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Regularity and convergence in variation for the laws of shot noise series and of related processes

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Regularity and convergence in variation for the laws of shot noise series and of related processes

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Abstract

We study the regularity of random series representing shoit noise series or Poisson integrals. We give conditions for the absolute continuity of their law with respect to Lebesgue measure and for their continuity for total variation norm. We deal also with the solution of drifted EDS driven by a Lévy process. We show the regularity of the law of the solution with respect to the drift term.

Keywords: Absolute continuity, Convergence in variation, shot noise series, Poisson integrals.
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1 Shot noise series and Poisson integrals

Let $(\Delta_i)_{i\geq 1}$ be an independent and identically distributed (i.i.d.) sequence of random variables with common law σ without atom in 0 and let $(T_i)_{i\geq 1}$ be a sequence of partial sums of independent $\mathcal{E}(\alpha)$-random variables, independent of $(\Delta_i)_{i\geq 1}$. Given a measurable function $h : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$, we are interested in the regularity of the law of the random series

$$I = \sum_{i=1}^{+\infty} h(T_i, \Delta_i). \tag{1}$$

Namely, we give conditions on h and on the law σ for the absolute continuity (with respect to the Lebesgue measure λ) of the law of (1) on $\mathbb{R}$ and on $\mathbb{R} \setminus \{0\}$. We study also the continuity of the law for the total variation norm with respect to the integrand h. Typically, our conditions are stated in terms of image measure $(\lambda \otimes \sigma)h^{-1}$ and we refer to Section 4 in [DLS] and to [ABP, D] for sufficient explicit conditions on h and on σ.

The interest of such random series lies for instance in their connection with shot noise series and more generally with Lévy-type integrals. Indeed, the series (1) can be seen as the total effect of a repeated signal Δ_i under a "filter" h. More precisely, as explained in [R], $h(T_i, \Delta_i)$ can be viewed as an effect at time zero of a shot Δ_i that happened T_i time units ago. In this case, it is standard to suppose that $t \mapsto |h(t, x)|$ is non-increasing for all x, so that the magnitude of the effect decreases as the time elapsed from the moment of the shot increases. In this setting, the series (1) represents the total shot noise at time zero (note from Remark 4.1 in [R] that the monotonicity condition is not required for the convergence of (1)).

When N is a Poisson measure with control measure ν, the series (1) is a representation for Poisson integral. More precisely, let $Z(t) = \int_0^t \int_{|x|\geq 1} xN(ds, dx)$ be the Poisson point process associated to N. Note $(T_i)_{i\geq 0}$ the sequence of ordered jump-times of Z in $B_1^c = \{|x| \geq 1\}$, that is $T_0 = 0$ and $T_i = \inf\{t > T_{i-1} \mid |\Delta Z_t| > 1\}$ for $i \geq 1$. The variable T_i is a sum of i independent $\mathcal{E}(1/\nu(B_1^c))$-distributed random variables (see [S, p. 137]). In particular, its law is equivalent to $\lambda_{\mathbb{R}_+}$. Moreover, it is well known that the $\Delta_i := \Delta Z_{T_i}$ form an i.i.d. sequence with law $\bar{\nu}_1 := \nu(\cdot \cap B_1^c)/\nu(B_1^c)$, independent of the jump-times $(T_i)_{i\geq 0}$. In this case, take

$$h(t, x) = f(t, x)\mathbf{1}_{|x|\geq 1}, \quad \alpha = 1/\nu(B_1^c), \quad \sigma = \bar{\nu}_1 \tag{2}$$

and the series in (1) becomes a representation for

$$\int_0^{+\infty} \int_{|x|\geq 1} f(t, x)N(dt, dx). \tag{3}$$

When $\nu(\mathbb{R} \setminus \{0\}) < +\infty$, we can consider Poisson integral directly on $\mathbb{R}^+ \times (\mathbb{R} \setminus \{0\})$ rather than on $\mathbb{R}_+ \times B_1^c$. In this case, the T_i's are sums of i independent $\mathcal{E}(1/\nu(\mathbb{R} \setminus \{0\}))$-distributed random variables and $\sigma = \nu/\nu(\mathbb{R} \setminus \{0\})$.

When $h(s, x) = f(s, x)\mathbf{1}_{|x|\geq 1}\mathbf{1}_{[0,t]}(s)$, the Poisson integrals is taken on $[0, t] \times B_1^c$ and there are almost surely (a.s.) a finite number of summand in (1).

Moreover when $f(t, x) = xg(t)$, the Poisson integral in (3) is the non-compensated part of the Lévy-type integrals

$$\int_0^{+\infty} g(s)dY_s = \int_0^{+\infty} \int_{|x|<1} xg(s)\tilde{N}(ds, dx) + \int_0^{+\infty} \int_{|x|\geq 1} xg(s)N(ds, dx) \tag{4}$$

where Y_t is the Lévy process $\int_0^t \int_{|x|<1} x\tilde{N}(ds, dx) + \int_0^t \int_{|x|\geq 1} xN(ds, dx)$. Since the integrals in the right hand side of (4) are independent, the regularity of the law of $\int_0^{+\infty} \int_{|x|\geq 1} xg(s)N(ds, dx)$ ensures the regularity of $\int_0^{+\infty} g(s)dY_s$. Note that if there were a Gaussian part in the Lévy-type integral, its contribution would ensure the regularity of the law. The importance of the contribution of the Poisson integrals in the law occurs only when Y has no Gaussian component.

In [R], Rosinski gives (necessary and sufficient) condition for the almost sure convergence of (1), namely

$$\int_0^{+\infty}\int_{\mathbb{R}\setminus\{0\}}(|h(t,x)|^2\wedge 1)dt\sigma(dx)<+\infty \tag{5}$$

and the following limit exists

$$a:=\lim_{s\to+\infty}\int_0^s\int_{\mathbb{R}\setminus\{0\}}h(t,x)\mathbf{1}_{|h(t,x)|\leq 1}dt\sigma(dx). \tag{6}$$

For instance, the series in (1) is well defined for $h\in L^1(\lambda\otimes\sigma)$ and

$$E[|I|]\leq\int_0^{+\infty}\int|h(t,x)|dt\sigma(dx).$$

Moreover, [R] recalls the law of the shot noise series (1) is infinitely divisible with characteristic function

$$\phi_I(u)=\exp\left(iau+\int_0^{+\infty}\int_{(\mathbb{R}\setminus\{0\})}(e^{iuh(t,x)}-1-iuh(t,x)\mathbf{1}_{|h(t,x)|\leq 1})dt\sigma(dx)\right). \tag{7}$$

In our study of shot noise series I, a key argument in the sequel is the conditioning by T_2. In fact, we shall consider (without restriction) that the probability space $(\Omega,\mathcal{F},\mathbb{P})$ is disintegrated as follows:

$$\left(\bar{\Omega}_2\times[0,T_2(\bar{\omega})]\times(\mathbb{R}\setminus\{0\}),\mathcal{F}_2^*\times\mathcal{B}([0,T_2(\bar{\omega})])\times\mathcal{B}(\mathbb{R}\setminus\{0\}),\bar{\mathbb{P}}_2\otimes\lambda^{\bar{\omega}}_{[0,T_2(\bar{\omega})]}\otimes\sigma\right) \tag{8}$$

where we note $\mathcal{F}_2^*=\sigma(T_i,\Delta T_i\mid i\geq 2)$ and $\bar{\lambda}_{[0,T_2(\bar{\omega})]}=\frac{1}{T_2(\bar{\omega})}\lambda_{[0,T_2(\bar{\omega})]}$ for the normalized Lebesgue measure on $[0,T_2(\bar{\omega})]$. For any random functional F on this space and every Borelian A, we have:

$$\mathbb{P}(F\in A)=\mathbb{P}(F(\bar{\omega},T_1,\Delta_1)\in A)=\bar{\mathbb{E}}_2\left[\frac{1}{T_2(\bar{\omega})}\int_{[0,T_2(\bar{\omega})]\times(\mathbb{R}\setminus\{0\})}\mathbf{1}_{\{F(\bar{\omega},t,x)\in A\}}dt\sigma(dx)\right]$$

where $\bar{\mathbb{E}}_2$ stands for the expectation with respect to $\bar{\mathbb{P}}_2$.

The paper is organized as follows. Section 2 is devoted to the study of the absolute continuity of the law of shot noise series I in (1) and of the truncated shot noise series:

$$I(t)=\sum_{\substack{n\geq 1\\ T_n\leq t}}f(T_n,\Delta_n). \tag{9}$$

Note that, according to our interpretation, this corresponds to the total effect of random shot than happened less than t units of times before zero. In Section 3, we study the convergence in variation of the laws of I and of $I(t)$. With the same argument based on the disintegrated probability space (8), we deal finally in Section 4 with the solutions of drifted stochastic differential equations (SDE) driven by a Lévy process (without Gaussian part). In [NS], Nourdin and Simon shows the regularizing effect of the drift term on the law of the solution. We continue their study showing the regularizing effect actually goes further on the law of the solution. Namely, the law of the solution depends continuously on the drift term with respect to the norm of total variation (see Prop. 4.1). In the sequel, we note $\mathcal{L}(X)$ for the law of a random variable X and $\mu_1 \asymp \mu_2$ means the measures μ_1 and μ_2 are equivalent, that is $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$.

2 Absolute continuity of the law of shot noise series

In this section, we give conditions for the absolute continuity of the law of I in (1). Note that, it is easy to see that the law of $I(t)$ in (9) has an atom in 0 since on the non-negligible event $\{T_1 > t\}$, the sum in (9) is empty and $I(t) = 0$. However,we will give sufficient condition for the absolute continuity of the laws on $\mathbb{R} \setminus \{0\}$. This is interesting in particular for $I(t)$ since in this case we give enlightment on the laws of processes defined as Poisson integrals on $[0, t]$.

For I, the sum in (1) is not empty and there is no obvious atom in 0. Since the law of T_n is equivalent to $\lambda_{\mathbb{R}_+}$, the condition

$$(\lambda_{\mathbb{R}_+} \otimes \sigma)h^{-1} \ll \lambda. \tag{10}$$

ensures the absolute continuity of the law of each summands $h(T_n, \Delta_n)$ in (1). But since the summands are not independent, (10) does not imply directly the absolute continuity of the law of I. However, this is sufficient as proved in 1) of the next proposition because of the conditional independence of the summands.

Proposition 2.1 *1) Under the condition* (10)*, the law of I has a density.*
2) Moreover, under the weaker condition

$$((\lambda_{\mathbb{R}_+} \otimes \sigma)h^{-1})_{|\mathbb{R}\setminus\{0\}} \ll \lambda, \tag{11}$$

the law of I is absolutely continuous with respect to λ on $\mathbb{R} \setminus \{0\}$, with possibly an atom at 0*.*

Remark 2.1
- Note that when $\sigma \ll \lambda$, Theorem 4.3 in [DLS] gives a sufficient condition for (10) to hold: it is sufficient that h is almost everywhere (a.e.) Fréchet differentiable and $(\lambda_{\mathbb{R}+} \otimes \lambda)\{(t, x) \mid \det Dh(t, x) = 0\} = 0$.
- When the shot noise series is of the form $I(t)$ in (9), that is $h(s, x) = 0$ for all $s > t$, the condition (10) does not, of course, hold: $(\lambda_{\mathbb{R}_+} \otimes \sigma)h^{-1}\{0\} = +\infty$ while $\lambda\{0\} = 0$. In this case, as explained previously, $I(t)$ has obviously an atom in 0.

- Anyway, (11) is a sufficient condition for the absolute continuity of the law of I on $\mathbb{R} \setminus \{0\}$. In particular, when the random series is of the form $I(t)$, this gives a positive result for the regularity of the law of $I(t)$ apart the atom in 0.
- With the choices in (2), I becomes the Poisson integral in (3) and Prop. 2.1 gives condition for the regularity of the law of Poisson integrals.

Proof. Working on the disintegrated probability space in (8), the law μ of I rewrites as a mixture of conditional measures $\mu_{\bar{\omega}}$:

$$\mu = \int_{\bar{\Omega}_2} \mu_{\bar{\omega}} \, \bar{\mathbb{P}}_2(d\bar{\omega}). \tag{12}$$

It is thus enough to show that for, $\bar{\mathbb{P}}_2$-almost all $\bar{\omega}$, $\mu_{\bar{\omega}}$ is absolutely continuous. Note that the conditional law of T_1 is uniform on $[0, T_2(\bar{\omega})]$:

$$\mathcal{L}(T_1|\mathcal{F}_2^*) = \mathcal{L}(T_1|T_2(\bar{\omega})) = \bar{\lambda}_{[0,T_2(\bar{\omega})]}.$$

Conditioning on $\mathcal{F}_2^*$, we have

$$\mu_{\bar{\omega}} = \mathcal{L}(I|\mathcal{F}_2^*)(\bar{\omega}) = \mathcal{L}\big(h(T_1', \Delta_1) + \Sigma_2(\bar{\omega})\big)$$

where $T_1' \stackrel{\mathcal{L}}{=} \mathcal{L}(T_1|\mathcal{F}_2^*)(\bar{\omega})$ is uniform and $\Sigma_2(\bar{\omega}) = \sum_{n\geq 2} h(T_n, \Delta_n)$ is known when $\bar{\omega} \in \bar{\Omega}_2$ is given. The law $\mu_{\bar{\omega}}$ is absolutely continuous iff $\mathcal{L}(h(T_1', \Delta_1)) = (\bar{\lambda}_{[0,T_2(\bar{\omega})]} \otimes \sigma)h^{-1}$ is. The condition (10) is sufficient for the absolute continuity of $\mu_{\bar{\omega}}$ for $\bar{\mathbb{P}}_2$-almost all $\bar{\omega}$. With (12), this proves 1).

For 2), consider the sequence of random variables $N_i(\omega) = \min\{n \geq N_{i-1} | f(T_n, \Delta_n) \neq 0\}$, with the convention $\min\{\emptyset\} = +\infty$. They are stopping times for the σ-algebras $\sigma(T_i - T_{i-1}, \Delta_i \mid i \leq k)$, $k \in \mathbb{N} \setminus \{0\}$.

Let $A_0 = \{N_1 = +\infty\}$, $A_1 = \{N_1 < N_2 = +\infty\}$, $A_2 = \{N_2 < +\infty\}$ and $\mu_k := \mathbb{P}(I \in \cdot | A_k)$ for $0 \leq k \leq 2$. Split the law μ of I as follows

$$\mu = \mathbb{P}(A_0)\mu_0 + \mathbb{P}(A_1)\mu_1 + \mathbb{P}(A_2)\mu_2. \tag{13}$$

On A_0, the sum defining I in (1) is empty, so that $\mu_0 = \delta_0$.
On A_1, the sum defining I in (1) reduces to $h(T_{N_1}, \Delta_{N_1})$. For any $A \in \mathcal{B}(\mathbb{R})$ with $\lambda(A) = 0$, we have:

$$\begin{aligned}
\mu_1(A) &= \mathbb{P}(h(T_{N_1}, \Delta_{N_1}) \in A \mid A_1) \\
&= \sum_{n_1=1}^{+\infty} \mathbb{P}(h(T_{n_1}, \Delta_{n_1}) \in A \setminus \{0\}, h(T_k, \Delta_k) = 0 \ \forall k \neq n_1)/\mathbb{P}(A_1) \\
&\leq \sum_{n_1=1}^{+\infty} \mathbb{P}(h(T_{n_1}, \Delta_{n_1}) \in A \setminus \{0\})/\mathbb{P}(A_1).
\end{aligned} \tag{14}$$

But $\mathcal{L}(h(T_{n_1}, \Delta_{n_1})) \asymp (\lambda_{\mathbb{R}_+} \otimes \sigma)h^{-1}$, the condition (11) entails $\mu_1 \ll \lambda$.
Next on A_2, we start by conditioning by $N_2 = n_2$

$$\mu_2 = \sum_{n_2=2}^{+\infty} \frac{\mathbb{P}(N_2 = n_2)}{\mathbb{P}(A_2)} \mu_{n_2}$$

where $\mu_{n_2} = \mathcal{L}(I|N_2 = n_2)$. Next, we condition by T_{n_2} :

$$\mu_{n_2} = \int \tilde{\mu}_{n_2} \, d\mathbb{P}_{T_{n_2}}$$

where the measure $\tilde{\mu}_{n_2}$ is the law of

$$\sum_{k=1}^{n_2} h(T'_k, \Delta'_k) + \sum_{k=n_2}^{+\infty} h(T_{n_2} + T'_k, \Delta_k) = h(T'_{N_1}, \Delta'_{N_1}) + \sum_{k=n_2}^{+\infty} h(T_{n_2} + T'_k, \Delta_k)$$

where $\mathcal{L}(T'_{N_1}, \Delta'_{N_1}) = \mathcal{L}((T_{N_1}, \Delta_{N_1})|N_2 = n_2, T_{n_2})$ and for $k \geq n_2$, $T'_k = T_k - T_{n_2}$ is independent of $\sigma(T_k, \Delta_k; k \leq n_2 - 1)$. Thus $\tilde{\mu}_{n_2}$ is absolutely continuous with respect to λ if $\mathcal{L}(h(T_{N_1}, \Delta_{N_1})|N_2 = n_2, T_{n_2})$ is. But for $A \in \mathcal{B}(\mathbb{R})$ with $\lambda(A) = 0$, we have

$$\begin{aligned}
&\mathbb{P}(h(T_{N_1}, \Delta_{N_1}) \in A|N_2 = n_2, T_{n_2}) \\
&= \sum_{n_1=1}^{n_2-1} \mathbb{P}(h(T_{n_1}, \Delta_{n_1}) \in A, N_1 = n_1|N_2 = n_2, T_{n_2}) \\
&= \sum_{n_1=1}^{n_2-1} \mathbb{P}(f(T_{n_1}, \Delta_{n_1}) \in A \setminus \{0\}, h(T_k, \Delta_k) = 0 \ \forall k < n_2, k \neq n_1|N_2 = n_2, T_{n_2}) \\
&\leq \sum_{n_1=1}^{n_2-1} \mathbb{P}(h(T_{n_1}, \Delta_{n_1}) \in A \setminus \{0\}|N_2 = n_2, T_{n_2}) \\
&= \sum_{n_1=1}^{n_2-1} \mathbb{P}(h(T'_{n_1}, \Delta_{n_1}) \in A \setminus \{0\}) \qquad (15)
\end{aligned}$$

where $\mathcal{L}(T_{n_1}|N_2 = n_2, T_{n_2}) = \mathcal{L}(T'_{n_1})$ is the n_1-th uniform order statistics on $[0, T_{n_2}]$. But since $h(T'_{n_1}, \Delta_{n_1}) \simeq (\lambda_{[0,T_{n_2}(\bar{\omega})]} \otimes \sigma)h^{-1}$, condition (11) entails

$$\mathbb{P}(h(T_{N_1}, \Delta_{N_1}) \in A|N_2 = n_2, T_{n_2}) = 0$$

and finally in this case also: $\mu_2 \ll \lambda$. With (13), this ends the proof of 2). □

For integrand $h(t, x) = xg(t)$, Prop. 2.1 can be specialized as follows for the non-compensated part in Lévy-type integrals (4):

Corollary 2.1 *Let $g : \mathbb{R}^+ \to \mathbb{R}$ and consider the Poisson integrals I in* (1) *but with integrand $f(t, x) = xg(t)$.*

1) The law of I is absolutely continuous with respect to λ on $\mathbb{R}\setminus\{0\}$ if :

$$\lambda_{\mathbb{R}_+} g^{-1} \ll \lambda. \tag{16}$$

2) The law of I has an atom at 0 and is absolutely continuous with respect to λ on $\mathbb{R}\setminus\{0\}$ if :

$$(\lambda_{\mathbb{R}_+} g^{-1})_{|\mathbb{R}\setminus\{0\}} \ll \lambda.$$

Remark 2.2
- Like in Remark 2.1, note that a sufficient condition for (16) is given by Th. 4.2 in [DLS]: it is enough that g is differentiable a.e. with $g'(t) \neq 0$ a.e.
- When $h(t,x) = xg(t)\mathbf{1}_{|x|\geq 1}$, we study the Poisson integral in the Lévy-Itô decomposition of a Lévy-type integral, cf. (4).

Proof. 1) Follow the same lines as in the proof of 1) in Prop. 2.1. Plugging $h(T_1', \Delta_1) = \Delta_1 g(T_1')$ in (2), the proof reduces to the absolute continuity of the law of $\Delta_1 g(T_1')$. But this is true because, from [DLS], $g(T_1')$ is absolutely continuous under (16) and because a product XY of independent random variables X, Y has an absolutely continuous law whenever X has and $\mathbb{P}(Y=0)=0$.

2) The atom in 0 comes from the Remark 2.1. Next, we follow similarly the same lines as in the proof of 2) in Prop. 2.1. Plugging $h(T_{n_1}, \Delta_{n_1}) = \Delta_{n_1} g(T_{n_1})$ in (14) and $h(T'_{n_1}, \Delta_{n_1}) = \Delta_{n_1} g(T'_{n_1})$ in (15), the proof reduces to the absolute continuity of the law of $\Delta_{n_1} g(T_{n_1})$ and of $\Delta_{n_1} g(T'_{n_1})$. We conclude like in 1) because a product XY of independent random variables X, Y has also an absolutely continuous on $\mathbb{R}\setminus\{0\}$ law when $\mathcal{L}(X)_{|\mathbb{R}\setminus\{0\}} \ll \lambda$ and $\mathbb{P}(Y=0)=0$. □

3 Regularity in variation of the law of shot noise series

In this section, we study further the law of shot noise series: we show the convergence in variation of the laws with respect to the "filter" h. When the laws have densities, this convergence is equivalent to the convergence in $L^1(\mathbb{R})$ of the densities. First, we deal in Section 3.2 with series I in (1) related to Poisson interals on $\mathbb{R}_+ \times (\mathbb{R}\setminus\{0\})$. Next, we consider truncated series $I(t)$ in (9) related to Poisson integrals on $[0,t]\times\mathbb{R}$ in Section 3.3. Since there is necessarily an atom in 0 in this second case, the arguments and the conditions change. We begin with some useful results on convergence in variation in Section 3.1. Our key argument to derive convergence in variance will be Lemma 3.3 and Lemma 3.4.

3.1 On the convergence variation

In the sequel $\|\mu\|$ stands for the total variation of a signed measure μ and $\xrightarrow{var}$ denotes the related convergence. We shall use the following elementary results:

Lemma 3.1 *Let X_n and X be random variables such that $\mathcal{L}(X_n) \xrightarrow{var} \mathcal{L}(X)$. Let Y be independent of $(X_n)_n$ and of X such that $\mathbb{P}(Y=0)=0$. Then $\mathcal{L}(X_nY) \xrightarrow{var} \mathcal{L}(XY)$.*

Proof. For any Borelian A, we have $\mathbb{P}(X_nY \in A) = \mathbb{E}_Y[\mathbb{P}_{X_n}(A_Y)]$ where $\mathbb{E}_Y$ is the expectation with respect to Y, $\mathbb{P}_{X_n}$ is the law of X_n and for any $y \neq 0$, $A_y := \{a/y | a \in A\}$. We thus have

$$|\mathbb{P}(X_nY \in A) - \mathbb{P}(XY \in A)| \leq \mathbb{E}_Y[|\mathbb{P}_{X_n}(A_Y) - \mathbb{P}_X(A_Y)|].$$

For any countable partition $(A_p)_{p\in\mathbb{N}\setminus\{0\}}$ of $\mathbb{R}$, we have

$$\sum_{p=1}^{+\infty} |\mathbb{P}(X_nY \in A_p) - \mathbb{P}(XY \in A_p)| \leq \mathbb{E}_Y\left[\sum_{p=1}^{+\infty} |\mathbb{P}_{X_n}(A_{p,Y}) - \mathbb{P}_X(A_{p,Y})|\right].$$

Since for all $y \neq 0$, the $A_{p,y}$'s are disjoint, taking supremum with respect to any partition $(A_p)_{p\in\mathbb{N}\setminus\{0\}}$, we derive

$$\begin{aligned}
\|\mathcal{L}(X_nY) - \mathcal{L}(XY)\| &\leq \mathbb{E}_Y\left[\sup_{(A_p)_p} \sum_{p=1}^{+\infty} |\mathbb{P}_{X_n}(A_{p,Y}) - \mathbb{P}_X(A_{p,Y})|\right] \\
&\leq \mathbb{E}_Y\left[\|\mathcal{L}(X_n) - \mathcal{L}(X)\|\right] \\
&= \|\mathcal{L}(X_n) - \mathcal{L}(X)\|
\end{aligned}$$

which concludes the proof of the lemma. □

Lemma 3.2 *If $\mu_n \xrightarrow{var} \mu$ when $n \to +\infty$, then for any $i \geq 1$, $\mu_n^{*i} \xrightarrow{var} \mu^{*i}$.*

Proof. By an immediate induction, it is enough to prove the result for $i = 2$. We have

$$\begin{aligned}
\|\mu_n * \mu_n - \mu * \mu\| &\leq \|\mu_n * (\mu_n - \mu)\| + \|(\mu_n - \mu) * \mu\| \\
&\leq \|\mu_n\|\|\mu_n - \mu\| + \|\mu_n - \mu\|\|\mu\| \\
&= 2\|\mu_n - \mu\|\|.
\end{aligned}$$

□

Moreover, we shall interpret the law of shot noise series as measure image and we shall use the following result (with $p = q = 2$) from [ABP] for which we introduce the Sobolev space $W^{2,2}(\lambda_{\mathbb{R}_+} \otimes \sigma) =$

$$\left\{f : \mathbb{R}_+ \otimes (\mathbb{R}\setminus\{0\}) \to \mathbb{R} \;\Big|\; \int |f(t,x)|^2 \lambda(dt)\sigma(dx) < +\infty, \int |Df(t,x)|^2 \lambda(dt)\sigma(dx) < +\infty\right\}.$$

Lemma 3.3 [ABP, Corollary 4] *Let G_n, $G \in W^{q,q}_{loc}(\mathbb{R}^p, \mathbb{R}^p)$ where $q \geq p$ and let the mappings G_n converge to G with respect to the Sobolev norm $\|\cdot\|_{q,q}$ on every ball. Assume that $E \subset \{\det DG \neq 0\}$ is a set of finite Lebesgue measure. Then for any measure $\mu \ll \lambda$, we have*

$$\mu_{|E} G_n^{-1} \xrightarrow{var} \mu_{|E} G^{-1}, \quad n \to +\infty.$$

In dimension one, Davydov gives more explicit conditions in [D]:

Lemma 3.4 [D, Theorem 1] *Suppose that the functions f_n and f are absolutely continuous on $[a,b]$ then $\lambda f_n^{-1} \xrightarrow{var} \lambda f^{-1}$ if the following conditions holds true:*

1. $f_n(a) \longrightarrow f(a)$, $n \to +\infty$,
2. $\|f_n' - f'\|_{L^1([a,b])} \longrightarrow 0$, $n \to +\infty$,
3. $f' \neq 0$ *a.e. on* $[a,b]$.

3.2 Convergence of shot noise series

In this section, we give conditions for the continuity for total variation norm of the law of shot noise series (1) with respect to h.

Proposition 3.1 *Let σ be a measure such that $\sigma \ll \lambda$ and let h_n, h in $W^{2,2}(\lambda_{\mathbb{R}_+} \otimes \sigma)$ satisfying* (6) *and* $\lim_{n\to+\infty} a_n = a$. *Suppose*

$$h_n \to h \text{ in } W^{2,2}(\lambda_{\mathbb{R}_+} \otimes \bar{\nu}_1), \tag{17}$$

and $\det Dh(t,x) \neq 0$ $\lambda \otimes \sigma$ *a.e. Then* $\mathcal{L}(I_n) \xrightarrow{var} \mathcal{L}(I)$ *when* $n \to +\infty$.

Proof. From (17), for any subsequence (n') there is some further $(n'') \subset (n')$ such that $h_{n''}(t,x) \to h(t,x)$ $\lambda \otimes \sigma$-a.e. But since moreover

$$\begin{aligned} |e^{iuy} - 1 - iuy\mathbf{1}_{|y|\le 1}| &\le 2\mathbf{1}_{|y|>1} + (e^u - 1 - u)y^2\mathbf{1}_{|y|\le 1} \\ &\le (2\mathbf{1}_{|y|>1} + (e^u - 1 - u)\mathbf{1}_{|y|\le 1})y^2 \end{aligned} \tag{18}$$

we derive from (7) that $\phi_{I-I_{n''}}(u) \to 1$ for all fixed u. We have thus $I_{n''} - I \xrightarrow{\mathcal{L}} \delta_0$ and $I_n \xrightarrow{\mathbb{P}} I$.

Since in particular $h_n \to h$ in $L^2(\lambda_{\mathbb{R}^+} \otimes \sigma)$ and T_1 has a bounded density, we have also $h_n(T_1, \Delta_1) \to h(T_1, \Delta_1)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Together with $I_n \xrightarrow{\mathbb{P}} I$, we derive

$$\Sigma_2^n := \sum_{k\ge 2} h_n(T_k, \Delta_k) \xrightarrow{\bar{\mathbb{P}}} \Sigma_2 := \sum_{k\ge 2} h(T_k, \Delta_k).$$

For any subsequence $(n') \subset (n)$, there is some further subsequence $(n'') \subset (n')$ and $\bar{\Omega}_0$ with $\bar{\mathbb{P}}_2(\bar{\Omega}_0) = 1$ such that for every $\bar{\omega} \in \bar{\Omega}_0$, the convergence $\Sigma_2^{n''}(\bar{\omega}) \to \Sigma_2(\bar{\omega})$ holds true.

Next, from Lemma 3.3, the condition (17) and $\det Dh(t,x) \neq 0$ a.e. imply $(\lambda_{[a,b]} \otimes \sigma)h_n^{-1} \xrightarrow{var} (\lambda_{[a,b]+} \otimes \sigma)h^{-1}$ for any interval $[a,b]$ and thus also

$$\tilde{\mu}_{n,1} := \mathcal{L}(h_n(U, \Delta_1)) \xrightarrow{var} \tilde{\mu}_1 := \mathcal{L}(h(U, \Delta_1)) \tag{19}$$

for any uniform random variable U independent of Δ_1. Like in the proof of the first part of Proposition 2.1, we disintegrate the probability space and we derive for the total variation of measures:

$$\|\mu_n-\mu\| = \int_{\bar\Omega} \|\mu_{n,\bar\omega}-\mu_{\bar\omega}\|\bar{\mathbb{P}}_2(d\bar\omega) = \int_{\bar\Omega_0} \|\mu_{n,\bar\omega}-\mu_{\bar\omega}\|\bar{\mathbb{P}}_2(d\bar\omega). \tag{20}$$

In the sequel, we study $\|\mu_{n,\bar\omega}-\mu_{\bar\omega}\|$ for $\bar\omega\in\bar\Omega_0$. Note $\tau_{n,\bar\omega}(x)=x+\Sigma_2^n(\bar\omega)$ the translation of $\Sigma_2^n(\bar\omega)$. The measure $\mu_{n,\bar\omega}$ is the law of $f_n(T_1',\Delta_1)+\Sigma_2^n$ where $T_1'\stackrel{\mathcal{L}}{=}\mathcal{L}(T_1|\bar\omega)$ is uniformly distributed on $[0,T_2(\bar\omega)]$. It rewrites $\mu_{n,\bar\omega}=\tilde\mu_{n,1}\tau_{n,\bar\omega}^{-1}$. Letting $\tau_{\bar\omega}(x)=x+\Sigma_2(\bar\omega)$, we have

$$\begin{aligned}
\|\mu_{n,\bar\omega}-\mu_{\bar\omega}\| &\le \|\tilde\mu_{n,1}\tau_{n,\bar\omega}^{-1}-\tilde\mu_1\tau_{\bar\omega}^{-1}\| \\
&\le \|\tilde\mu_{n,1}\tau_{n,\bar\omega}^{-1}-\tilde\mu_1\tau_{n,\bar\omega}^{-1}\| + \|\tilde\mu_1\tau_{n,\bar\omega}^{-1}-\tilde\mu_1\tau_{\bar\omega}^{-1}\| \\
&\le \|\tilde\mu_{n,1}-\tilde\mu_1\| + \|\tilde\mu_1\tau_{n,\bar\omega}^{-1}-\tilde\mu_1\tau_{\bar\omega}^{-1}\|. \qquad (21)
\end{aligned}$$

From (19), the first term in (21) goes to 0. Next, for the subsequence (n''), since for all $\bar\omega\in\bar\Omega_0$, $\Sigma_2^{n''}(\bar\omega)\to\Sigma_2(\bar\omega)$, since the operator of translation is continuous in $L^1(\mathbb{R})$, and since, from $\det Dh(t,x)\neq 0$ a.e., $\tilde\mu_1$ has a density (see the first point in Remark 2.1), the second term in (21) goes to 0. This yields $\mu_{n'',\bar\omega}\xrightarrow{var}\mu_{\bar\omega}$ when $n''\to+\infty$ for all $\bar\omega\in\bar\Omega_0$.

Finally from the disintegration (20), for any $(n')\subset(n)$, there is some $(n'')\subset(n')$ such that $\mu_{n''}\xrightarrow{var}\mu$. This proves $\mu_n\xrightarrow{var}\mu$. □

Remark 3.1

- The main condition in the proof of Prop. 3.1 is in fact

$$(\lambda_{\mathbb{R}_+}\otimes\sigma)h_n^{-1}\xrightarrow{var}(\lambda_{\mathbb{R}_+}\otimes\sigma)h^{-1}. \tag{22}$$

  From Lemma 3.3, (17) is an explicit condition for (22). The same remark applies for (23) in Cor. 3.1 and for (24) in Prop. 3.2 below.

- The following conditions on h_n are sufficient to apply Prop. 3.1: $h_n\in L^1(\lambda_{\mathbb{R}_+}\otimes\sigma)\cap W^{2,2}(\lambda_{\mathbb{R}_+}\otimes\sigma)$, $n\ge 1$ and

$$h_n\to h \text{ in } L^1(\lambda_{\mathbb{R}_+}\otimes\sigma) \text{ and in } W^{2,2}(\lambda_{\mathbb{R}_+}\otimes\sigma).$$

When $h_n(t,x)=xg_n(t)$, we can adapt the proof of Proposition 3.1 for the shot noise series $I_n=\sum_{k\ge1}\Delta_k g_n(T_k)$ with more specific conditions. When furtherly $h_n(t,x)=xg_n(t)\mathbf{1}_{B_1^c}(x)$, the series I_n become Poisson integrals $\int_{\mathbb{R}_+}\int_{B_1^c}xg_n(t)N(dt,dx)$, the following result thus applies for the Poisson integral in the Lévy-Itô decomposition of a Lévy-type stochastic integral (4). Note that here the measure σ is not supposed to be absolutely continuous with respect to λ.

Corollary 3.1 *Consider the shot noise series $I_n=\sum_{k\ge1}\Delta_k g_n(T_k)$ for $g_n\in W^{1,1}(\lambda_{\mathbb{R}_+})$. Suppose*

$$g_n\to g \text{ in } W^{1,1}(\lambda_{\mathbb{R}_+}) \tag{23}$$

and $g'(t)\neq 0$ a.e. Then $\mathcal{L}(I_n)\xrightarrow{var}\mathcal{L}(I)$ when $n\to+\infty$.

Proof. We adapt the proof of Prop. 3.1. Since from (18), we have also

$$|e^{iuy} - 1 - iuy\mathbf{1}_{|y|\le 1}| \le (2\mathbf{1}_{|y|>1} + (e^u - 1 - u)\mathbf{1}_{|y|\le 1})|y|$$

the condition (23) entails $\phi_{I_{n''}-I} \to 1$ and $I_n \xrightarrow{\mathbb{P}} I$ like previously.
Similarly since T_1 has a bounded density, $h_n(T_1, \Delta_1) \to h(T_1, \Delta_1)$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ comes from $h_n \to h$ in $L^1(\lambda_{\mathbb{R}^+} \otimes \sigma)$ and we derive

$$\Sigma_2^n := \sum_{k\ge 2} \Delta_k g_n(T_k) \xrightarrow{\bar{\mathbb{P}}} \Sigma_2 := \sum_{k\ge 2} \Delta_k g(T_k).$$

For any subsequence (n'), there is some further $(n'') \subset (n')$ and $\bar{\Omega}_0$ with $\bar{\mathbb{P}}(\bar{\Omega}_0) = 1$ such that $\Sigma_2^{n''}(\bar{\omega}) \to \Sigma_2(\bar{\omega})$ for every $\bar{\omega} \in \bar{\Omega}_0$.

Next from Lemma 3.4, the condition (23) implies $\lambda_{[a,b]} g_n^{-1} \xrightarrow{var} \lambda_{[a,b]} g^{-1}$ for all interval $[a, b]$ and together with Lemma 3.1:

$$\tilde{\mu}_{n,1} := \mathcal{L}(\Delta_1 g_n(U)) \xrightarrow{var} \tilde{\mu}_1 := \mathcal{L}(\Delta_1 g(U))$$

for any uniform random variable U independent of Δ_1. The rest of the proof follows the same lines as that of Prop. 3.1. □

3.3 Convergence for truncated shot noise series

When we consider truncated shot noise series $I(t)$ with $h_n(s, x) = f_n(s, x)\mathbf{1}_{[0,t]}(s)$ (see (9)), we can not adapt the proof of Prop. 3.1 but the convergence in variation of the laws still holds true. Moreover, when $h_n(s, x) = f_n(s, x)\mathbf{1}_{[0,t]}(s)\mathbf{1}_{B_1^c}(x)$, the related series in (9) are Poisson integrals on $[0, t] \times B_1^c$ and the following result applies to Poisson integrals in the Lévy-Itô decomposition of Lévy-type integrals on $[0, t]$, see (4).
We shall use the following elementary result:

Lemma 3.5 *Conditionally to $A_i = \{T_i \le t < T_{i+1}\}$, the vector $(T_1, \dots, T_i)$ is the uniform order statistics, i.e.: its law is given by the density* $\frac{i!}{t^i}\mathbf{1}_{0\le t_1\le t_2\le \dots \le t_i \le t}$.

Proposition 3.2 *Let σ be a law such that $\sigma \ll \lambda$, and for some fixed $t > 0$, let $f_n \in W^{2,2}(\lambda_{[0,t]} \otimes \sigma)$ such that the shot noise series $I_n(t)$ are well defined. Suppose*

$$f_n \to f \text{ in } W^{2,2}(\lambda_{[0,t]} \otimes \sigma) \tag{24}$$

such that $Df(s, x) \ne 0$ $\lambda \otimes \sigma$-a.e. Then $\mathcal{L}(I_n(t)) \xrightarrow{var} \mathcal{L}(I(t))$ when $n \to +\infty$.

Like for Prop. 3.1, the condition (24) gives an explicit condition for

$$(\lambda_{[0,t]} \otimes \sigma) f_n^{-1} \xrightarrow{var} (\lambda_{[0,t]} \otimes \sigma) f^{-1}$$

which is the real requirement in the following proof. Moreover, explicit conditions for the existence of $I_n(t)$ are given in (5) and (6) with $h(s, x) = f(s, x)\mathbf{1}_{[0,t]}(s)$.

Proof. Let μ_n (resp. μ) stands for $\mathcal{L}(I_n(t))$ (resp. for $\mathcal{L}(I(t))$). Note $A_0 = \{T_1 > t\}$ and, for $i \geq 1$, $A_i = \{T_i \leq t < T_{i+1}\}$ and $\mu_{n,i}$ (resp. μ_i) the conditional law of $I_n(t)$ (resp. of $I(t)$) under A_i. We have:

$$\mu_n = \mathbb{P}(A_0)\delta_0 + \sum_{i=1}^{+\infty} \mathbb{P}(A_i)\mu_{n,i} \quad \text{and} \quad \mu = \mathbb{P}(A_0)\delta_0 + \sum_{i=1}^{+\infty} \mathbb{P}(A_i)\mu_i.$$

We have for all $p \geq 1$:

$$\|\mu_n - \mu\| \leq \sum_{i=1}^{+\infty} \mathbb{P}(A_i)\|\mu_{n,i} - \mu_i\| \leq \sum_{i=1}^{p} \mathbb{P}(A_i)\|\mu_{n,i} - \mu_i\| + 2\sum_{i=p+1}^{+\infty} \mathbb{P}(A_i).$$

Since $\sum_{i=0}^{+\infty} \mathbb{P}(A_i) = 1$ is a convergent series, it is enough to show $\mu_{n,i} \xrightarrow{var} \mu_i$ when $n \to +\infty$ for all $i \geq 1$.

Note that conditionally to A_i, $I_n(t)$ and $I(t)$ rewrites

$$I_n(t) = \sum_{k=1}^{i} f_n(T_k, \Delta_k) \quad \text{and} \quad I(t) = \sum_{k=1}^{i} f(T_k, \Delta_k).$$

Using Lemma 3.5 (and commutativity of addition), conditionally to A_i, $I_n(t)$ and $I(t)$ have the same law as

$$\sum_{k=1}^{i} f_n(U_k, \Delta_k) \quad \text{and} \quad \sum_{k=1}^{i} f(U_k, \Delta_k)$$

where U_k $(1 \leq k \leq i)$ are i.i.d. uniform random variables on $[0, t]$. By independence, the law of $\sum_{k=1}^{i} f_n(U_k, \Delta_k)$ is the convolution of the law of $f_n(U_k, \Delta_k)$, $1 \leq k \leq i$, that is

$$\mathcal{L}\left(\sum_{k=1}^{i} f_n(U_k, \Delta_k)\right) = \left((t^{-1}\lambda_{[0,t]} \otimes \sigma)f_n^{-1}\right)^{*i} \tag{25}$$

and similarly

$$\mathcal{L}\left(\sum_{k=1}^{i} f(U_k, \Delta_k)\right) = \left((t^{-1}\lambda_{[0,t]} \otimes \sigma)f^{-1}\right)^{*i}. \tag{26}$$

From [ABP], (24) implies $(\lambda_{[a,b]} \otimes \sigma)f_n^{-1} \xrightarrow{var} (\lambda_{[a,b]} \otimes \sigma)f^{-1}$ for any interval $[a, b]$. Finally, from the expressions (25) and (26) and from Lemma 3.2, we derive $\mu_{n,i} \xrightarrow{var} \mu_i$ for all $i \geq 1$. This concludes the proof of Prop. 3.2. $\square$

In the case of integrands $f(s, x) = xg(s)$, Proposition 3.2 rewrites as follows:

Corollary 3.2 *For some fixed $t > 0$, let $I_n(t)$ be the shot noise series with kernels $f_n(s, x) := xg_n(s)$. Suppose g_n and g are absolutely continuous with $g_n(0) \to g(0)$ and $g_n' \to g'$ in $L^1([0, t])$ when $n \to +\infty$. Then the laws $\mathcal{L}(I_n(t))$ converge in total variation to $\mathcal{L}(I(t))$.*

Proof. Following the same strategy as in the proof of Proposition 3.2, it is enough to show for all $i \geq 1$ that

$$\mathcal{L}\left(\sum_{k=1}^{i} \Delta_k g_n(U_k)\right) \xrightarrow{var} \mathcal{L}\left(\sum_{k=1}^{i} \Delta_k g(U_k)\right), \quad n \to +\infty.$$

But from [D], for all $k \geq 1$, $\mathcal{L}(g_n(U_k)) \xrightarrow{var} \mathcal{L}(g(U_k))$ when $n \to +\infty$. Lemma 3.1 implies then $\mathcal{L}(\Delta_k g_n(U_k)) \xrightarrow{var} \mathcal{L}(\Delta_k g(U_k))$ for all $k \geq 1$. Finally we conclude like in Proposition 3.2 by independence with Lemma 3.2. □

4 Lévy-type integrals

In this section, we consider the solution of the one-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t a(X_s)ds + Z_t \tag{27}$$

where Z is a real Lévy process without Gaussian part and with Lévy measure ν:

$$Z_t = bt + \int_{[0,t]\times B_1} x\tilde{N}(ds,dx) + \int_{[0,t]\times B_1^c} xN(ds,dx)$$

and a is a function of class C^1 with bounded derivative. Here N is a Poisson measure on $\mathbb{R}_+ \times (\mathbb{R}\setminus\{0\})$ with compensator $\lambda \otimes \nu$, $\tilde{N}$ is the compensated measure and $b \in \mathbb{R}$. In [NS], Nourdin and Simon study the absolute continuity of the law of X. They prove that for all $t > 0$:

$$\mathcal{L}(X_t) \ll \lambda \Longleftrightarrow \mathcal{L}(X_1) \ll \lambda \Longleftrightarrow \nu \text{ is infinite.}$$

when a is locally monotone in x_0, that is there is $\alpha > 0$ such that for $x_0 - \alpha < y < x_0 < z < x_0 + \alpha$, we have $a(y) < a(x_0) < a(z)$ (in the increasing case) or $a(z) < a(x_0) < a(y)$ (in the decreasing case). In a sense, they show that the drift term $a(X_s)ds$ in the SDE (27) has a regularizing effect for a large class of drift functions a. They wonder also if this effect concerns the regularity of the densities.

In the sequel, we give a first answer to this question: we show that the law of X_t depends continuously on the drift coefficient a for the total variation norm. Under the condition of [NS], this convergence rewrites as the regularity in $L^1(\mathbb{R})$ of the densities with respect to the drift function. The proof is based on a simple application of the method of statification of Davydov *et al.* (see [DLS]): it relies on a suitable transformation of one jump of the driving Lévy process Z. The precise statement of the result is the following:

Proposition 4.1 *Assume the Lévy measure ν is infinite and let X_n be the solution of the SDE*

$$dX_{n,t} = a_n(X_{n,t})dt + dZ_t, \quad X_{n,0} = x_{n,0}. \tag{28}$$

Suppose a_n and a are derivable with bounded derivatives and the convergence $\dot{a}_n \to \dot{a}$ is uniform on bounded sets. Suppose also $a_n(y)$ and $\dot{a}_n(y)$ are both continuous functions of the couple (n, y) and that $x_{n,0} \to x_0$. Suppose moreover a is locally monotone. Then for each $t > 0$, the law of $X_{n,t}$ converges in variation to that of X_t.

Remark 4.1 Since $\dot{a}$ and $\dot{a}_n$ are globally Lipschtiz, it is well known that there are unique stong solutions to (27) and to (28) defined on $\mathbb{R}_+$. The continuity of $\dot{a}_n(y)$ with respect to (n, y) ensures the continuity of the solution of related ordinary differential equation (ODE) with respect to the parameter n (see [P, p. 185]).

Proof. For simplicity, we prove the result for $t = 1$ and we note $K := \|\dot{a}\|_\infty < +\infty$. The setting follows that of Nourdin and Simon in [NS]. Let $\varepsilon > 0$ be fixed. Considering $-X$ if necessary, we suppose X is locally increasing, let $\alpha > 0$ such that for $x_0 - \alpha < y < x_0 < z < x_0 + \alpha$, we have $a(y) < a(x_0) < a(z)$. Let

$$A_1 = \sup_{-\alpha/2 \le y \le \alpha/2} |a(y)|, \qquad A_2 = \sup_{-A_3 \le y \le A_3} |a(y)|$$

where $A_3 = K(|x_0| + A_1)e^K + \alpha/2$. By the right-continuity of Z, there is $\gamma_0 > 0$ such that

$$\mathbb{P}\left(\sup_{t \le \gamma_0} |Z_t| < \alpha/6\right) \ge 1 - \varepsilon/6$$

Let $\gamma = \gamma_0 \wedge (\alpha/(3A_2)) \wedge 1$ and let T_n be the sequence of jumping times $\Delta_n := \Delta Z_{T_n}$ of Z into $[\eta, \alpha/6]$. Since $\nu([0,1]) = +\infty$, there is some $\eta > 0$ such that

$$\mathbb{P}(T_2 < \gamma) > 1 - \varepsilon/6.$$

Following [NS], we consider $\bar{Z}_t = Z_t - \Delta_1 \mathbf{1}_{T_1 \le t}$ which can be constructed on the disintegrated probability space

$$\left(\bar{\Omega} \times [0, T_2(\bar{\omega})], \bar{\mathcal{F}} \times \mathcal{B}([0, T_2(\bar{\omega})]), \bar{\mathbb{P}} \otimes \bar{\lambda}_{[0,T_2(\bar{\omega})]}\right)$$

where $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is the canonical space associated to $(\Delta_1, \bar{Z})$. Let

$$\bar{\Omega}_1 = \left\{\sup_{t \le \gamma} |\bar{Z}_s| < \alpha/3, T_2 < \gamma\right\} \supset \left\{\sup_{t \le \gamma} |Z_s| < \alpha/6, T_2 < \gamma\right\}$$

Note that $\bar{\Omega}_1 \in \bar{\mathcal{F}}$ and $\bar{\mathbb{P}}(\bar{\Omega}_1^c) \le \varepsilon/3$.

In the sequel, we consider $Y = X - Z$. It is solution of the ODE $dY_t = a(Y_t + Z_t)dt$, $Y_0 = x_0$. Consider also $\bar{X}$ for the process defined by the SDE $d\bar{X}_t = a(\bar{X}_t)dt + d\bar{Z}_t$ and $\bar{Y} = \bar{X} - \bar{Z}$.
Note that for $t \le \gamma$:

$$|Y_t| \;\; \le \;\; |x_0| + \int_0^t |a(Z_s)|ds + \int_0^t K|Y_s|ds$$

$$\leq \quad |x_0| + A_1 + \int_0^t K|Y_s|ds$$

Applying the Gronwall's inequality, we have

$$|Y_t| \leq K(|x_0| + A_1)e^{Kt},$$

so that for $t \leq \gamma$: $|Y_t + Z_t| \leq A_3$ and from the ODE defining Y_t:

$$|Y_t - x_0| \leq A_2\gamma \leq \alpha/3.$$

Next for $t < T_1$, we have

$$|X_t(\bar{\omega}) - x_0| \leq |Y_t(\bar{\omega}) - x_0| + |\bar{Z}_t(\bar{\omega})| \leq \alpha/3 + \alpha/3 < \alpha \tag{29}$$

and for $T_1 \leq t < T_2 \leq \gamma$, we have

$$|X_t(\bar{\omega}) - x_0| \leq |Y_t(\bar{\omega}) - x_0| + |\bar{Z}_t(\bar{\omega})| + |\Delta_1| \leq \alpha/3 + \alpha/3 + \alpha/6 < \alpha. \tag{30}$$

Note μ_n for the law of $X_{n,1}$, we have

$$\begin{aligned} \|\mu_n - \mu\| &\leq \int_{\bar{\Omega}} \|\bar{\mu}_{n,\bar{\omega}} - \bar{\mu}_{\bar{\omega}}\| d\bar{\mathbb{P}} \\ &\leq 2\mathbb{P}(\bar{\Omega}_1^c) + \int_{\bar{\Omega}_1} \|\bar{\mu}_{n,\bar{\omega}} - \bar{\mu}_{\bar{\omega}}\| d\bar{\mathbb{P}} \\ &\leq (2/3)\varepsilon + \int_{\bar{\Omega}_1} \|\bar{\mu}_{n,\bar{\omega}} - \bar{\mu}_{\bar{\omega}}\| d\bar{\mathbb{P}} \end{aligned} \tag{31}$$

where

$$\bar{\mu}_{n,\bar{\omega}} = \bar{\lambda}_{[0,T_2(\bar{\omega})]} X_{n,1}(\bar{\omega}, \cdot)^{-1}$$

and $X_{n,1}(\bar{\omega}, \cdot) = Y_{n,1}(\bar{\omega}, \cdot) + Z_1(\bar{\omega}, \cdot)$ with $Y_n = X_n - Z$ the solution of the ODE: $dY_{n,t} = a_n(Y_{n,t} + Z_t)dt$, $Y_{n,0} = x_{n,0}$ that is:

$$Y_n(t) = x_{n,0} + \int_0^t a_n(Y_{n,s} + Z_s)ds. \tag{32}$$

Note that $Z_1(\bar{\omega}, \cdot)$ actually does not depend on T_1 since Z jumps at least twice in $[\eta, \alpha/6]$ ($T_2 \leq \gamma \leq 1$) and by the Lévy-Itô decomposition, the terminal value Z_1 is independent of the first jumping time T_1. Noting $\tau_{Z_1(\bar{\omega})}$ for the translation $x \mapsto x + Z_1(\bar{\omega})$, we have

$$\begin{aligned} \bar{\lambda}_{[0,T_2(\bar{\omega})]} X_{n,1}(\bar{\omega}, \cdot)^{-1} &= \bar{\lambda}_{[0,T_2(\bar{\omega})]} Y_{n,1}(\bar{\omega}, \cdot)^{-1} \tau_{Z_1(\bar{\omega})}^{-1} \\ \bar{\lambda}_{[0,T_2(\bar{\omega})]} X_1(\bar{\omega}, \cdot)^{-1} &= \bar{\lambda}_{[0,T_2(\bar{\omega})]} Y_1(\bar{\omega}, \cdot)^{-1} \tau_{Z_1(\bar{\omega})}^{-1} \end{aligned}$$

and it remains to show for all $\bar{\omega} \in \bar{\Omega}_1$

$$\bar{\lambda}_{[0,T_2(\bar{\omega})]} Y_{n,1}(\bar{\omega}, \cdot)^{-1} \xrightarrow{var} \bar{\lambda}_{[0,T_2(\bar{\omega})]} Y_1(\bar{\omega}, \cdot)^{-1} \tag{33}$$

for which we shall apply Lemma 3.4, for all $\bar{\omega} \in \bar{\Omega}_1$, with $[a, b] = [0, T_2(\bar{\omega})]$, $f_n = Y_{n,1}(\bar{\omega}, \cdot)$ and $f = Y_1(\bar{\omega}, \cdot)$.

The main point is given in [NS, Prop. 2]: it is shown that $Y_{n,1}$ and Y_1 depend differentiably on T_1 with derivatives given by

$$\frac{dY_{n,1}}{dT_1} = (a_n(X_{n,T_1^-}) - a_n(X_{n,T_1})) \exp\left(\int_{T_1}^{1} \dot{a}_n(X_{n,s})ds\right) \tag{34}$$

and the same for dY_1/dT_1.

Seing n as a parameter in the ODE (32), it is well known that, when $\dot{a}_n(y)$ is continuous in the couple (n, y), $Y_{n,t}$ depends continuously of (n, t) (see [P, p. 185]). Then, it is easy to see $M_1(\bar{\omega}) = \sup_{n\in\mathbb{N},t\in[0,1]} |Y_{n,t}|$ is finite. Indeed, if $M_1(\bar{\omega}) = +\infty$, then for all p there are $t_p \in [0, 1]$ and $n_p \in \mathbb{N}$ such that $|Y_{n_p,t_p}| \geq p$. But extracting $t_{p'} \to t_\infty \in [0, 1]$,

- either $(n_{p'})_{p'}$ has an accumulation point n_a so that for a further subsequence (p''), $n_{p''} = n_a$. Taking the limit $p'' \to +\infty$ in $|Y_{n_a,t_{p''}}| \geq p''$ gives $|Y_{n_a,t_\infty}| = +\infty$, which contradicts Y_{n_a} is bounded.
- either $(n_{p'})_{p'}$ is not bounded and for a further subsequence (p''), $n_{p''} \to +\infty$. Taking the limit $p'' \to +\infty$ in $|Y_{n_{p''},t_{p''}}| \geq p''$ gives $|Y_{t_\infty}| = +\infty$, which contradicts Y is bounded.

For $\bar{\omega} \in \bar{\Omega}_1$ and any $n \in \mathbb{N}$, we have for all $t \leq 1$:

$$|Y_{n,t} + Z_t| \leq M_1(\bar{\omega}) + |\Delta_1(\bar{\omega})| + |\bar{Z}_t(\bar{\omega})| =: M_2(\bar{\omega}).$$

In particular, we have $|X_{n,t}| \leq M_2(\bar{\omega})$ for all $t \leq 1$. Next, since we have:

$$|Y_{n,t} - Y_t| \leq |x_{n,0} - x_0| + \sup_{x\in[-M_2(\bar{\omega}),M_2(\bar{\omega})]} |a_n(x) - a(x)| + K \int_0^t |Y_{n,s} - Y_s|ds$$

using Gronwall's inequality, we derive:

$$|X_{n,t} - X_t| = |Y_{n,t} - Y_t| \leq \left(|x_{n,0} - x_0| + \sup_{x\in[-M_2(\bar{\omega}),M_2(\bar{\omega})]} |a_n(x) - a(x)|\right) e^{Kt}. \tag{35}$$

The first condition of Lemma 3.4 is satisfied since with $t = 1$ above yields $Y_{n,1} \to Y_1$, for all $T_1 \in [0, T_2(\bar{\omega})]$.

Since $X_{n,T_1^-}, X_{n,T_1}, X_{T_1^-}, X_{T_1}$ are all bounded by $M_2(\bar{\omega})$, the uniform convergence of a_n to a on $[-M_2(\bar{\omega}), M_2(\bar{\omega})]$ together with (35) entails:

$$\lim_{n\to+\infty} a_n(X_{n,T_1^-}) - a_n(X_{n,T_1}) = a(X_{T_1^-}) - a(X_{T_1})$$

uniformly in T_1. Similarly, the uniform convergence $\dot{a}_n \to \dot{a}$ on $[-M_2(\bar{\omega}), M_2(\bar{\omega})]$, the uniform convergence $X_{n,s} \to X_s$ in (35) and $\|\dot{a}\|_\infty = K$ together imply

$$\lim_{n\to+\infty} \int_{T_1}^1 \dot{a}_n(X_{n,s})ds = \int_{T_1}^1 \dot{a}(X_s)ds$$

uniformly in T_1. We derive now from (34): almost surely the following convergence holds true uniformly in T_1:

$$\lim_{n\to+\infty} \frac{dY_{n,1}}{dT_1} = \frac{dY_1}{dT_1}. \tag{36}$$

But since the convergence is uniform in T_1, the same convergence as in (36) actually holds true in $L^1([0,1])$. This shows condition 2 in Lemma 3.4 is satisfied. Moreover, from (29) and (30), $X_{T_1^-}, X_{T_1} \in]x_0 - \alpha, x_0 + \alpha[$, so that the local monotony of a in x_0 implies $dY_1/dT_1 > 0$, taking care of condition 3, see (34). Applying Lemma 3.4, we have (33) for all $\bar{\omega} \in \bar{\Omega}_1$ and also

$$\bar{\lambda}_{[0,T_2(\bar{\omega})]} X_{n,1}(\bar{\omega},\cdot)^{-1} \xrightarrow{var} \bar{\lambda}_{[0,T_2(\bar{\omega})]} X_1(\bar{\omega},\cdot)^{-1}.$$

Taking limit in (31), we have $\overline{\lim}_n \|\mu_n - \mu\| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the result follows. □

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