(g.(v_1^* \ldots v_{m+1}^* \otimes v)\right) = c\left(v_1^* \circ g^{-1} \ldots v_{m+1}^* \circ g^{-1} \otimes g(v)\right) \\ &= \sum_i < g(v), v_i^* \circ g^{-1} > \; v_1^* \circ g^{-1} \ldots \widehat{v_i^* \circ g^{-1}} \ldots v_{m+1}^* \circ g^{-1} \\ &= \sum_i < v, v_i^* > \; v_1^* \circ g^{-1} \ldots \widehat{v_i^* \circ g^{-1}} \ldots v_{m+1}^* \circ g^{-1} = g.c(f). \end{aligned}$$

$\square$

**Remark.** One may of course check directly that if $g \in GL$, $f \in E$, $r \in R$ then
(i) $\nabla(g.f) = g.\nabla f$, i.e. $\nabla(g \circ f \circ g^{-1}) = (\nabla f) \circ g^{-1}$;

(ii) $\Delta(g.r) = g.\Delta(r)$, i.e. $\Delta(r \circ g^{-1}) = g \circ \Delta(r) \circ g^{-1}$.

For (i), we use the facts that $\nabla f = \mathrm{Tr}\ f'$ and that $g'(x)$ does not depand on $x$ (since $g$ is linear). We have

$$
\begin{aligned}
(g \circ f \circ g^{-1})'(x) &= g'\left(f \circ g^{-1}(x)\right) \times f'\left(g^{-1}(x)\right) \times (g^{-1})'(x) \\
&= g'(x) \times f'\left(g^{-1}(x)\right) \times g'(x)^{-1}
\end{aligned}
$$

and the result follows by taking the trace.

For (ii), since $\Delta(r)(x) = r(x)x$, we obtain

$$
\begin{aligned}
\left(g \circ \Delta(r) \circ g^{-1}\right)(x) &= g\left(r \circ g^{-1}(x)\ g^{-1}(x)\right) = r \circ g^{-1}(x)\ g\left(g^{-1}(x)\right) \text{ since } g \text{ is linear} \\
&= r \circ g^{-1}(x)\ x = \left(\Delta(r \circ g^{-1})\right)(x).
\end{aligned}
$$

Lemma 1.1 shows us that $E^0 := \mathrm{Ker}\ \nabla$ and $E^1 := \mathrm{Im}\ \Delta$ are $GL$-submodules of $E$.

**Lemma 1.2.** $E = E^0 \oplus E^1$.

**Proof.** $\forall\, r \in R_m$, $\nabla \circ \Delta(r) = \nabla(r\ id) = \displaystyle\sum_L \frac{\partial}{\partial x_L}(r\ x_L) = \sum_L \left(x_L \frac{\partial r}{\partial x_L} + r\right) = (N+m)r$.

Therefore $\frac{1}{N+m}\Delta$ is a section of $\nabla : E_{m+1} \to R_m$ and the split short exact sequence: $0 \to E^0_{m+1} \to E_{m+1} \underset{\frac{1}{N+m}\Delta}{\overset{\nabla}{\rightleftarrows}} R_m \to 0$ shows us that $E_{m+1} = E^0_{m+1} \oplus E^1_{m+1}$. □

Two direct sum decompositions $F = \underset{i}{\oplus} G_i$ and $F = \underset{j}{\oplus} H_j$ of a vector space $F$ are called compatible if the following equivalent assertions are satisfied:

(i) $\forall\ i,\ G_i = \displaystyle\bigoplus_j G_i \cap H_j$; (ii) $\forall\ j,\ H_j = \displaystyle\bigoplus_i G_i \cap H_j$; (iii) $F = \displaystyle\bigoplus_{i,j} G_i \cap H_j$.

It is clear that $E = \underset{m \geq 0}{\oplus} E_m$ and $E = \underset{n=0,1}{\oplus} E^n$ are compatible, so that $E = \displaystyle\bigoplus_{(m,n) \in \mathbb{N} \times \{0,1\}} E^n_m$, where $E^n_m := E_m \cap E^n$ is a $GL$-module. In fact:

**Theorem 1.1.** (i) The $GL$-representations $E^n_m$, $(m,n) \in \mathbb{N} \times \{0,1\}$, are irreducible and pairwise non isomorphic;
(ii) If $N \geq 3$, the restricted $SL$-representations are still pairwise non isomorphic;
(iii) If $N = 2$, the restricted $SL$-representations $E^0_m$, $m \in \mathbb{N}$, are still pairwise non isomorphic, but the restricted $SL$-representations $E^0_m$ and $E^1_{m+2}$ are now isomorphic.

**Remarks.** 1. According to their definition of irreducible representations, some readers may prefer to except the case $(m,n) = (0,1)$ where $E^1_0$ is the null space.
2. A $GL$-representation is irreducible if and only if its restricted $SL$-representation is. Furthermore, it is well known that two irreducible $GL$-representations are isomorphic if and only if their restrictions to $SL$ and to $\mathbb{C}^* \leq GL$ are isomorphic.

**Proof.** Let us begin to show that the restricted $SL$-representations are irreducible.

The case $m = 0$ being obvious since $E_0^0 \simeq V$ and $E_0^1 = \{0\}$, let us assume that $m \geq 1$. One could show directly that $E_m^n$ is an irreducible $SL$-module by choosing a maximal torus of $SL$ and by studying the weights of the representation $E_m^n$. Also, if $n = 1$, the $SL$-morphism $\Delta$ shows us that $E_m^1 \simeq R_{m-1} \simeq S^{m-1}V^*$ and this last $SL$-module is well-known to be irreducible. But if $n = 0$, we did not find such an easy argument.

However, $E_m \simeq S^mV^* \otimes V$ and $E_m = E_m^0 \oplus E_m^1$. Therefore, to show that $E_m^0$ and $E_m^1$ are irreducible, it is sufficient to show that $S^mV^* \otimes V$ is the direct sum of exactly two irreducible representations. For technical reasons, we will rather show that the dual representation $(S^mV^* \otimes V)^* \simeq S^mV \otimes V^*$ is the sum of two irreducible ones.

The main points are the following ones (see [12]):

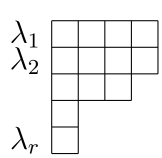
• The representations $S^mV$ and $V^*$ are irreducible;

• Any irreducible representation of $SL$ is isomorphic to some Weyl module $\mathbb{S}_\lambda V$, where $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of an integer $d \geq 1$ ($d = \lambda_1 + \ldots + \lambda_r$ with $\lambda_1 \geq \ldots \geq \lambda_r \geq 1$) and $\mathbb{S}_\lambda$ denotes the Schur functor associated to $\lambda$;

• The Weyl module $\mathbb{S}_\lambda V$ is an irreducible $SL$-module and it is nonzero if and only if $r \leq N$ (i.e. the partition has at most $N$ parts). Moreover, if $\lambda = (\lambda_1, \ldots, \lambda_r)$ (resp. $\mu = (\mu_1, \ldots, \mu_s)$) is the partition of an integer $d$ (resp. $e$), the $SL$-modules $\mathbb{S}_\lambda V$ and $\mathbb{S}_\mu V$ are isomorphic if and only if $\lambda_k - \mu_k$ is constant, independant of $k$, for $1 \leq k \leq N$, where we agree that $\lambda_k = 0$ (resp. $\mu_k = 0$) if $k > r$ (resp. $k > s$). Therefore, if we want a unique Schur functor for each representation, we can restrict to those $\lambda$ with $\lambda_N = 0$ (i.e. partitions of an integer $d$ in at most $N - 1$ parts) which we will call reduced;

• If $\lambda$ (resp. $\mu$) is the partition of an integer $d$ (resp. $e$), the plethysm relations for $SL$ tells us that $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \simeq \bigoplus_\nu N_{\lambda,\mu,\nu}\, \mathbb{S}_\nu V$ (as $SL$-modules) where the sum is over all partitions $\nu$ of $d + e$ and the multiplicities $N_{\lambda,\mu,\nu}$ are computed by the Littlewood-Richardson rule. In fact, we will use this rule in the special case where $\mu$ is of the shape $\mu = (\underbrace{1, \ldots, 1}_{k})$, i.e. $\mathbb{S}_\mu V \simeq \bigwedge^k V$. Then, the Littlewood-Richardson rule simplifies in the simpler Pieri formula which asserts that $\mathbb{S}_\lambda V \otimes \bigwedge^k V \;\simeq\; \bigoplus_\nu \mathbb{S}_\nu V$ where the sum is over all partitions $\nu$ whose Young diagram is obtained from that of $\lambda$ by adding $k$ boxes, with no two in the same row.

Let us recall that the Young diagram of the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the following picture



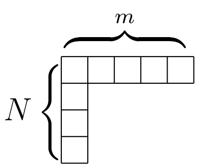

with $\lambda_i$ boxes in the $i$-th row, the rows of boxes lined up on the left. In our case, $S^mV \simeq \mathbb{S}_\lambda V$ where $\lambda = (m)$ is represented by the single row with $m$ boxes: $\overbrace{\square\square\square\square\square}^{m}$

Furthermore $\bigwedge^N V$ is the trivial representation (it would not be the case if $SL$ was replaced by $GL$), so that the non degenerate pairing

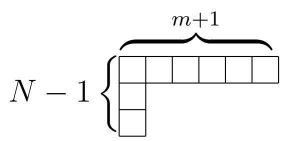
$$\begin{array}{ccccc} V & \otimes & \bigwedge^{N-1} V & \to & \bigwedge^N V \simeq \mathbb{C} \\ u & \otimes & v & \mapsto & u \wedge v \end{array}$$

shows us that $V^* \simeq \bigwedge^{N-1} V$.

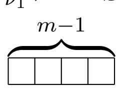
Therefore $S^m V \otimes V^* \simeq S^m V \otimes \bigwedge^{N-1} V \simeq \mathbb{S}_{\nu_1} V \oplus \mathbb{S}_{\nu_2} V$ where $\nu_1 = (m, \underbrace{1, \ldots, 1}_{N-1})$ and $\nu_2 = (m+1, \underbrace{1, \ldots, 1}_{N-2})$ are represented by the hooks:



and



so that we have shown the irreducibility of the $E_m^n$.

Finally, $\mathbb{S}_{\nu_1} V \simeq \mathbb{S}_{\nu_1'} V$ where $\nu_1' = (m-1)$ is represented by the single row with $m-1$ boxes:  so that $\nu_1'$ and $\nu_2$ are the reduced partitions such that

$$(E_m^0)^* \simeq \mathbb{S}_{\nu_1'} V \text{ and } (E_m^1)^* \simeq \mathbb{S}_{\nu_2} V.$$

This shows (ii) and (iii).
It remains to show that if $N = 2$ the $GL$-representations $E_m^0$ and $E_{m+2}^1$ are non isomorphic. But this is clear since their restriction to $\mathbb{C}^* \subset GL$ are non isomorphic. Indeed, $\lambda \in \mathbb{C}^*$ acts on $E_m^0$ as the dilatation of ratio $\lambda^{m-1}$ and on $E_{m+2}^1$ as the dilatation of ratio $\lambda^{m+1}$.
□

If $N = 2$, the next result provides us a $SL$-isomorphism between $E_m^0$ and $R_{m+1}$:

**Lemma 1.3.** If $N = 2$, the map $\begin{array}{cccc} \alpha : & R & \to & E^0 \\ & r & \mapsto & (\frac{\partial r}{\partial x_2}, -\frac{\partial r}{\partial x_1}) \end{array}$ is a $SL$-morphism.

**Proof.** If $g \in GL$, an easy computation shows that

$$\alpha(r \circ g^{-1}) = (\det g)^{-1}\, g \circ \alpha(r) \circ g^{-1}, \text{ i.e. } \alpha(g.r) = (\det g)^{-1}\, g.\alpha(r)$$

which proves that $\alpha$ is a $SL$-morphism (but not a $GL$-morphism !). □

## II. THE SPACE E AS A LIE ALGEBRA.

In this section, it is important to stress that even if the endomorphism $f \in E$ is usually written as the line-vector $f = (f_1, \ldots, f_N)$ in the literature (and in this paper !), it should actually be thought of as the column-vector $f = {}^t(f_1, \ldots, f_N)$. Indeed, this point is fundamental to grasp the Lie bracket formula on $E$ given further. In the proof of lemma 2.2 below, it is also necessary to know that id $\in E$ is a column-vector and that $s'$ is a line-vector (when $s \in R$), in order to understand that id $\times s'$ is a $N \times N$ matrix and that

$s' \times \mathrm{id}$ is a scalar.

Let us denote by $Der\ R$ the set of $\mathbb{C}$-derivations of $R$, i.e. the set of complex linear maps $D\ :\ R \to R$ satisfying $D(rs) = D(r)s + rD(s)$ for all $r, s \in R$. We will analyze the Lie algebra structure of $E$ (which comes from the isomorphism $E \simeq Der\ R$). In th. 2.1, we compute $[E_m^n, E_p^q]$.

The isomorphism $\psi\ :\ E \to DerR,\ f = (f_1, \dots, f_N) \mapsto D_f = \sum_{1 \le L \le N} f_L \frac{\partial}{\partial x_L}$ comes from the usual identification between $e_L$ and the derivation $\frac{\partial}{\partial x_L}\ :\ R \to R$. By pulling-back the Lie bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ defined on $DerR$, we obtain the Lie bracket $[f, g] = g'(x) \times f(x) - f'(x) \times g(x) = g' \times f - f' \times g$ defined on $E$, where we agree that $f' = f'(x) = \left(\frac{\partial f_L}{\partial x_M}\right)_{1 \le L, M \le N}$ is the Jacobian matrix of $f$.

Let us show this formula. If $f = (f_1, \dots, f_N)$ and $g = (g_1, \dots, g_N) \in E$, we have:

$$[D_f, D_g]\, x_M = \sum_{L=1}^{N} f_L \frac{\partial g_M}{\partial x_L} - \sum_{L=1}^{N} f_L \frac{\partial f_M}{\partial x_L} = D_h\, x_M, \qquad 1 \le M \le N$$

where $h = g' \times f - f' \times g$, hence $[D_f, D_g] = D_h$.

If $\mathfrak{a}$, $\mathfrak{b}$ are additive subgroups of a Lie algebra $\mathfrak{g}$, the subgroup generated by all brackets $[a, b]$, $(a, b) \in \mathfrak{a} \times \mathfrak{b}$ is denoted by $[\mathfrak{a}, \mathfrak{b}]$. If $\mathfrak{a}$, $\mathfrak{b}$ are subspaces, then $[\mathfrak{a}, \mathfrak{b}]$ also.

**Remarks.** 1. We have $[E_m, E_n] \subset E_{m+n-1}$, therefore, if we set $L_m = E_{m+1}$, then $E = \bigoplus_{m \ge -1} L_m$ is a graduated Lie algebra. However, even if we will always use this fact, we prefer to use our old notation $E_m$ instead of the new one $L_m$ which would have the big drawback of introducing a tiresome shift between degrees as polynomial endomorphisms and degrees as graduated objects.
2. We may have defined the Lie bracket on $E \simeq SV^* \otimes V$ by using the contraction map $c\ :\ SV^* \otimes V \to SV^*$. We let the reader check that the Lie bracket is then given by the following map:

$$\begin{array}{ccccccccccc} (S^{m+1}V^* & \otimes & V) & \otimes & (S^{n+1}V^* & \otimes & V) & \to & S^{m+n+1}V^* \otimes V \\ (t & \otimes & u) & \otimes & (z & \otimes & v) & \mapsto & t\, c(z \otimes u) \otimes v - z\, c(t \otimes v) \otimes u. \end{array}$$

This allows us to see directly that $\forall\ g \in GL,\ \forall\ u, v \in E,\ g.[u, v] = [g.u, g.v]$ which is otherwise the following computation:

$$\begin{aligned} g.[u, v] &= g\, \circ\, \big(v'(x) \times u(x) - u'(x) \times v(x)\big) \circ g^{-1} \\ &= \big(g'\left(v(x)\right) \times v'(x) \times u(x) - g'\left(u(x)\right) \times u'(x) \times v(x)\big) \circ g^{-1} \\ &= g'(v \circ g^{-1}(x)) \times v'(g^{-1}(x)) \times u(g^{-1}(x)) \\ &\qquad - g'(u \circ g^{-1}(x)) \times u'(g^{-1}(x)) \times v(g^{-1}(x)) \\ &= g'(v \circ g^{-1}(x)) \times v'(g^{-1}(x)) \times (g^{-1})'(x) \times g\left(u(g^{-1}(x))\right) \\ &\qquad - g'(u \circ g^{-1}(x)) \times u'(g^{-1}(x)) \times (g^{-1})'(x) \times g\left(v(g^{-1}(x))\right) \\ &= (g \circ v \circ g^{-1})'(x)\ \times\ g\ \circ u \circ g^{-1}(x) - (g \circ u \circ g^{-1})'(x)\ \times\ g\ \circ v \circ g^{-1}(x) \\ &= [g \circ u \circ g^{-1}, g \circ v \circ g^{-1}] = [g.u, g.v]. \end{aligned}$$

We have used the fact that if $g$ is linear (i.e. $g \in E_1$), then the Jacobian matrix $g' = g'(x)$ of $g$ has constant coefficients. In particular, for all $u = u(x) \in E$, we have $g'(u(x)) = g'(x)$ and for all $v = v(x) \in E$, we have $g(v(x)) = g'(x) \times v(x) = g'(u(x)) \times v(x)$. Therefore, if $g \in GL$, we have $(g^{-1})'(x) \times g(u(x)) = g^{-1}(g(u(x))) = u(x)$ and this explains the equality $(g^{-1})'(x) \times g(u(g^{-1}(x))) = u(g^{-1}(x))$.

As a consequence, if $\mathfrak{a}$, $b$ are $GL$-submodules of $E$, then $[\mathfrak{a}, \mathfrak{b}]$ also.

3. The Lie algebra structure on $E$ contains in some sense the $GL$-module structure. Indeed, let us denote by $\mathfrak{g}l = E_1$ the Lie algebra of $GL$.

**Lemma 2.1.** The Lie algebra representation: $\begin{array}{ccccc} \mathfrak{g}l & \times & E & \to & E \\ (g & , & f) & \mapsto & [f,g] \end{array}$ is the one associated with the Lie group representation: $\begin{array}{ccccc} GL & \times & E & \to & E \\ (g & , & f) & \mapsto & g \circ f \circ g^{-1}. \end{array}$

**Proof.** If $f \in E$, we just want to show that the differential at the point $id$ of the map $GL \to E$, $g \mapsto g \circ f \circ g^{-1}$ is the map $\mathfrak{g}l \to E$, $g \mapsto [f,g]$.
If $g \in \mathfrak{g}l$, it is enough to compute the differential at the origin of the following map (defined on a small centered neighborhood of the origin):

$$\begin{array}{rccc} \theta : & ]-\varepsilon, \varepsilon[ & \to & E \\ & t & \mapsto & (id + t\ g) \circ f \circ (id + t\ g)^{-1} \end{array}$$

We have $\theta(t) = (id + t\ g) \circ f \circ (id - t\ g + o(t)) = (id + t\ g) \circ (f - t\ f' \times g + o(t))$
$= f + t\ (g \circ f - f' \times g) + o(t) = f + t\ (g' \times f - f' \times g) + o(t)$
since $g$ is linear, so $\theta'(0) = [f,g]$. □

If $\mathfrak{s}l$ (resp. $\mathfrak{c}$) denotes the Lie subalgebra of $\mathfrak{g}l$ corresponding to the subgroup $SL$ (resp. $\mathbb{C}^*$) of $GL$, then the decomposition $E_1 = E_1^0 \oplus E_1^1$ is the same as the classical Levy decomposition $\mathfrak{g}l = \mathfrak{s}l \oplus \mathfrak{c}$. Furthermore, if $W$ is a representation of a reductive Lie algebra $\mathfrak{g}$, it is well known that $\mathfrak{g}.W$ is equal to the sum of the non trivial irreducible subrepresentations of $W$ (indeed, if we assume in addition that $W$ is irreducible, it is clear that $\mathfrak{g}.W = \{0\}$ if $W$ is trivial and that $\mathfrak{g}.W = W$ otherwise). If $m \geq 2$ and $n = 0, 1$, we know that:
• the $\mathfrak{s}l$-representation $E_m^n$ is irreducible and non trivial;
• the $\mathfrak{c}$-representation $E_m^n$ corresponds to the $\mathbb{C}^*$-representation of $E_m^n$ where $\lambda \in \mathbb{C}^*$ acts on $E_m^n$ as the dilatation of ratio $\lambda^{m-1}$.

As a result:

**Corollary 2.1.** $[E_1^0, E_1^0] = [E_1^0, E_1] = [E_1, E_1] = E_1^0$;
$[E_1^1, E_1^0] = [E_1^1, E_1^1] = [E_1^1, E_1] = \{0\}$;
$[E_1^0, E_m^n] = [E_1^1, E_m^n] = [E_1, E_m^n] = E_m^n$ for $m \geq 2$ and $n = 0, 1$.

The next result will show that $E^0$ and $E^1$ are Lie subalgebra of $E$.

**Lemma 2.2.** (i) $\forall\ f, g \in E,\ \nabla([f,g]) = (\nabla g)' \times f - (\nabla f)' \times g$;
(ii) $\forall\ (r,s) \in R_m \times R_n,\ [r\,\mathrm{id}, s\,\mathrm{id}] = (n-m)\ rs\,\mathrm{id}$.

**Proof.** We have:

$$\nabla(g'\times f) = \sum_L \frac{\partial}{\partial x_L}(g'.f)_L = \sum_L \frac{\partial}{\partial x_L}\left(\sum_M \frac{\partial g_L}{\partial x_M} f_M\right)$$

$$= \sum_{L,M} \frac{\partial^2 g_L}{\partial x_L \partial x_M} f_M \ + \ \sum_{L,M} \frac{\partial g_L}{\partial x_M}\ \frac{\partial f_M}{\partial x_L}$$

$$= \sum_M \frac{\partial}{\partial x_M}(\nabla g)\ f_M \ + \ \sum_L (g'\times f')_{L,L} = (\nabla g)'\times f \ + \ \mathrm{Tr}(g'\times f').$$

and (i) follows.

We have $(s\,\mathrm{id})' = \mathrm{id}\times s' \ + \ s\ I_N$ where $I_N$ is the $N\times N$ identity matrix, so that $(s\,\mathrm{id})'\times(r\,\mathrm{id}) = r\,.\,\mathrm{id}\ (s'\times\mathrm{id}) \ + \ rs\,\mathrm{id} = n\ rs\,\mathrm{id} \ + \ rs\,\mathrm{id} = (n+1)\ rs\,\mathrm{id}$ and (ii) follows. We have used the fact that $s'\times id = n\ s$ by the Euler formula. $\square$

**Corollary 2.2.** $E^0$ and $E^1$ are Lie subalgebra of $E$.

Even if $E = E^0\oplus E^1$ as vector spaces, the sum is not direct as Lie algebra since $[E^0, E^1]\neq\{0\}$. In fact, $E^0$ and $E^1$ are even not Lie ideals of $E$:

**Theorem 2.1.** Let $m, n\geq 1$.

(i) $[E^0_m, E^0_n] = E^0_{m+n-1}$;

(ii) $[E^1_m, E^1_n] = E^1_{m+n-1}$ if $m\neq n$;
$= \{0\}$ if $m = n$;

(iii) $[E_m, E_n] = E_{m+n-1}$ if $m$ or $n\geq 2$;
$= E^0_1$ if $m = n = 1$;

(iv) $[E^0_m, E^1_n] = E_{m+n-1}$ if $m, n\geq 2$;
$= E^1_n$ if $m = 1$, $n\geq 2$;
$= E^0_m$ if $m\geq 2$, $n = 1$;
$= \{0\}$ if $m = n = 1$.

**Proof.** We recall that $[x^\alpha e_L, x^\beta e_M] = \frac{\partial}{\partial x_M}(x^\alpha)x^\beta e_L - \frac{\partial}{\partial x_L}(x^\beta)x^\alpha e_M$.

(i) Since $[E^0_m, E^0_n]$ is a sub $GL$-module of the irreducible $GL$-module $E^0_{m+n-1}$, it is sufficient to show that $[E^0_m, E^0_n]\neq\{0\}$. Indeed, $x_2^m e_1\in E^0_m$, $x_1^n e_2\in E^0_n$ and $[x_2^m e_1, x_1^n e_2] = m x_1^n x_2^{m-1} e_1 - n x_1^{m-1} x_2^n\ e_2\neq 0$.

(ii) Point (ii) of lemma 2.2 shows us that $[E^1_m, E^1_n] = \{0\}$ if and only if $m = n$. When $m\neq n$, since $[E^1_m, E^1_n]$ is a sub $GL$-module of the irreducible $GL$-module $E^1_{m+n-1}$, we can conclude to the equality.

(iv) We can assume that $m, n\geq 2$. Let us set $u = [x_2^m e_1,\ x_1^{n-1}\ \mathrm{id}]\in[E^0_m, E^1_n]$.

$$u = [x_2^m e_1,\ x_1^{n-1}\sum_{L\geq 1} x_L e_L\,] = [x_2^m e_1, x_1^n e_1] \ + \ [x_2^m e_1, x_1^{n-1} x_2 e_2] \ + \ \sum_{L\geq 3}[x_2^m e_1, x_1^{n-1} x_L e_L]$$

$$= (m-n)x_1^{n-1}x_2^m e_1 - (n-1)x_1^{n-2}x_2^{m+1}e_2 - (n-1)x_1^{n-2}x_2^m\sum_{L\geq 3} x_L e_L$$

$$= (m-n)x_1^{n-1}x_2^m e_1 - (n-1)x_1^{n-2}x_2^m \sum_{L\geq 2} x_L e_L$$

$$= (m-1)x_1^{n-1}x_2^m e_1 - (n-1)x_1^{n-2}x_2^m \,\mathrm{id} \qquad \text{so that } u \notin E^1_{m+n-1} \text{ and}$$

$$\begin{aligned}\nabla u &= (n-1)(m-1)x_1^{n-2}x_2^m - (n-1)(N+m+n-2)x_1^{n-2}x_2^m \\ &= -(n-1)(N+n-1)x_1^{n-2}x_2^m \neq 0 \qquad \text{so that } u \notin E^0_{m+n-1}.\end{aligned}$$

Since $[E^0_m, E^1_n]$ is a sub $GL$-module of $E_{m+n-1}$, we must have $[E^0_m, E^1_n] = E_{m+n-1}$.

(iii) It is a consequence of (i), (ii) and (iv). $\square$

## III. NOTATIONS AND PRELIMINARY RESULTS.

### 1. Jets.

If $r \in \widetilde{R}$ and $a \in \mathbb{A}^N$, we will distinguish between the (classical) $n$-jet of $r$ at $a$: $\mathfrak{J}_{n,a}\, r := \sum\limits_{0\leq k\leq n} \frac{1}{k!}\, D^k_a r \,.\, x^k$ and the centered $n$-jet of $r$ at $a$: $J_{n,a}\, r := \sum\limits_{1\leq k\leq n} \frac{1}{k!}\, D^k_a r \,.\, x^k$.
Of course, $D^k_a r$ denotes the $k$-th differential of $r$ at the point $a$ and we recall that $D^k_a r \,.\, x^k = \sum\limits_{\substack{\alpha\in\mathbb{N}^N \\ |\alpha|=k}} \binom{k}{\alpha} \frac{\partial^k r}{\partial x^\alpha}(a)\, x^\alpha$, where $\binom{k}{\alpha} = \frac{k!}{\alpha!} = \frac{k!}{\alpha_1!\ldots\alpha_N!}$.
The classical and centered $n$-jets are related by the formula $\mathfrak{J}_{n,a}\, r = r(a) + J_{n,a}\, r$.

By the same way, if $f \in \widetilde{E}$ and $a \in \mathbb{A}^N$, we will denote by

$$\mathfrak{J}_{n,a}\, f := \sum_{0\leq k\leq n} \frac{1}{k!}\, D^k_a f \,.\, x^k \qquad (\text{resp. } J_{n,a}\, f := \sum_{1\leq k\leq n} \frac{1}{k!}\, D^k_a f \,.\, x^k)$$

the classical (resp. centered) $n$-jet of $f$ at the point $a$. If $f = (f_1, \ldots, f_N)$, we could also have set $\mathfrak{J}_{n,a}\, f = \Big(\mathfrak{J}_{n,a}\, f_1, \ldots, \mathfrak{J}_{n,a}\, f_N\Big)$ and $J_{n,a}\, f = \Big(J_{n,a}\, f_1, \ldots, J_{n,a}\, f_N\Big)$.
If $a$ is the origin of the affine space $\mathbb{A}^N$, we will simply write $\mathfrak{J}_n$ (resp. $J_n$) instead of $\mathfrak{J}_{n,0}$ (resp. $J_{n,0}$). We will denote by $\mathfrak{J}_n(R)$ (resp. $J_n(R)$) the space of classical (resp. centered) $n$-jets of polynomials in $N$ indeterminates and by $\mathfrak{J}_n(E)$ (resp. $J_n(E)$) the space of classical (resp. centered) $n$-jets of polynomial endomorphisms of $\mathbb{A}^N$.
Observe that $\mathfrak{J}_n(R)$ (resp. $J_n(R)$, resp. $\mathfrak{J}_n(E)$, resp. $J_n(E)$) are naturally isomorphic to $R_{\leq n} := \bigoplus\limits_{k\leq n} R_k$ (resp. $R_{1\,\leq\,.\,\leq\,n} := \bigoplus\limits_{1\leq k\leq n} R_k$, resp. $E_{\leq n} := \bigoplus\limits_{k\leq n} E_k$, resp. $E_{1\,\leq\,.\,\leq\,n} := \bigoplus\limits_{1\leq k\leq n} E_k$).

**Remark.** The algebraic and analytic $n$-jet spaces are naturally isomorphic so that we will write $\mathfrak{J}_n(R) = \mathfrak{J}_n(\widetilde{R})$, $J_n(R) = J_n(\widetilde{R})$, $\mathfrak{J}_n(E) = \mathfrak{J}_n(\widetilde{E})$, $J_n(E) = J_n(\widetilde{E})$.

Furthermore, one could easily check that the Jacobian map $Jac : E \to R$ (or $Jac : \widetilde{E} \to \widetilde{R}$) induces a map $\mathfrak{J}_n(E) = \mathfrak{J}_n(\widetilde{E}) \to \mathfrak{J}_{n-1}(R) = \mathfrak{J}_{n-1}(\widetilde{R})$ that we will still call $Jac$. Therefore, the Jacobian of an endomorphism $n$-jet is naturally a $(n-1)$-jet.

Finally, let us recall that $J_n(E) = J_n(\widetilde{E})$ is naturally a semi-group and that the formula $D_a(f\circ g) = D_{g(a)}(f)\circ D_a(g)$ for differentials is generalized by the formula $J_{n,a}(f\circ g) =$

$J_{n,g(a)}(f) \circ J_{n,a}(g)$ for centered $n$-jets (the latter formula generalizes the former since $D_a(g)$ is identified with $J_{1,a}(g)$).

Let $J_n(E)^*$ be the group of invertible centered $n$-jets. We recall that $j \in J_n(E)$ is invertible if and only if $j_{(1)} \in GL \iff Jac\, j$ is an invertible element of $\mathfrak{J}_{n-1}(R) \iff (Jac\ j)(0) \neq 0$.

## 2. Monoidal subgroups.

Our first lemma is obtained by an easy computation.

**Lemma 3.1.** If $S$ is any sub-semigroup of $\mathbb{N}$, then $E_{1+S}$ is a sub-semigroup of $E$.

**Proof.** Since $E_{1+S} = (\mathbb{C}[X]_{1+S})^N$, it is enough to show that $\prod\limits_{1 \leq L \leq N} f_L^{\alpha_L} \in \mathbb{C}[X]_{1+S}$ for any $x^\alpha \in \mathbb{C}[X]_{1+S}$ and any $f_1, \ldots, f_N \in \mathbb{C}[X]_{1+S}$. By decomposing each $f_L$ as a sum of homogeneous polynomials, it is enough to show that $\prod\limits_{1 \leq j \leq r} g_j^{\beta_j} \in \mathbb{C}[X]_{1+S}$ for any homogeneous polynomials $g_1, \ldots, g_r \in \mathbb{C}[X]_{1+S}$ and for any $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^r$ such that $|\beta| = \sum\limits_j \beta_j \in 1 + S$. Let $s_j \in S$ be such that $\deg g_j = 1 + s_j$. We have

$\deg \prod\limits_j g_j^{\beta_j} = \sum\limits_j \beta_j(1 + s_j) = \sum\limits_j \beta_j + \sum\limits_j \beta_j s_j \in 1 + S + S \subset 1 + S.$ □

It is well known that any nonempty finite subset of a group which is stable by composition is a subgroup. Our second lemma is the generalization of this result for algebraic groups.

**Lemma 3.2.** Any nonempty closed subset of an algebraic group which is stable by composition is a subgroup.

**Proof.** Let $G$ be the algebraic group and $H$ the subset. For any $h \in H$, the map $m_h : H \to H$, $k \mapsto hk$ being an injective endomorphism, it is surjective (see prop. 17.9.6 p. 80 in [15] for the original idea, but the precise result is proven in [4], [7], [5], [8] or [24]), so that $1 \in H$ and $h^{-1} \in H$. □

Our last lemma is an obvious consequence of the first two.

**Lemma 3.3.** If $S$ is any sub-semigroup of $\mathbb{N}$, then $J_n(A_{1+S})$ is a subgroup of $J_n(A)$.

**Proposition 3.1.** If $S$ is any sub-semigroup of $\mathbb{N}$, then $A_{1+S}$ is a subgroup of $A$.

**Proof.** It easily follows from lemma 3.3. Indeed, if $f \in A_{1+S}$, we want to show that $f^{-1} \in A_{1+S}$. However, for any $n \geq 1$, $J_n(f) \in J_n(A_{1+S})$, so that $J_n(f^{-1}) = J_n(f)^{-1} \in J_n(A_{1+S})$. This is sufficient for showing that $f^{-1} \in A_{1+S}$. □

**Example.** If $S = \mathbb{N}$, then $A_{1+\mathbb{N}}$ is the group of automorphisms fixing the origin. If $S = 2\mathbb{N}$, then $A_{1+2\mathbb{N}}$ is the group of odd automorphisms, i.e. automorphisms $f$ satisfying $f(-x) = -f(x)$. More generally, if $n \geq 2$ and $\omega_n := e^{\frac{2\pi i}{n}}$, then $A_{1+n\mathbb{N}}$ is the group of automorphisms $f$ satisfying $f(\omega_n x) = \omega_n f(x)$.

## 3. A useful lemma in representation theory.

**Lemma 3.4.** Let $G$ be a connected reductive complex algebraic group and let $W$ be a finite dimensional $G$-reprensentation which does not contain the trivial representation. Then, any $G$-stable subgroup of $(W,+)$ is a vector subspace.

**Proof.** We will argue by induction on dim $W$. If dim $W = 0$, there is nothing to prove. Let us now assume that dim $W > 0$ and that $F$ is a $G$-stable subgroup of $W$. Of course, we may assume that $\mathrm{Span}(F) = W$. Let $\mathbb{T}$ be a maximal torus of $G$. If $\mathbb{T}^*$ is the character group of $\mathbb{T}$ (the set of algebraic group-morphisms $\chi : \mathbb{T} \to \mathbb{C}^*$) and $\mathbb{T}_{\mathbb{Q}} := \{t \in \mathbb{T},\ \forall\, \chi \in \mathbb{T}^*,\ \chi(t) \in \mathbb{Q}\}$ is the subset of rational points of $\mathbb{T}$, it is a classical fact that $\mathbb{T}^* \simeq \mathbb{Z}^m$ and $\mathbb{T}_{\mathbb{Q}} \simeq (\mathbb{Q}^*)^m$ (as groups). If $\chi \in \mathbb{T}^*$, $W_\chi := \{u \in W,\ \forall\, t \in \mathbb{T},\ t.u = \chi(t)\, u\}$ will denote the eigenspace of $W$ associated to the eigenvalue $\chi$. Since $W = \bigoplus_{\chi \in \mathbb{T}^*} W_\chi$, any $u \in W$ can be uniquely written $u = \sum_{\chi \in \mathbb{T}^*} u_\chi$, $u_\chi \in W_\chi$. By representation theory, $W$ is a non trivial $\mathbb{T}$-module. Hence, there exists a non trivial $\psi \in \mathbb{T}^*$ for which $W_\psi \neq \{0\}$.

<u>Main claim.</u> $F \cap W_\psi \neq \{0\}$.

Since $\mathrm{Span}(F) = W$, there exists $u \in F$ with $u_\psi \neq 0$. Let $u = \sum_{k=1}^{n} u_{\chi_k}$ be the decomposition of $u$ in sum of eigenvectors where $\chi_1, \ldots, \chi_n$ are distinct and $\chi_1 = \psi$. The maps $\chi_{k|\mathbb{T}_{\mathbb{Q}}}$, $1 \leq k \leq n$, are still distinct ($\mathbb{T}_{\mathbb{Q}}$ being a dense subset of $\mathbb{T}$). We now use the fact that if $G$ is any group and $K$ any field, then the set $\mathrm{Hom}(G, K^*)$ of all group-morphisms $G \to K^*$ is a linearly independant subset of $K^G$ the space of all $K$-valued functions on $G$ (see lemma 16.1 of [16]). Therefore, there exist $t_1, \ldots, t_n \in \mathbb{T}_{\mathbb{Q}}$ such that the $n \times n$ matrix $M := (\chi_k(t_l))_{1 \leq k,l \leq n}$ is invertible. Let $r = {}^t(r_1, \ldots, r_n) \in \mathbb{Q}^n$ be such that $M.r = {}^t(1, 0, \ldots, 0)$ and let $\mu$ be a nonzero integer such that $\mu r_1, \ldots, \mu r_n$ are integers (we can just take for $\mu$ the least common multiple of the denominators of the $r_k$). Let us check that $v := \mu \sum_{k=1}^{n} r_k\ t_k.u \in F \cap W_\psi$. Indeed $v = \sum_{k=1}^{n} (\mu r_k) t_k.u \in F$ and

$$v = \mu \sum_{k=1}^{n} r_k\ t_k.\left(\sum_{l=1}^{n} u_{\chi_l}\right) = \mu \sum_{k=1}^{n} r_k \sum_{l=1}^{n} \chi_l(t_k) u_{\chi_l} = \mu \sum_{l=1}^{n} \left(\sum_{k=1}^{n} r_k \chi_l(t_k)\right) u_{\chi_l}$$

$= \mu\ u_{\chi_1} = \mu\ u_\psi \in W_\psi \setminus \{0\}$ and the claim is proven.

Let us now show that $F$ contains the $G$-subrepresentation $W_1 := \underset{g \in G}{Span}\, g.v$.
We have $\forall\, t \in \mathbb{T}$, $t.v = \psi(t) v \in F$ and $\psi : \mathbb{T} \to \mathbb{C}^*$ is onto since non trivial. Therefore $\lambda v \in F$ for any $\lambda \in \mathbb{C}$. Any $w \in W_1$ can be written $w = \sum_{k=1}^{r} \lambda_k\ g_k.v$, where $\lambda_k \in \mathbb{C}$, $g_k \in G$. The equality $w = \sum_{k=1}^{r} g_k.(\lambda_k v)$ shows us that $w \in F$.

If $W_2$ is a $G$-subrepresentation of $W$ such that $W = W_1 \oplus W_2$, it is clear that $W_2$ does not contain the trivial $G$-representation and that $W_2 \cap F$ is a $G$-stable subgroup of $W_2$. Therefore, by induction hypothesis, it is a subspace of $W_2$. It is easy to show that the subspace $W_1 \oplus (W_2 \cap F)$ is equal to $F$. $\square$

**Corollary 3.1.** If $m \geq 2$, the $SL$-stable subgroups of $E_m$ are $\{0\}, E_m, E_m^0$ and $E_m^1$.

**Proof.** The $SL$-representations $E_m^0$ and $E_m^1$ are irreducible, non trivial and non isomorphic. $\square$

## 4. Initial groups.

Let $\mathbb{N}_{<n} := \{0, 1, \dots, n-1\}$. If $G \leq J_n(E)^*$ and $k \in \mathbb{N}_{<n}$, we define the $k$-th initial group of $G$ by $H_k(G) := \{f_{(k+1)},\ f \in G,\ J_k f = J_k(\mathrm{id})\}$. It turns out that $H_0(G) = J_1(G)$ is a multiplicative subgroup of $GL$, whereas $H_k(G)$ is an additive subgroup of $E_{k+1}$ for $k \geq 1$. Indeed, if $u_2, u_2 \in E_{k+1}$ and $j_m = \mathrm{id} + u_m \in J_{k+1}(E)^*$ for $m = 1, 2$, then $j_1 \circ j_2^{\pm 1} = \mathrm{id} + u_1 \pm u_2$.
Furthermore, let $f, g \in \widetilde{A}$ be such that $J_m f = \mathrm{id} + u$, $u \in E_m$ and $J_n g = \mathrm{id} + v$, $v \in E_n$. If $[f, g] := f \circ g \circ f^{-1} \circ g^{-1} \in \widetilde{A}$, it is shown in [3] that $J_{m+n-1}\,[f, g] = \mathrm{id} - [u, v]$. Therefore, we get:

**Lemma 3.5.** If $k, l > 0$ are such that $k + l < n$, then $[H_k(G), H_l(G)] \subset H_{k+l}(G)$.

These $H_k(G)$ look like the initial ideals used in Gröbner bases theory. They satisfy an analogous fundamental property (see lemma 15.5 in [10]):

**Lemma 3.6.** If $G_1 \leq G_2 \leq J_n(E)^*$, then $G_1 = G_2 \iff \forall\, k \in \mathbb{N}_{<n},\ H_k(G_1) = H_k(G_2)$ .

**Proof.** If $G_1 \neq G_2$, let $k$ be the biggest integer such that there exists $f \in G_2 \setminus G_1$ with $J_k f = J_k(\mathrm{id})$. Since $H_k(G_1) = H_k(G_2)$, we may write $J_{k+1} f = J_{k+1} g$ with $g \in G_1$. But then $f \circ g^{-1} \in G_2 \setminus G_1$ and $J_{k+1}\, f \circ g^{-1} = J_{k+1}(\mathrm{id})$; a contradiction. $\square$

In the sequel, we will always assume that $G$ is $SL$-invariant. Therefore, $H_k(G)$ is $SL$-invariant too and using cor. 3.1, we get $H_k(G) = \{0\}$, $E_{k+1}$, $E_{k+1}^0$ or $E_{k+1}^1$ for $k \geq 1$. This incites us to set $H_k^l(G) := H_k(G) \cap E_{k+1}^l$ for $l = 0, 1$ and $k \in \mathbb{N}_{<n}$, $k \geq 1$. It is clear that $H_k^l(G) = \{0\}$ or $E_{k+1}^l$ and that $H_k(G) = \underset{l=0,1}{\oplus} H_k^l(G)$. Therefore, the $H_k(G)$ for $k \geq 1$ are encoded by the sets $\mathcal{I}_l(G) := \{0\} \cup \{k \in \mathbb{N}_{<n},\ k \geq 1,\ H_k^l(G) \neq \{0\}\}$. Lemma 3.5 and th. 2.1 imply the following result:

**Lemma 3.7.** If $k, l \geq 0$ are such that $k + l < n$, then $k, l \in \mathcal{I}_0(G) \Longrightarrow k + l \in \mathcal{I}_0(G)$ and $k \in \mathcal{I}_0(G)$, $l \in \mathcal{I}_1(G) \Longrightarrow k + l \in \mathcal{I}_1(G)$.

**Corollary.** If $G \leq J_n(E)^*$ is $SL$-invariant, then:
(i) $\mathcal{I}_0(G) = \mathbb{N}_{<n} \iff 1 \in \mathcal{I}_0(G)$;
(ii) $\mathcal{I}_0(G) = \mathcal{I}_1(G) = \mathbb{N}_{<n} \iff 1 \in \mathcal{I}_0(G) \cap \mathcal{I}_1(G)$.

If $I \subset \mathbb{N}$, $< I >$ will denote the sub-semigroup of $\mathbb{N}$ generated by $I$. Let $\mathcal{T}$ be the set of subsets $I$ of $\mathbb{N}_{<n}$ which are the traces of sub-semigroups of $\mathbb{N}$, i.e. such that $I =< I > \cap \mathbb{N}_{<n}$. Lemma 3.7 shows us that $\mathcal{I}_0(G) \in \mathcal{T}$.

**Proposition 3.2.** $J_n(A) = J_n(T) = \{f \in J_n(E), Jac\, f \in \mathbb{C}^*\}$.

**Proof.** If $G_1 := J_n(A)$, $G_2 = J_n(T)$ and $G_3 := \{f \in J_n(E), Jac\, f \in \mathbb{C}^*\}$, it is clear that $G_2 \leq G_1 \leq G_3$, so it is enough to show that $G_2 = G_3$. If $k \in \mathbb{N}_{<n}$, $k \geq 1$ and $u \in H_k(G_3) \subset E_{k+1}$, then $f := \mathrm{id} + u \in J_{k+1}(E)$ satisfies $Jac\, f = 1$. However, $Jac\, f = 1 + \nabla u$, so that $\nabla u = 0$, $u \in E^0_{k+1}$ and $H^1_k(G_3) = \{0\}$. This shows that $\mathcal{I}_1(G_3) = \{0\}$. But $\mathrm{id} + x_2^2 e_1 \in G_2$, so that $x_2^2 e_1 \in H^0_1(G_2)$, $1 \in \mathcal{I}_0(G_2)$ and $\mathcal{I}_0(G_2) = \mathbb{N}_{<n}$. Finally, it is clear that $\mathcal{I}_0(G_2) = \mathcal{I}_0(G_3) = \mathbb{N}_{<n}$, that $\mathcal{I}_1(G_2) = \mathcal{I}_1(G_3) = \{0\}$ and that $H_0(G_2) = H_0(G_3) = GL$, so that $G_2 = G_3$ by lemma 3.6. $\square$

In some sense, at the level of $n$-jets, the equality $J_n(A) = \{f \in J_n(E), Jac\, f \in \mathbb{C}^*\}$ solves the Jacobian problem (see [19], [6] and [11]) and the equality $J_n(A) = J_n(T)$ solves the tameness problem for algebraic automorphisms (see [18], [20], [21], [25] and [26]).

**Proposition 3.3.** $J_n(\widetilde{A}) = J_n(\widetilde{T}) = J_n(E)^*$.

**Proof.** If $G_1 := J_n(\widetilde{A})$, $G_2 = J_n(\widetilde{T})$ and $G_3 := J_n(E)^*$, it is clear that $G_2 \leq G_1 \leq G_3$, so it is enough to show that $G_2 = G_3$. Since $f := (e^{x_2} x_1, x_2, \ldots, x_N) \in \widetilde{T}$, we get $x_1 x_2\, e_1 \in H_1(G_2)$, so that $H_1(G_2) = E_2$, $1 \in \mathcal{I}_0(G_2) \cap \mathcal{I}_1(G_2)$ and $\mathcal{I}_0(G_2) = \mathcal{I}_1(G_2) = \mathbb{N}_{<n}$. Finally, it is clear that $\mathcal{I}_0(G_2) = \mathcal{I}_0(G_3) = \mathbb{N}_{<n}$, that $\mathcal{I}_1(G_2) = \mathcal{I}_1(G_3) = \mathbb{N}_{<n}$ and that $H_0(G_2) = H_0(G_3) = GL$, so that $G_2 = G_3$ by lemma 3.6. $\square$

## IV. PROOFS OF THEOREMS A,B,C.

Let $J_n(A)_{id} := \{f \in J_n(A),\ J_1 f = \mathrm{id}\}$ and let $\mathcal{S}$ be the set of subgroups of $J_n(A)_{id}$ which are $SL$-invariant. If $I \in \mathcal{T}$, we set $\mathcal{G}(I) := J_n(A_{1+<I>}) \cap J_n(A)_{id}$.

**Lemma 4.1.** (i) If $G_1 \leq G_2$ belong to $\mathcal{S}$, then $G_1 = G_2 \iff \mathcal{I}_0(G_1) = \mathcal{I}_0(G_2)$.
(ii) If $I \in \mathcal{T}$, then $\mathcal{G}(I) \in \mathcal{S}$ and $\mathcal{G}(I)$ is the subgroup generated by the $g \circ f^{[k]} \circ g^{-1}$, $g \in SL$, $k \in I$, where $f^{[k]} := \mathrm{id} + x_2^{k+1} e_1$. If $J \subset \mathbb{N}_{<n}$ satisfies $I =< J > \cap \mathbb{N}_{<n}$, then $\mathcal{G}(I)$ is also generated by the $g \circ f^{[k]} \circ g^{-1}$, $g \in SL$, $k \in J$.

**Proof.** (i). Since $H_0(G) = \{\mathrm{id}\}$ and $\mathcal{I}_1(G) = \{0\}$ for $G \in \mathcal{S}$, it is a direct consequence of lemma 3.6.
(ii). The fact that $\mathcal{G}(I) \in \mathcal{S}$ is obvious. Let $G_1 :=< g \circ f^{[k]} \circ g^{-1}, g \in SL, k \in J >$, $G_2 := \mathcal{G}(I)$ and let us show that $G_1 = G_2$ by applying the last point.
We clearly have $G_1 \leq G_2$ and $\mathcal{I}_0(G_2) = I$. It remains to show that $\mathcal{I}_0(G_1) = I$. The relation $G_1 \leq G_2$ implies $\mathcal{I}_0(G_1) \subset \mathcal{I}_0(G_2) = I$. On the converse, since $J \subset \mathcal{I}_0(G_1)$ and $\mathcal{I}_0(G_1) \in \mathcal{T}$, we have $< J > \cap \mathbb{N}_{<n} = I \subset \mathcal{I}_0(G_1)$. $\square$

If $I \in \mathcal{T}$, it is clear that $\mathcal{I}_0(\mathcal{G}(I)) = I$. It turns out that if $G \in \mathcal{S}$, the equality $\mathcal{G}(\mathcal{I}_0(G)) = G$ is also true, but does not look so clear for us. Indeed, if $I = \mathcal{I}_0(G)$, $G_1 = G$ and

$G_2 = \mathcal{G}(\mathcal{I}_0(G))$, it is clear that $G_1, G_2 \in \mathcal{S}$ and that $\mathcal{I}_0(G_1) = \mathcal{I}_0(G_2) = I$. Unfortunately, we cannot apply right now point (i) of lemma 4.1, since we do not know yet that $G_1 \leq G_2$ or $G_2 \leq G_1$.

**Theorem 4.1.** The map $\mathcal{I}_0 : \mathcal{S} \to \mathcal{T}$, $G \mapsto \mathcal{I}_0(G)$ is bijective with inverse the map $I \mapsto \mathcal{G}(I)$.

**Proof.** The main point is to show that for any $G \in \mathcal{S}$ we have $G = \mathcal{G}(\mathcal{I}_0(G))$. If we set $I = \mathcal{I}_0(G)$, it is sufficient to show that $G = \mathcal{G}(I)$, i.e. (by (ii) of lemma 4.1) $G =< g \circ f^{[k]} \circ g^{-1},\, g \in SL,\, k \in I >$. We argue by induction on $n$.
If $n = 1$, it is obvious. If $n \geq 2$ and if $J = \mathcal{I}_0(J_{n-1}G)$, then by induction hypothesis we have $J_{n-1}G =< g \circ f^{[k]} \circ g^{-1},\, g \in SL,\, k \in J >$.
For each such $k$, there exists $u^{[k]} \in G$ such that $J_{n-1}f^{[k]} = J_{n-1}u^{[k]}$. Therefore, if we set $h^{[k]} := (f^{[k]})^{-1} \circ u^{[k]} \in J_n(A)$, then $J_{n-1}h^{[k]} = \mathrm{id}$ and $f^{[k]} \circ h^{[k]} \in G$.

First case. $n-1 \in I$, i.e. $I = J \cup \{n-1\}$.
This implies $H_{n-1}(G) = E_n^0$, so that for any $u \in E_n^0$, $\mathrm{id} + u \in G$. Therefore, $f^{[n-1]} \in G$ and $\forall\, k \in J$, $f^{[k]}$ and $h^{[k]} \in G$. It is clear that $G = \mathcal{G}(I)$.

Second case. $n-1 \notin I$, i.e. $I = J$.
This implies $H_{n-1}(G) = \{0\}$ and $n-1 \notin< J >$. It is enough to show that $h^{[k]} = \mathrm{id}$ for each $k \in J$. This comes from the next assertion. First and foremost, let us recall that $\mathcal{G}(J)_{\mathbb{Q}}$ denotes the set of elements belonging to $\mathcal{G}(J)$ whose coordinates belong to $\mathbb{Q}$ and that $\mathcal{G}(\{0, n-1\})$ denotes the subgroup of $J_n(A)_{id}$ whose elements are of the shape $h = \mathrm{id} + h_{(n)}$ where $h_{(n)} \in E_n^0$. It follows that $\mathcal{G}(\{0, n-1\})$ is included into the center of $J_n(A)_{id}$. Indeed, if $f \in J_n(A)_{id}$ and $h = \mathrm{id} + h_{(n)} \in \mathcal{G}(\{0, n-1\})$, $f \circ h = h \circ f = f + h_{(n)}$. In particular, the groups $\mathcal{G}(J)$ and $\mathcal{G}(\{0, n-1\})$ commute.

Assertion. $\forall\, f \in \mathcal{G}(J)_{\mathbb{Q}}$, $\forall\, h \in \mathcal{G}(\{0, n-1\})$, $f \circ h \in G \Longrightarrow h = \mathrm{id}$.
If the assertion is false, let $k$ be the biggest integer such that there exists a counterexample $(f, h)$ with $J_k f = \mathrm{id}$. Since $h_{(n)} \neq 0$ and since $E_n^0$ is an irreducible $SL$-module, there exist $g_1, \ldots, g_r \in SL_{\mathbb{Q}}$ such that the $g_i \circ h_{(n)} \circ g_i^{-1}$, $1 \leq i \leq r$ constitue a $\mathbb{C}$-basis of $E_n^0$. Indeed, if $W$ is an irreducible $SL$-module of dimension $r$ and if $w \in W$ is nonzero, there exist $g_1, \ldots, g_r \in SL_{\mathbb{Q}}$ such that the $g_i.w$, $1 \leq i \leq r$ constitute a $\mathbb{C}$-basis of $W$: the map $\varphi : (SL)^r \to \bigwedge^d W$, $(g_i)_{1 \leq i \leq r} \mapsto \bigwedge_i g_i.w$ being nonzero, it has to be nonzero on $(SL_{\mathbb{Q}})^r$ since $SL_{\mathbb{Q}}$ is (Zariski) dense in $SL$. However, $\dim E_{k+1}^0 < \dim E_n^0$, so that the $g_i \circ f_{(k+1)} \circ g_i^{-1}$, $1 \leq i \leq r$, are $\mathbb{C}$-linearly dependant. These last elements belonging to $\left(E_{k+1}^0\right)_{\mathbb{Q}}$, they are even $\mathbb{Q}$-linearly dependant, showing the existence of integers $m_i$, non all zero, such that $\sum_{1 \leq i \leq r} m_i \;\; g_i \circ f_{(k+1)} \circ g_i^{-1} = 0$. If $a_1, \ldots, a_r \in J_n(A)$, let us agree that $\prod_{i=1}^{r} a_i$ denotes the composition $a_1 \circ a_2 \circ \ldots \circ a_r$ in that order. If we set

$$\widetilde{f} := \prod_{i=1}^{r} g_i \circ f^{m_i} \circ g_i^{-1} \in \mathcal{G}(J)_{\mathbb{Q}} \text{ and } \widetilde{h} := \prod_{i=1}^{r} g_i \circ h^{m_i} \circ g_i^{-1} \in \mathcal{G}(\{0, n-1\}), \text{ then}$$

- $\widetilde{f} \circ \widetilde{h} = \prod_{i=1}^{r} g_i \circ (f \circ h)^{m_i} \circ g_i^{-1} \in G;$

- $(\widetilde{f})_{(k+1)} = \sum_{i=1}^{r} m_i \, g_i \circ f_{(k+1)} \circ g_i^{-1} = 0$, so that $J_{k+1}\widetilde{f} = \mathrm{id}$;
- $(\widetilde{h})_{(n)} = \sum_{i=1}^{r} m_i \, g_i \circ h_{(n)} \circ g_i^{-1} \neq 0$, so that $\widetilde{h} \neq \mathrm{id}$.

This is a contradiction. □

**Corollary 4.1.** If $G \in \mathcal{S}$, then $\mathcal{I}_0(G) = \{k \in \mathbb{N}_{<n}, \ \exists\, f \in G, \ f_{(k+1)} \neq 0\}$.

**Remark.** We could give a more simple proof of the theorem using cor. 4.1. Unfortunately, we were not able to prove it without using the theorem.

**Theorem 4.2.** Any group $G$ such that $SL \leq G \leq J_n(A)$ is equal to some $\mathcal{G}(I) \rtimes K$, where $SL \leq K \leq GL$.

**Proof.** It is sufficient to show that $H_0(G) \leq G$. If $l \in H_0(G)$, let us show by contradiction that $n$ is the biggest integer $k$ for which there exists some $f \in G$ satisfying $J_k f = l$.
If we had $k < n$, then $f = l + f_{(k+1)} + \ldots$ where $f_{(k+1)} \neq 0$.
We begin to show that $H_k(G \cap J_n(A)_{id}) \neq \{0\}$.
Since $f = l \circ (\mathrm{id} + l^{-1} \circ f_{(k+1)} + \ldots)$, $a := l^{-1} \circ f_{(k+1)} \in E^0_{k+1}$ and since $a$ is a nonzero element of the irreducible non trivial $SL$ representation $E^0_{k+1}$, there exists $u \in SL$ such that $a \neq u \circ a \circ u^{-1}$.
If $g := f \circ u \circ f^{-1}$, one can check that $g_{(k+1)} = l \circ (a - u \circ a \circ u^{-1}) \circ u \circ l^{-1} \neq 0$, while $g \in G$ and $Jac\, g = 1$. Therefore $h := g_{(1)}^{-1} \circ g \in G \cap J_n(A)_{id}$ and $h_{(k+1)} \neq 0$, so that $H_k(G \cap J_n(A)_{id}) \neq \{0\}$.
Since $H_k(G \cap J_n(A)_{id}) = E^0_{k+1}$, there exists $\widetilde{h} \in G \cap J_n(A)_{id}$ such that $J_{k+1}\widetilde{h} = \mathrm{id} - a$.
Therefore $\widetilde{f} := f \circ \widetilde{h} \in G$ and $J_{k+1}\widetilde{f} = l$, a contradiction. □

**Remark.** We recall that any group $K$ such that $SL \leq K \leq GL$ is equal to some $\det^{-1} \widetilde{K}$ where $\det : GL \to \mathbb{C}^*$ and $\widetilde{K} \leq \mathbb{C}^*$.

**Corollary 4.2.** Any group $G$ such that $GL \leq G \leq J_n(A)$ is equal to some $\mathcal{G}(I) \rtimes GL$, i.e. $J_n(A_{1+<I>})$.

**Corollary 4.3.** Any group $G$ such that $SL \leq G \leq J_n(A)_1$ is equal to some $\mathcal{G}(I) \rtimes SL$, i.e. $J_n(A_{1+<I>}) \cap J_n(A)_1$.

**Corollary 4.4.** If $n \geq 2$ and $j \in J_n(A)_1$, the following assertions are equivalent:
(i) $< SL, j >= J_n(A)_1$;
(ii) $j_{(2)} \neq 0$.

**Proof.** If we set $G := < SL, j >$, then $G = J_n(A)_1$ if and only if $\mathcal{I}_0(G) = \mathbb{N}_{<n}$, which is still equivalent to $1 \in \mathcal{I}_0(G)$, i.e. $H_1(G) \neq \{0\}$, i.e. $j_{(2)} \neq 0$. □

**Proof of th. B.** Assume that $J_n(A) \leq G \leq J_n(\widetilde{A})$. Since $1 \in \mathcal{I}_0(G)$, lemma 3.7 implies that $\mathcal{I}_1(G)$ is equal to some $\{0\} \cup \{k, k+1, \ldots, n-1\}$ where $1 \leq k \leq n$.

Applying lemma 3.6 with $G_1 :=< J_n(A), g^{[k]} >$, where $g^{[k]} := (1 + x_1^k)\mathrm{id} \in J_n(\widetilde{A})$, and $G_2 := J_{n,k}^{-1}(J_k(A))$, one could show as in lemma 4.1 that $J_{n,k}^{-1}(J_k(A)) =< J_n(A), g^{[k]} >$. Remark: if $k = n$, we agree that $g^{[n]} = \mathrm{id}$. Using these preliminaries, let us show that $G =< J_n(A), g^{[k]} >$. As above, the proof is by induction on $n$. The case $n = 1$ being obvious, we may assume that $n \geq 2$.

First case. $k < n$, i.e. $n - 1 \in \mathcal{I}_1(G)$.
By induction hypothesis, the groups $G$ and $H :=< J_n(A), g^{[k]} >$ coincide at the level of $n-1$ jets, i.e. $J_{n-1}G = J_{n-1}H$. However, since $G$ and $H$ both contain the group $\{\mathrm{id} + u,\ u \in E_n\} \leq J_n(\widetilde{A})$, it is clear that the last equality can be lifted up at the level of $n$-jets to show that $G = H$.

Second case. $k = n$, i.e. $\mathcal{I}_1(G) = \{0\}$.
Since $J_n(A) \leq G$ and $\mathcal{I}_1(J_n(A)) = \mathcal{I}_l(G) = \{0\}$, we get $J_n(A) = G$ by lemma 3.6. □

**Corollary 4.5.** If $n \geq 2$ and $j \in J_n(E)^*$, the following assertions are equivalent:
(i) $< J_n(A), j >= J_n(E)^*$;
(ii) $J_2\, j \notin J_2(A)$.

**Proof.** If we set $G :=< J_n(A), j >$, then $G = J_n(E)^*$ if and only if $\mathcal{I}_1(G) = \mathbb{N}_{<n}$, which is still equivalent to $1 \in \mathcal{I}_1(G)$, i.e. $H_1^1(G) \neq \{0\}$, i.e. $J_2\, j \notin J_2(A)$. □

**Proof of th. C.** Let $f \in A \backslash GA$. Let us show that there exists $a \in \mathbb{A}^N$ such that if we set $g := f \circ \tau_a$ (where $\tau_a = \mathrm{id} + a$ is the translation of vector $a$), then the quadratic part $g_{(2)}$ of $g$ is nonzero. Since $f = (f_1, \ldots, f_N)$ is not affine, there exists a component $f_L$ of $f$ such that deg $f_L \geq 2$. Therefore, it is sufficient to show that if $p \in R = \mathbb{C}[x_1, \ldots, x_N]$ satisfies deg $p \geq 2$, then there exists $a \in \mathbb{A}^N = \mathbb{C}^N$ such that $q(x) := p(a + x)$ satisfies $q_{(2)} \neq 0$. But it is clear that there exist integers $L, M$ such that $\dfrac{\partial^2 p}{\partial x_L \partial x_M}$ is a nonzero polynomial. Therefore, there exists $a \in \mathbb{C}^N$ such that $\dfrac{\partial^2 p}{\partial x_L \partial x_M}(a) \neq 0$. By Taylor formula, we have $q(x) = p(a + x) = \displaystyle\sum_{\alpha \in \mathbb{N}^N} \frac{\partial^\alpha p}{\partial x^\alpha}(a)\frac{x^\alpha}{\alpha!}$ so that $q_{(2)} \neq 0$. By replacing $g$ (where $g \in G$ satisfies $g_{(2)} \neq 0$) by $h \circ g$ (where $h$ is a well chosen affine map), we may assume moreover that $g(0) = 0$ and that $Jac\ g = 1$. Now, by cor. 4.5, we have $< SL, J_n(g) >= J_n(A)_1$ and it is clear that $\mathfrak{J}_n(G) = \mathfrak{J}_n(A)$. □

## V. PROOF OF THEOREM D.

### 1. The Algebraic case.

We have seen in prop. 3.2 above that for any $j \in \mathfrak{J}_n(E)$ whose Jacobian is a nonzero constant there exists a tame automorphism $f$ such that $j = \mathfrak{J}_n(f)$. The following generalization is equivalent to the algebraic case of th. D:

**Theorem 5.1 (interpolation of n-jets by an algebraic tame automorphism).**

Let $n \geq 1$, let $u^{[1]}, \dots, u^{[m]}$ be distinct points of $\mathbb{A}^N$ and let $j^{[1]}, \dots, j^{[m]} \in \mathfrak{J}_n(E)$ be $n$-jets whose Jacobians are nonzero constants. The two following assertions are equivalent:

(i) $\exists\, f \in T,\ \mathfrak{J}_{n,u^{[k]}}\, f = j^{[k]},\ 1 \leq k \leq m;$ $\quad$ (ii) $\begin{cases} 1.\ \text{the points } j^{[k]}(0)_{1\leq k\leq m} \text{ are distinct;} \\ 2.\ \exists\, \lambda \in \mathbb{C}^*,\ Jac\, j^{[k]} = \lambda,\ 1 \leq k \leq m. \end{cases}$

**Proof.** (i) $\Longrightarrow$ (ii). We have $f(u^{[k]}) = j^{[k]}(0)$, so that (1) comes from the injectivity of $f$. Since $f$ is a polynomial automorphism, $Jac\ f \equiv \lambda \in \mathbb{C}^*$, i.e. $\forall\ a \in \mathbb{A}^N,\ \det\ f'(a) = \lambda$ and we get $Jac\ j^{[k]} = \det\ (j^{[k]})'(0) = \det\ f'(u^{[k]}) = \lambda$.

(ii) $\Longrightarrow$ (i). It is enough to prove that given: $u^{[1]}, \dots, u^{[m]}$ distinct points of $\mathbb{A}^N$; $v^{[1]}, \dots, v^{[m]}$ distinct points of $\mathbb{A}^N$; $\lambda \in \mathbb{C}^*$; $j^{[1]}, \dots, j^{[m]}$ centered $n$-jets of $J_n(E)$ such that $Jac\ j^{[k]} = \lambda$ (for $1 \leq k \leq m$) there exists $f \in T$ such that $f(u^{[k]}) = v^{[k]}$ and $J_{n,u^{[k]}}\, f = j^{[k]}$ (for $1 \leq k \leq m$).

Let $G$ be the group of tame automorphisms $f$ such that $f(u^{[k]}) = u^{[k]}$ (for $1 \leq k \leq m$) and such that $Jac\ f = 1$ and let $J := J_n(A)_1$. Using lemma 5.1 below, it is sufficient to show that the group-morphism $\varphi\ :\ G \to J^m,\ f \mapsto \big(J_{n,u^{[k]}}\, f\big)_{1\leq k\leq m}$ is onto. This is a direct consequence of lemma 5.2 below. $\square$

**Lemma 5.1.** If $u^{[1]}, \dots, u^{[m]}$ and $v^{[1]}, \dots, v^{[m]}$ are two families of $m$ pairwise distinct points of $\mathbb{A}^N$ and if $\lambda \in \mathbb{C}^*$, then there exists a tame automorphism $f$ with Jacobian equal to $\lambda$ such that $f(u^{[k]}) = v^{[k]}$ for $1 \leq k \leq m$.

**Proof.** It is proven as a watermark in [17] that $T$ acts $m$-transitively on $\mathbb{A}^N$. It is also a consequence of th. 2 of [27] asserting that if $X_1$, $X_2$ are smooth closed algebraic subsets of $\mathbb{A}^N$ of dimension $d$ with $N \geq 2d+2$, then any isomorphism from $X_1$ to $X_2$ can be extended into a tame automorphism of $\mathbb{A}^N$ (see also § 5.3 of [11] for an overview). Therefore, if we set $w^{[k]} := k\, e_N \in \mathbb{A}^N$ (for $1 \leq k \leq m$), there exist $g, h \in T$ such that $g(u^{[k]}) = w^{[k]}$ and $h(w^{[k]}) = v^{[k]}$ (for $1 \leq k \leq m$). If we set $\mu := \lambda/(Jac\ g\ \times\ Jac\ h) \in \mathbb{C}^*$ and $d_\mu := (\mu x_1, x_2, \dots, x_N) \in T$, then $f := h \circ d_\mu \circ g$ satisfies the required conditions. $\square$

**Lemma 5.2.** If $u^{[0]}, \dots, u^{[m]}$ are $m+1$ pairwise distinct points of $\mathbb{A}^N$, let $G_0$ be the group of tame automorphisms $f$ satisfying $f(u^{[k]}) = u^{[k]}$ for $0 \leq k \leq m$, $J_{n,u^{[k]}}\, f = \mathrm{id}$ for $1 \leq k \leq m$ and $Jac\ f = 1$. As above, let $J := J_n(A)_1$ be the group of centered $n$-jets of automorphisms with Jacobian equal to 1. Then, the group-morphism $\psi\ :\ G_0 \to J$, $f \mapsto J_{n,u^{[0]}}\, f$ is onto.

**Proof.** Let us set $u = (\underbrace{1, \dots, 1}_{N}) \in \mathbb{A}^N$. Since there exists a tame automorphism sending $u^{[k]}$ on $k\, u$ (for $0 \leq k \leq m$), we may assume that $u^{[k]} = k\, u$ (for $0 \leq k \leq m$). Using cor. 4.4, it is sufficient to show that: (i) $\mathrm{id} + x_2^2\, e_1 \in \mathrm{Im}\ \psi$ $\quad$ and $\quad$ (ii) $SL \subset \mathrm{Im}\ \psi$.

Proof of (i). Let $p(\xi) \in \mathbb{C}[\xi]$ be such that $p(\xi) \equiv \xi^2 \bmod \xi^{n+1}$ and $p(k+\xi) \equiv 0 \bmod \xi^{n+1}$, $1 \leq k \leq m$. Then $f := \mathrm{id} + p(x_2)e_1 \in G_0$ and $\psi(f) = J_n\, f = \mathrm{id} + x_2^2 e_1$.

Proof of (ii). We know that $SL$ is generated by the elementary transvections $t_{\alpha,L,M} := \mathrm{id} + \alpha\, x_M\, e_L$ (where $\alpha \in \mathbb{C}$ and $L \neq M \in \{1, \dots, N\}$). It is enough to show that $t_{\alpha,L,M} \in \mathrm{Im}\ \psi$.

Let $p(\xi) \in \mathbb{C}[\xi]$ be such that $p(\xi) \equiv \alpha\xi \bmod \xi^{n+1}$ and $p(k+\xi) \equiv 0 \bmod \xi^{n+1}$, $1 \le k \le m$. Then $f := \mathrm{id} + p(x_M)e_L \in G_0$ and $\psi(f) = J_n\, f = t_{\alpha,L,M}$. $\square$

**2. The Analytic case.**

We have seen in prop. 3.3 above that for any $j \in J_n(E)^*$, there exists a tame analytic automorphism $f$ such that $j = J_n\, f$. The following generalization is equivalent to the analytic case of th. D:

**Theorem 5.2 (interpolation of n-jets by an analytic tame automorphism).** Let $n \ge 1$, let $u^{[1]}, \dots, u^{[m]}$ be distinct points of $\mathbb{A}^N$, let $v^{[1]}, \dots, v^{[m]}$ be points of $\mathbb{A}^N$ and let $j^{[1]}, \dots, j^{[m]} \in J_n(E)^*$ be invertible centered $n$-jets. The two following assertions are equivalent:

(i) $\exists\, f \in \widetilde{T},\ \mathfrak{J}_{n,u^{[k]}}\, f = v^{[k]} + j^{[k]},\ 1 \le k \le m$; (ii) the points $\big(v^{[k]}\big)_{1\le k\le m}$ are distinct.

**Proof.** We follow the same path as in the algebraic case. The implication (i) $\Longrightarrow$ (ii) is obvious and (ii) $\Longrightarrow$ (i) is a consequence of the following lemma. $\square$

**Lemma 5.3.** If $u^{[0]}, \dots, u^{[m]}$ are $m+1$ distinct points of $\mathbb{A}^N$, let $\widetilde{G}$ be the group of tame analytic automorphisms $f$ such that $f(u^{[k]}) = u^{[k]}$, $0 \le k \le m$ and $J_{n,u^{[k]}}\, f = \mathrm{id}$, $1 \le k \le m$. Let $\widetilde{J} := J_n(E)^*$ be the group of invertible centered $n$-jets. Then, the group-morphism $\widetilde{\psi} :\ \widetilde{G} \to \widetilde{J},\ f \mapsto J_{n,u^{[0]}}\, f$ is onto.

**Proof.** We may assume that $u^{[k]} = k\,u$ $(0 \le k \le m)$ where $u = (\underbrace{1,\dots,1}_{N}) \in \mathbb{A}^N$. Using cor. 4.5, it is enough to show that: (i) $J_n(A) \subset \mathrm{Im}\ \widetilde{\psi}$ and (ii) $\mathrm{id} + x_1x_2\, e_1 \in \mathrm{Im}\ \widetilde{\psi}$.

Proof of (i). We already know that $J_n(A)_1 \subset \psi(G) \subset \widetilde{\psi}(\widetilde{G})$. Therefore, it is sufficient to show that for any $\lambda \in \mathbb{C}^*$, $d_\lambda := (\lambda x_1, x_2, \dots, x_N) \in \mathrm{Im}\ \widetilde{\psi}$. Let us choose $\mu \in \mathbb{C}$ such that $e^\mu = \lambda$ and let us choose $p(\xi) \in \mathbb{C}[\xi]$ such that $p(\xi) \equiv \mu \bmod \xi^{n+1}$ and $p(k+\xi) \equiv 0 \bmod \xi^{n+1}$, $1 \le k \le m$. Then $f := (e^{p(x_2)}x_1, x_2, \dots, x_N) \in \widetilde{G}$ and $\widetilde{\psi}(f) = J_n\, f = d_\lambda$.

Proof of (ii). Let $p(\xi) \in \mathbb{C}[\xi]$ be such that $p(\xi) \equiv \ln(1+\xi) \bmod \xi^{n+1}$ or equivalently $p(\xi) \equiv \displaystyle\sum_{1\le k\le n} (-1)^{k+1}\frac{\xi^k}{k} \bmod \xi^{n+1}$ and $p(k+\xi) \equiv 0 \bmod \xi^{n+1}$, $1 \le k \le m$. Then $f := (e^{p(x_2)}x_1, x_2, \dots, x_N) \in \widetilde{G}$ and $\widetilde{\psi}(f) = J_n\, f = \mathrm{id} + x_1x_2\, e_1$. $\square$

## VI. CONSEQUENCES ON VARIABLES.

We recall that $f_1 \in R$ is called a variable, if there exist $f_2, \dots, f_N \in R$ such that $(f_1, \dots, f_N)$ is an algebraic automorphism.

**Theorem 6.1.** If $n \ge 1$ and $j_L \in \mathfrak{J}_n(R)$ for $1 \le L \le N$, the following assertions are equivalent:

(i) the linear parts $\mathcal{L}(j_L)$ of the $j_L$, $1 \leq L \leq N-1$ are linearly independant;
(ii) there exists $j_N \in \mathfrak{J}_n(R)$ such that $(j_1, \ldots, j_N) \in \mathfrak{J}_n(A)$;
(iii) there exists $j_N \in \mathfrak{J}_n(R)$ such that $Jac(j_1, \ldots, j_N) = 1 \in \mathfrak{J}_{n-1}(R)$.

Furthermore, if these assertions are satisfied and if we choose any linear form $l \in V^*$ such that $\mathcal{L}(j_1), \ldots, \mathcal{L}(j_{N-1}), l$ is a basis of $V^*$, then there exists a unique $p \in \mathfrak{J}_{n-1}(R)$ such that $j_N := lp \in \mathfrak{J}_n(R)$ satisfies $Jac(j_1, \ldots, j_N) = 1 \in \mathfrak{J}_{n-1}(R)$.

**Proof.** (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) is obvious. Let us now choose $l$ such that $\mathcal{L}(j_1), \ldots, \mathcal{L}(j_{N-1}), l$ is a basis of $V^*$. Let us show that there exists a unique $p \in \mathfrak{J}_{n-1}(R)$ such that $j_N := lp$ satisfies $Jac(j_1, \ldots, j_N) = 1 \in \mathfrak{J}_{n-1}(R)$. If $\varphi : \mathfrak{J}_{n-1}(R) \to \mathfrak{J}_{n-1}(R)$ is the finite dimensional linear endomorphism defined by $\varphi(p) = Jac(j_1, \ldots, j_{N-1}, lp)$, it is sufficient to show that $\varphi$ is an automorphism, which is equivalent to saying that Ker $\varphi = \{0\}$. If $p \neq 0 \in$ Ker $\varphi$, let $h \neq 0$ be the homogeneous part of smallest degree of $p$. Let $l_1, \ldots, l_{N-1}$ be the linear parts of $j_1, \ldots, j_{N-1}$. The equality $Jac(j_1, \ldots, j_{N-1}, lp) = 0$ implies $Jac(l_1, \ldots, l_{N-1}, lh) = 0$ which is absurd by the following lemma. $\square$

**Lemma 6.1.** If $l_1, \ldots, l_N$ is a basis of $V^*$, then the map $\psi : h \mapsto Jac(l_1, \ldots, l_{N-1}, l_N h)$ is a linear automorphism of $R$.

**Proof.** Injectivity. It is well known that $h \in$ Ker $\psi \iff$ the family $l_1, \ldots, l_{N-1}, l_N h$ is $\mathbb{C}$-algebraically dependant (see [22], [14] or [11]). Therefore, we may assume that $l_L = x_L$ for all $L$, so that $\psi(h) = 0 \iff \dfrac{\partial(x_N h)}{\partial x_N} = 0 \iff x_N h \in \mathbb{C}[x_1, \ldots, x_{N-1}] \iff h = 0$.
Surjectivity. For any $n \geq 0$, $\psi$ induces a linear endomorphism of the finite dimensional subspace $R_{\leq n}$ which is injective hence surjective. $\square$

The next result on variables, already proven in [9], is an easy consequence of th. 6.1.

**Theorem 6.2.** If $n \geq 1$, then $j \in \mathfrak{J}_n(R)$ is the $n$-jet of a variable if and only if $j_{(1)} \neq 0$.

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