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A complete market model with Poisson and Brownian components

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A complete market model with Poisson and Brownian components

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Abstract

We present a complete market model with jumps, using a martingale constructed from a Brownian motion and a Poisson process that are mutually excluding eachother. The chaotic calculus relative to this martingale is developed to obtain a Clark formula aimed at the computation of hedging strategies.

Key words: Market completeness, normal martingales, hedging strategies, Malliavin calculus, chaos representation property.
Mathematics Subject Classification (1991): 90A09, 60H30, 60H07, 60G44.

1 Introduction

In this paper we present a complete market model that considers jumps in dynamics of asset prices. The sum of a Brownian motion and a Poisson process has been used to model discontinuous asset prices, cf. [7], however it does not have the predictable representation property, thus it does not yield market completeness. A model having market completeness with jumps has been presented in [3] using the standard Azéma martingale studied in [4] (which is a normal martingale according to the definition of [2]), but this process does not seem to be natural in applications.

We suggest to use a modification $(M_t)_{t\in\mathbf{R}_+}$ of another normal martingale also introduced in [4], which consists in a combination of a Brownian motion and a Poisson process. This process does have the chaos representation property because its Brownian part vanishes whenever it jumps. The martingale of [4] is originally a normal martingale and this constrains the height of jumps to be linked to the intensity of the Poisson component. For this reason we use a modification of this process which can have jumps of arbitrary heights, according to a deterministic function of time. As a consequence, the predictable quadratic variation of our process is no longer equal to dt. The Poisson process and Brownian motion are included as particular cases in our approach, as well as the representation formulas obtained in [6] using the Wiener chaos associated to a continuous martingale with deterministic angle bracket.

In Sect. 2 we define the martingale $(M_t)_{t\in\mathbf{R}_+}$ which is a combination of a Poisson process and a Brownian motion, and satisfies a structure equation. Sect. 3 presents the chaos representation property. Sect. 4 deals with the predictable representation property and the Clark formula with simplified proofs, using a gradient operator that plays a role in the computation of hedging strategies. Changes of probability and the Girsanov theorem for the process $(M_t)_{t\in\mathbf{R}_+}$ are considered in Sect. 5, with an application to the pricing of European options in Sect. 6.

2 Solution of a deterministic structure equation

Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $\alpha : \mathbb{R}_+ \longrightarrow]0,\infty[$ be two deterministic functions, with $\phi \in \cap_{p\geq 1} L^p(\mathbb{R}_+, \alpha_t^2 dt)$. Let $i_t = 1_{\{\phi_t=0\}}$, and

$$\lambda_t = (1-i_t)\alpha_t^2/\phi_t^2 = \begin{cases} \alpha_t^2/\phi_t^2 & \text{if } \phi_t \neq 0, \\ 0 & \text{if } \phi_t = 0, \quad t\in\mathbb{R}_+. \end{cases}$$

Let $(B_t)_{t\in\mathbf{R}_+}$ be a standard Brownian motion, and $(N_t)_{t\in\mathbf{R}_+}$ a Poisson process with intensity $\nu_t = \int_0^t \lambda_s ds$, $t\in\mathbb{R}_+$, i.e. $(\nu_t)_{t\in\mathbf{R}_+}$ is the unique continuous deterministic function such that $(N_t-\nu_t)_{t\in\mathbf{R}_+}$ is a martingale. We assume that the processes $(B_t)_{t\in\mathbf{R}_+}$ and $(N_t)_{t\in\mathbf{R}_+}$ are independent and that $\lim_{t\to\infty}\nu_t = \infty$ and $\nu_t<\infty$, $\forall t\in\mathbb{R}_+$. Let $(X_t)_{t\in\mathbf{R}_+}$ denote the process defined as

$$dX_t = i_t dB_t + \frac{\phi_t}{\alpha_t}\left(dN_t - \lambda_t dt\right), \quad t\in\mathbb{R}_+, \quad X_0 = 0, \tag{1}$$

which satisfies the classical structure equation

$$d[X,X]_t = dt + \frac{\phi_t}{\alpha_t} dX_t. \tag{2}$$

It is known from [4] that $(X_t)_{t\in\mathbf{R}_+}$ is in fact the unique solution of (2). Relation (2) implies that the process $(X_t)_{t\in\mathbf{R}_+}$ has predictable quadratic variation $d\langle X,X\rangle_t = dt$. From $(X_t)_{t\in\mathbf{R}_+}$ we construct a martingale $(M_t)_{t\in\mathbf{R}_+}$ with predictable quadratic variation $d\langle M,M\rangle_t = \alpha_t^2 dt$, as

$$dM_t = \alpha_t dX_t, \quad t\in\mathbb{R}_+, \quad M_0 = 0, \tag{3}$$

i.e.

$$dM_t = i_t\alpha_t dB_t + \phi_t\left(dN_t - \lambda_t dt\right), \quad t\in\mathbb{R}_+, \quad M_0 = 0. \tag{4}$$

We are using $(M_t)_{t\in\mathbf{R}_+}$ instead of $(X_t)_{t\in\mathbf{R}_+}$ because the parameters $(\phi_t)_{t\in\mathbf{R}_+}$ and $(\alpha_t)_{t\in\mathbf{R}_+}$ have independent roles in the definition of $(M_t)_{t\in\mathbf{R}_+}$: $(\phi_t)_{t\in\mathbf{R}_+}$ controls the height of jumps, and $(\alpha_t)_{t\in\mathbf{R}_+}$ is relative to the continuous part of $(M_t)_{t\in\mathbf{R}_+}$. At first it could seem more general to set $dM_t = \beta_t dX_t$, however this is not the case, because the functions $(\phi_t)_{t\in\mathbf{R}_+}$ and $(\alpha_t)_{t\in\mathbf{R}_+}$ are sufficient to completely characterize $(M_t)_{t\in\mathbf{R}_+}$ as a solution of (5).

Proposition 1 *The martingale $(M_t)_{t\in\mathbf{R}_+}$ satisfies the deterministic structure equation*

$$d[M,M]_t = \alpha_t^2 dt + \phi_t dM_t, \qquad t \in \mathbb{R}_+, \tag{5}$$

with $d\langle M,M\rangle_t = \alpha_t^2 dt$, $t \in \mathbb{R}_+$.

Proof. Using the relations $d[B,N]_t = 0$ and $i_t\phi_t = 0$, $t \in \mathbb{R}_+$, we have:

$$\begin{aligned} d[M,M]_t &= i_t\alpha_t^2 dt + \phi_t^2 dN_t \\ &= i_t\alpha_t^2 dt + \phi_t\left(dM_t - i_t\alpha_t dB_t + (1-i_t)\frac{\alpha_t^2}{\phi_t}dt\right) \\ &= \alpha_t^2 dt + \phi_t dM_t, \quad t \in \mathbb{R}_+. \quad \square \end{aligned}$$

3 Chaos representation property

Let $(\mathcal{F}_t)_{t\in\mathbf{R}_+}$ be the filtration generated by $(M_t)_{t\in\mathbf{R}_+}$ which is the same as the filtration generated by $(X_t)_{t\in\mathbf{R}_+}$, since α is deterministic and does not vanish. This filtration is smaller than the filtration generated by the pair Brownian motion - Poisson process. We assume that we are working on a probability space $(\Omega,\mathcal{F},P)$ with $\mathcal{F} = \mathcal{F}_\infty$. We denote by $L^2(\mathbb{R}_+,\alpha_t^2 dt)^{\circ n}$ the space of symmetric functions on $\mathbb{R}_+^n$ that are square-integrable with respect to $\alpha_{t_1}^2\cdots\alpha_{t_n}^2 dt_1\cdots dt_n$, equipped with the $L^2(\mathbb{R}_+^n,\alpha_{t_1}^2\cdots\alpha_{t_n}^2 dt_1\cdots dt_n)$-norm. Given $f_1,\ldots,f_n \in L^2(\mathbb{R}_+,\alpha_t^2 dt)$, let also

$$f_1\circ\cdots\circ f_n : \mathbb{R}_+^n \longrightarrow \mathbb{R}$$

denote the symmetrization in n variables of the function

$$(t_1,\ldots,t_n) \mapsto f_1(t_1)\cdots f_n(t_n).$$

It is known, cf. [4], that the process $(X_t)_{t\in\mathbf{R}_+}$ has the chaos representation property, i.e. any $F \in L^2(\Omega,\mathcal{F},P)$ has a unique decomposition

$$F = E[F] + \sum_{n=1}^\infty n!\int_0^\infty\int_0^{t_n}\cdots\int_0^{t_2} g_n(t_1,\ldots,t_n)dX_{t_1}\cdots dX_{t_n}, \tag{6}$$

with $g_n \in L^2(\{0 \leq t_1 < \cdots < t_n\})$. Obviously a similar property holds for $(M_t)_{t\in\mathbb{R}_+}$. Let $I_n(f_n)$ denote the multiple stochastic integral of $f_n \in L^2(\mathbb{R}_+, \alpha_t^2 dt)^{\circ n}$ with respect to $(M_t)_{t\in\mathbb{R}_+}$, defined as

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM_{t_n}, \quad n \geq 1,$$

with the convention $I_0(f_0) = f_0$, $f_0 \in \mathbb{R}$. The multiple stochastic integrals satisfy the isometry and orthogonality properties

$$E[I_n(f_n)I_m(g_m)] = \begin{cases} (f_n, g_n)_{L^2(\mathbb{R}_+,\alpha_t^2 dt)^{\circ n}} & n = m, \\ 0, & n \neq m, \end{cases} \tag{7}$$

$f_n \in L^2(\mathbb{R}_+, \alpha_t^2 dt)^{\circ n}$, $g_m \in L^2(\mathbb{R}_+, \alpha_t^2 dt)^{\circ m}$.

Proposition 2 *The martingale $(M_t)_{t\in\mathbb{R}_+}$ has the chaos representation property: any $F \in L^2(\Omega, \mathcal{F}, P)$ has a unique decomposition*

$$F = E[F] + \sum_{n=1}^\infty I_n(f_n), \quad f_n \in L^2(\mathbb{R}_+, \alpha_t^2 dt)^{\circ n}, \quad n \geq 1.$$

Proof. It suffices to consider the expansion (6) of $F \in L^2(\Omega, \mathcal{F}, P)$ and to define $f_n \in L^2(\mathbb{R}_+, \alpha_t^2 dt)^{\circ n}$ as

$$f_n(t_1, \ldots, t_n) = (\alpha_{t_1} \cdots \alpha_{t_n})^{-1} g_n(t_1, \ldots, t_n), \quad t_1, \ldots, t_n \in \mathbb{R}_+^n, \quad n \geq 1. \quad \square$$

The action of the conditional expectation with respect to $\mathcal{F}_t$ on multiple stochastic integrals is

$$E[I_n(f_n) \mid \mathcal{F}_t] = I_n(f_n 1_{\{* \leq t\}}), \quad t \in \mathbb{R}_+, \quad f_n \in L^2(\mathbb{R}_+, \alpha_s^2 ds)^{\circ n}, \quad n \geq 1,$$

since $(M_t)_{t\in\mathbb{R}_+}$ is a martingale.

4 Gradient and divergence operators

In this section we introduce the gradient operator that will be used in the computation of hedging strategies. We denote by $\mathrm{Dom}(D)$ the set of $F \in L^2(\Omega, \mathcal{F}, P)$ whose decomposition $F = \sum_{n=0}^\infty I_n(f_n)$ satisfies

$$\sum_{n=1}^\infty n(n!) \|f_n\|^2_{L^2(\mathbb{R}_+,\alpha_t^2 dt)^{\circ n}} < \infty.$$

We let

$$D : \mathrm{Dom}(D) \longrightarrow L^2(\Omega \times \mathbb{R}_+, dP \times \alpha_t^2 dt)$$

denote the unbounded gradient operator defined as

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(*, t)), \quad dP \times dt - a.e.,$$

if $F \in \mathrm{Dom}(D)$ is written as $F = \sum_{n=0}^{\infty} I_n(f_n)$. We have

$$\begin{aligned}
\|DF\|^2_{L^2(\Omega \times \mathbb{R}_+, dP \times \alpha_t^2 dt)} &= \int_0^\infty E[(D_t F)^2] \alpha_t^2 dt \\
&= \int_0^\infty \sum_{n=1}^{\infty} n^2 (n-1)! \|f_n(*, t)\|^2_{L^2(\mathbb{R}_+, \alpha_t^2 dt)^{\circ(n-1)}} \alpha_t^2 dt \\
&= \sum_{n=1}^{\infty} n(n!) \|f_n\|^2_{L^2(\mathbb{R}_+, \alpha_t^2 dt)^{\circ n}}.
\end{aligned}$$

We denote by $\mathrm{Dom}(\delta)$ the set of $u \in L^2(\Omega \times \mathbb{R}_+, dP \times \alpha_t^2 dt)$ whose decomposition $u_t = \sum_{n=0}^{\infty} I_n(u_n(*, t))$, $t \in \mathbb{R}_+$, satisfies

$$\sum_{n=0}^{\infty} (n+1)! \|\tilde{u}_n\|^2_{L^2(\mathbb{R}_+, \alpha_t^2 dt)^{\circ(n+1)}} < \infty,$$

where $\tilde{u}_n$ denotes the symmetrization in $n+1$ variables of u_n. Let

$$\delta : \mathrm{Dom}(\delta) \longrightarrow L^2(\Omega, \mathcal{F}, P)$$

be the unbounded divergence operator defined as

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{u}_n), \quad dP - a.e.,$$

if $u \in \mathrm{Dom}(\delta)$ has the expression $u_t = \sum_{n=0}^{\infty} I_n(u_n(*, t))$, $t \in \mathbb{R}_+$. Let $\mathcal{S}$ denote the vector space generated by multiple stochastic integrals of the form $I_n(f_1 \circ \cdots \circ f_n)$, $f_1, \ldots, f_n \in \cap_{p \geq 1} L^p(\mathbb{R}_+, \alpha_t^2 dt)$, $n \in \mathbb{N}$.

Proposition 3 *The operators D and δ have the following properties:*

(i) D and δ are mutually adjoint in the following sense:

$$E[(DF, u)_{L^2(\mathbb{R}_+, \alpha_t^2 dt)}] = E[F \delta(u)], \qquad u \in \mathrm{Dom}(\delta), \quad F \in \mathrm{Dom}(D). \tag{8}$$

(ii) δ coincides with the stochastic integral with respect to $(M_t)_{t \in \mathbb{R}_+}$ on the square-integrable adapted processes $u \in L^2(\Omega \times \mathbb{R}_+, dP \times \alpha_t^2 dt)$:

$$\delta(u) = \int_0^\infty u(t) dM_t.$$

Proof. We first note that for F and u of the form $F = I_n(f_n)$ and $u_t = h_t I_m(g_m)$, (8) follows from the isometry relation (7). Hence by linearity, (8) holds on dense vector subspaces of $L^2(\Omega, \mathcal{F}, P)$ and $L^2(\Omega \times \mathbb{R}_+, dP \times \alpha_t^2 dt)$. As a consequence, D and δ are closable hence (8) holds on $\mathrm{Dom}(D)$ and $\mathrm{Dom}(\delta)$ by density. Concerning (ii) we write

$$u(t) = \sum_{n=0}^{\infty} I_n(u_n(*,t)) = \sum_{n=0}^{\infty} n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} u_n(t_1, \ldots, t_n, t) dM_{t_1} \cdots dM_{t_n},$$

with $u_n(*,t)$ symmetric, $t \in \mathbb{R}_+$. We have $u_n(*,t) = u_n(*,t) 1_{\{* \leq t\}}$, $t \in \mathbb{R}_+$, since u is $\mathcal{F}_t$-adapted, hence the symmetrization $\tilde{u}_n$ of u_n in $n+1$ variables coincides with $\frac{1}{n+1} u_n$ on $\{0 \leq t_1 < \cdots < t_n\}$. Consequently,

$$\begin{aligned}
\delta(u) &= \sum_{n=0}^{\infty} I_{n+1}(\tilde{u}_n) \\
&= \sum_{n=0}^{\infty} \frac{(n+1)!}{n+1} \int_0^\infty \int_0^{t_{n+1}} \cdots \int_0^{t_2} u_n(t_1, \ldots, t_n, t_{n+1}) dM_{t_1} \cdots dM_{t_{n+1}} \\
&= \int_0^\infty \sum_{n=0}^{\infty} n! \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} u_n(t_1, \ldots, t_n, t) dM_{t_1} \cdots dM_{t_n} dM_t \\
&= \int_0^\infty \sum_{n=0}^{\infty} I_n(u_n(*,t)) dM_t = \int_0^\infty u(t) dM_t. \quad \square
\end{aligned}$$

The next result states that $(M_t)_{t \in \mathbb{R}_+}$ has the predictable representation property, as a consequence of the chaos representation property.

Proposition 4 *Any $F \in \mathrm{Dom}(D) \subset L^2(\Omega, \mathcal{F}, P)$ has a representation*

$$F = E[F] + \int_0^\infty E[D_t F \mid \mathcal{F}_t] dM_t. \tag{9}$$

Proof. We write the chaos expansion of F:

$$\begin{aligned}
F &= E[F] + \sum_{n=1}^{\infty} n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM_{t_n} \\
&= E[F] + \sum_{n=1}^{\infty} n \int_0^\infty I_{n-1}(f_n(*,t) 1_{\{* \leq t\}}) dM_t \\
&= E[F] + \int_0^\infty E[D_t F \mid \mathcal{F}_t] dM_t. \quad \square
\end{aligned}$$

This formula is called the Clark formula in the case of Brownian motion, cf. [1], [9]. Combining Prop. 4 with (ii) of Prop. 3 we can also write

$$F = E[F] + \delta(E[D.F \mid \mathcal{F}.]).$$

Instead of the adapted projection $(E[D_tF \mid \mathcal{F}_t])_{t\in\mathbb{R}_+}$ we may also use the predictable projection $(E[D_tF \mid \mathcal{F}_{t^-}])_{t\in\mathbb{R}_+}$ defined by

$$E[D_tI_n(f_n) \mid \mathcal{F}_{t^-}] = I_n(f_n 1_{\{*<t\}}), \quad f_n \in L^2(\mathbb{R}_+, \alpha_s^2 ds)^{\circ n}, \quad t > 0, \ n \geq 1,$$

in fact this leads to the same representation since the adapted and predictable projections coincide in $L^2(\Omega \times \mathbb{R}_+, dP \times \alpha_t^2 dt)$. The following proposition gives the product rule for the operator D, which can be useful in practice for the computation of hedging strategies.

Proposition 5 *We have the product rule*

$$D_t(FG) = FD_tG + GD_tF + \phi_t D_tF D_tG, \qquad t \in \mathbb{R}_+, \tag{10}$$

for any F and G in the vector space $\mathcal{S}$.

Proof. Rewriting the multiplication formula for multiple stochastic integrals with respect to $(X_t)_{t\in\mathbb{R}_+}$, we obtain

$$I_n(f^{\circ n})I_1(g) = I_{n+1}(g \circ f^{\circ n}) + n(f,g)_{L^2(\mathbb{R}_+,\alpha_t^2 dt)} I_{n-1}(f^{\circ(n-1)}) + nI_n((\phi f g) \circ f^{\circ n-1}). \tag{11}$$

Consequently, (10) can be proved by replicating the proof of Prop. 1 in [10]. □

From the duality between D and δ, relation (11) implies

$$F\delta(hF) = \delta(hF) + (h, DF)_{L^2(\mathbb{R}_+,\alpha_t^2 dt)} + \delta(\phi h DF),$$

$h \in \cap_{p\geq 1} L^p(\mathbb{R}_+, \alpha_t^2 dt)$, $F \in \mathcal{S}$, which can be seen as a reformulation of (11).

Remark 1 *Although $D : L^2(\Omega, \mathcal{F}, P) \longrightarrow L^2(\Omega \times \mathbb{R}_+, dP \times \alpha_t^2 dt)$ is unbounded, the representation formula (9) of Prop. 4 can be extended from $F \in \mathrm{Dom}(D)$ to $F \in L^2(\Omega, \mathcal{F}_\infty, P)$ using the fact that the operator $F \mapsto E[D.F \mid \mathcal{F}.]$ has a continuous extension from $L^2(\Omega, \mathcal{F}, P)$ into the space of square-integrable adapted processes, cf. [8], [11].*

Proof. We prove the continuity of the composition of the adapted projection operator with D. Let $F = \sum_{n=0}^{\infty} I_n(f_n) \in Dom(\nabla)$ and $u(\cdot) = \sum_{n=0}^{\infty} I_n(u_n(*, \cdot)) \in L^2(\Omega \times \mathbb{R}_+, dP \times \alpha_t^2 dt)$. Let

$$\Delta_n = \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n \ : \ 0 \leq t_1 < \cdots < t_n\}, \qquad n \geq 1.$$

We have, using the symmetry of f_{n+1}:

$$
\begin{aligned}
&\left|(E[D.F \mid \mathcal{F}_\cdot], u)_{L^2(\Omega\times\mathbb{R}_+, dP\times\alpha_t^2 dt)}\right| \\
&= \left|\sum_{n=0}^\infty (n+1)\int_0^\infty n!(f_{n+1}(*,t)1_{\{*<t\}}, u_n(*,t))_{L^2(\mathbb{R}_+,\alpha_t^2 dt)^{\circ n}}\alpha_t^2 dt\right| \\
&\leq \sum_{n=0}^\infty (n+1)!\left|\int_0^\infty (f_{n+1}(*,t)1_{\{*<t\}}, u_n(*,t))_{L^2(\mathbb{R}_+,\alpha_t^2 dt)^{\circ n}}\alpha_t^2 dt\right| \\
&\leq \sum_{n=0}^\infty (n+1)!\left|\int_0^\infty \|(f_{n+1}(*,t)1_{\{*<t\}}\|_{L^2(\mathbb{R}_+,\alpha_s^2 ds)^{\circ n}}\|u_n(*,t)\|_{L^2(\mathbb{R}_+,\alpha_s^2 ds)^{\circ n}}\alpha_t^2 dt\right| \\
&\leq \sum_{n=0}^\infty (n+1)!\sqrt{n!}\left|\int_0^\infty \|f_{n+1}(*,t)1_{\Delta_n}(*)1_{\{*<t\}}\|_{L^2(\mathbb{R}_+,\alpha_s^2 ds)^{\circ n}}\|u_n(*,t)\|_{L^2(\mathbb{R}_+,\alpha_s^2 ds)^{\circ n}}\alpha_t^2 dt\right| \\
&\leq \sum_{n=0}^\infty (n+1)!\sqrt{n!}\left(\int_0^\infty \|(f_{n+1}(*,t)1_{\Delta_n}(*)1_{\{*<t\}}\|^2_{L^2(\mathbb{R}_+,\alpha_s^2 ds)^{\circ n}}\alpha_t^2 dt\right)^{1/2} \\
&\qquad\qquad \times\left(\int_0^\infty \|u_n(*,t)\|^2_{L^2(\mathbb{R}_+,\alpha_s^2 ds)^{\circ n}}\alpha_t^2 dt\right)^{1/2} \\
&\leq \sum_{n=0}^\infty (n+1)!\sqrt{n!}\|f_{n+1}1_{\Delta_{n+1}}\|_{L^2(\mathbb{R}_+,\alpha_s^2 ds)^{\circ(n+1)}}\|u_n\|_{L^2(\mathbb{R}_+,\alpha_s^2 ds)^{\circ(n+1)}} \\
&\leq \sum_{n=0}^\infty \sqrt{(n+1)!}\sqrt{n!}\|f_{n+1}\|_{L^2(\mathbb{R}_+,\alpha_s^2 ds)^{\circ(n+1)}}\|u_n\|_{L^2(\mathbb{R}_+,\alpha_s^2 ds)^{\circ(n+1)}} \\
&\leq \left(\sum_{n=0}^\infty (n+1)!\|f_{n+1}\|^2_{L^2(\mathbb{R}_+,\alpha_t^2 dt)^{\circ n}}\right)^{1/2}\left(\sum_{n=0}^\infty n!\|u_n\|^2_{L^2(\mathbb{R}_+,\alpha_t^2 dt)^{\circ(n+1)}}\right)^{1/2} \\
&\leq \|F\|_{L^2(\Omega)}\|u\|_{L^2(\Omega\times\mathbb{R}_+,dP\times\alpha_t^2 dt)}. \quad \square
\end{aligned}
$$

5 Changes of probability

Let Q be a probability measure which is equivalent to P. Then

$$
L_t = E\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right], \qquad t \in \mathbb{R}_+, \tag{12}
$$

is a strictly positive $\mathcal{F}_t$-martingale under P. Due to the predictable representation property of $(X_t)_{t\in\mathbb{R}_+}$ and $(M_t)_{t\in\mathbb{R}_+}$ (Prop. 4), there exists a predictable process $(\psi_t)_{t\in\mathbb{R}_+}$ such that

$$
dL_t = L_{t^-}\psi_t dM_t, \qquad t \in \mathbb{R}_+, \quad L_0 = 1.
$$

We have

$$
L_t = \exp\left(\int_0^t i_s\alpha_s\psi_s dB_s - \frac{1}{2}\int_0^t i_s\alpha_s^2\psi_s^2 ds + \int_0^t \lambda_s\psi_s ds\right)\prod_{T_k\leq t}(1+\phi_{T_k}\psi_{T_k}),
$$

where $(T_k)_{k\geq 1}$ denotes the sequence of jump times of $(N_t)_{t\in\mathbb{R}_+}$, hence the process $(\psi_t)_{t\in\mathbb{R}_+}$ satisfies $1+\phi_t\psi_t > 0$, $t \in \mathbb{R}_+$.

Proposition 6 *Under the probability Q, the process*

$$Z_t = M_t - \int_0^t \alpha_s^2 \psi_s ds, \quad t \in \mathbb{R}_+, \tag{13}$$

is a martingale which satisfies the structure equation

$$d[Z,Z]_t = \alpha_t^2 (1 + \phi_t \psi_t) dt + \phi_t dZ_t. \tag{14}$$

In particular, if $(\psi_t)_{t\in\mathbb{R}_+}$ is deterministic, then $(Z_t)_{t\in\mathbb{R}_+}$ has the chaos representation property under Q.

Proof. From the classical Girsanov theorem, cf. [5], the canonical decomposition of $(M_t)_{t\in\mathbb{R}_+}$ under Q is

$$dM_t = \left(dM_t - \frac{1}{L_{t^-}} d\langle L, M\rangle_t \right) + \frac{1}{L_{t^-}} d\langle L, M\rangle_t, \quad t \in \mathbb{R}_+,$$

i.e. $dZ_t = dM_t - \frac{1}{L_{t^-}} d\langle L, M\rangle_t$ is a martingale under Q, with $d\langle L, M\rangle_t = L_{t^-} \alpha_t^2 \psi_t dt$. (Under Q, the process

$$d\tilde{Z}_t = dM_t - \frac{1}{L_t} d[L,M]_t = \frac{1}{1 + \phi_t \psi_t} \left(dM_t - \psi_t \alpha_t^2 dt \right)$$

is also a martingale, since $L_t = L_{t^-}(1 + \phi_t \psi_t)$). We also have

$$dZ_t = i_t \alpha_t dB_t + \phi_t dN_t - \lambda_t \phi_t (1 + \phi_t \psi_t) dt.$$

Under the probability Q,

$$d\tilde{B}_t = dB_t - \frac{1}{L_{t^-}} d\langle L, B\rangle_t = dB_t - i_t \psi_t \alpha_t dt$$

is a standard Brownian motion, and

$$dN_t - \lambda_t dt - \frac{1}{L_{t^-}} d\langle L_t, N_t - \int_0^t \lambda_s ds \rangle = dN_t - \lambda_t (1 + \phi_t \psi_t) dt,$$

is a martingale under Q, i.e. $(N_t)_{t\in\mathbb{R}_+}$ is a Poisson process with intensity $\gamma_t dt$, with

$$\gamma_t = \lambda_t (1 + \phi_t \psi_t), \quad t \in \mathbb{R}_+.$$

Writing

$$dZ_t = i \alpha_t d\tilde{B}_t + \phi_t (dN_t - \lambda_t (1 + \phi_t \psi_t) dt),$$

we have

$$\begin{aligned}
d[Z,Z]_t &= i_t \alpha_t^2 dt + \phi_t^2 dN_t \\
&= i_t \alpha_t^2 dt + \phi_t (dZ_t - i_t \alpha_t d\tilde{B}_t + \lambda_t \phi_t (1 + \phi_t \psi_t) dt) \\
&= i_t \alpha_t^2 dt + (1 - i_t) \alpha_t^2 (1 + \phi_t \psi_t) dt + \phi_t dZ_t \\
&= \alpha_t^2 (1 + \phi_t \psi_t) dt + \phi_t dZ_t.
\end{aligned}$$

From the above discussion, if $(\psi_t)_{t\in\mathbf{R}_+}$ is deterministic, then the process $(Z_t)_{t\in\mathbf{R}_+}$ has the chaos representation property since it is a martingale that satisfies the deterministic structure equation (14).

$\square$

Prop. 6 can be obtained in a different way, applying separately the Girsanov theorem on the Wiener and Poisson spaces. The process $dB_t - i_t\psi_t\alpha_t dt$ is a standard Brownian motion under the probability

$$\exp\left(\int_0^\infty i_s\psi_s\alpha_s dB_s - \frac{1}{2}\int_0^\infty i_s\psi_s^2\alpha_s^2 ds\right) dP.$$

let $(\nu_t^{-1})_{t\in\mathbf{R}_+}$ denote the right-continuous inverse of $(\nu_t)_{t\in\mathbf{R}_+}$. The process $(N_{\nu_t^{-1}})_{t\in\mathbf{R}_+}$ is Poisson with intensity 1 under P, and has intensity $(1+\phi_t\psi_t)dt$ under the probability

$$\lim_{t\to\infty} \exp\left(\int_0^t (\phi\psi)(\nu_s^{-1})ds\right) \prod_{k=1}^{k=N_t} (1 + (\phi\psi)(\nu^{-1}(\nu_{T_k})))dP,$$

where $(\nu_{T_k})_{k\geq 1}$ denotes the jumps of $(N_{\nu_t^{-1}})_{t\in\mathbf{R}_+}$. Hence $(N_t)_{t\in\mathbf{R}_+}$ is a Poisson process with intensity

$$\frac{d}{dt}\left(\nu_t + \int_0^t (1-i_s)\frac{\psi_s\alpha_s^2}{\phi_s} ds\right) = (1+\phi_t\psi_t)\lambda_t dt$$

under the probability Q, and

$$i_t\alpha_t dB_t - i_t\psi_t\alpha_t dt + \phi_t(dN_t - \lambda_t(1+\phi_t\psi_t)dt) = dM_t - \psi_t\alpha_t^2 dt$$

is a martingale under Q.

6 Application : European call

6.1 The model

Let $(\mu_t)_{t\in\mathbf{R}_+}$ be a deterministic process such that $\int_0^t |\mu_s|ds < \infty$, $\forall t \in \mathbb{R}_+$, and let $\sigma : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a deterministic bounded function satisfying $1+\sigma_t\phi_t > 0$, $t\in\mathbb{R}_+$. Let $(S_t)_{t\in\mathbf{R}_+}$ denote the solution of the equation

$$dS_t = \mu_t S_t dt + \sigma_t S_{t^-} dM_t, \quad t\in\mathbb{R}_+,$$

with deterministic initial condition S_0. We have

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \sigma_s\alpha_s i_s dB_s + \int_0^t (\mu_s - \phi_s\lambda_s\sigma_s)ds - \frac{1}{2}\int_0^t i_s\sigma_s^2\alpha_s^2 ds\right) \\ &\quad \times \prod_{k=1}^{k=N_t} (1+\sigma_{T_k}\phi_{T_k}), \quad t\in\mathbb{R}_+, \end{aligned}$$

where $(T_k)_{k\geq 1}$ denotes the jump times of $(N_t)_{t\in\mathbf{R}_+}$. We assume that $(S_t)_{t\in\mathbf{R}_+}$ represents the price of a risky asset, and that $(r_t)_{t\in\mathbf{R}_+}$ is a deterministic non negative function which models the spot rate of a riskless asset.

As it is well known, if there exists a probability Q equivalent to P such that under Q, $\left(S_t \exp\left(-\int_0^t r_s ds\right)\right)_{t\in\mathbf{R}_+}$ is a martingale, the market is arbitrage free. Such probabilities are called equivalent martingale measure (EMM). Moreover, if the market is complete, the EMM is unique.

In the following we check that our model has no arbitrage, more precisely we construct explicitly the probability Q via the Girsanov theorem. Unlike in the continuous case, the law of $(S_t)_{t\in\mathbf{R}_+}$ under Q will also depend on $(\mu_t)_{t\in\mathbf{R}_+}$ by means of the intensity of the jump process.

Proposition 7 *Let us assume that* $1 + (r_t - \mu_t)\phi_t/(\sigma_t\alpha_t^2) > 0$, $t \in \mathbb{R}_+$, *and let* $(\psi_t)_{t\in\mathbf{R}_+}$ *be defined as*

$$\psi_t = \frac{r_t - \mu_t}{\sigma_t\alpha_t^2}, \quad t \in \mathbb{R}_+.$$

Then, the unique EMM is the probability Q *such that* $E\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right] = L_t$, $t \in \mathbb{R}_+$, *where* $dL_t = L_{t^-}\psi_t dM_t$, $t \in \mathbb{R}_+$, $L_0 = 1$.

Proof. We have $dS_t = S_{t^-}\left(\mu_t dt + \sigma_t dZ_t + \sigma_t\psi_t\alpha_t^2 dt\right)$ and

$$\mu_t + \sigma_t\psi_t\alpha_t^2 = r_t, \quad t \in \mathbb{R}_+,$$

hence $dS_t = S_{t^-}(r_t dt + \sigma_t dZ_t)$ where $(Z_t)_{t\in\mathbf{R}_+}$, defined in Prop. 6, is a martingale under Q. $\square$

If $(\psi_t)_{t\in\mathbf{R}_+}$ is defined as in Prop. 7, then

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \sigma_s\alpha_s i_s d\tilde{B}_s + \int_0^t (r_s - \phi_s\lambda_s\sigma_s(1+\phi_s\psi_s))ds - \frac{1}{2}\int_0^t i_s\sigma_s^2\alpha_s^2 ds\right) \\ &\quad \times \prod_{k=1}^{k=N_t}\left(1 + \sigma_{T_k}\phi_{T_k}\right), \quad t \in \mathbb{R}_+. \end{aligned}$$

where $\tilde{B}$ is a standard Q-Brownian motion. In the following, the probability Q will be the equivalent martingale measure constructed from the above proposition.

6.2 Pricing of a call

In order to price a European option we compute $E_Q\left[e^{-TR(T)}(S_T - K)^+\right]$, where $R_T = \frac{1}{T}\int_0^T r_s ds$. The process $(S_t)_{t\in\mathbf{R}_+}$ is a Markov process and the price at time t is given by

$$C(t,x) = E_Q\left[\exp\left(-\int_t^T r_s ds\right)(S_T - K)^+ \mid S_t = x\right].$$

Let

$$\mathrm{BS}(x,T;r,\sigma^2;K) = E[e^{-rT}(xe^{rT-\sigma^2 T/2+\sigma W_T} - K)^+]$$

denote the classical Black-Scholes function, where W_T is a Gaussian centered random variable with variance T. In the case of deterministic volatility $(\sigma_s)_{s\in\mathbb{R}_+}$ and interest rate $(r_s)_{s\in\mathbb{R}_+}$, the price of a call in the Black-Scholes model is $\mathrm{BS}(x,T;R_T,\frac{1}{T}\Sigma_T;K)$ with $\Sigma_T = \int_0^T \sigma_s^2 ds$. Let $\Gamma_t^\sigma = \int_0^t i_s\alpha_s^2\sigma_s^2 ds$ denote the variance of $\int_0^t i_s\alpha_s\sigma_s dB_s$, $t\in\mathbb{R}_+$, and $\Gamma_t = \int_0^t \gamma_s ds$, $t\in\mathbb{R}_+$, denote the intensity of $(N_t)_{t\in\mathbb{R}_+}$ under Q, where $\gamma_t = \lambda_t(1+\phi_t\psi_t)$, $t\in\mathbb{R}_+$.

Proposition 8 *The expectation* $E_Q\left[\exp\left(-\int_0^T r_s ds\right)(S_T-K)^+\right]$ *can be computed as*

$$\begin{aligned}
E_Q\left[\exp\left(-\int_0^T r_s ds\right)(S_T-K)^+\right] &= \exp\left(-\Gamma_T\right)\sum_{k=0}^\infty \frac{1}{k!}\int_0^T\cdots\int_0^T \\
& BS\left(S_0\exp\left(-\int_0^T \phi_s\gamma_s\sigma_s ds\right)\prod_{i=1}^{i=k}\left(1+\sigma_{t_i}\phi_{t_i}\right), T; R_T, \frac{\Gamma_T^\sigma}{T}; K\right) \\
& \times\gamma_{t_1}\cdots\gamma_{t_k} dt_1\cdots dt_k.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
& E_Q\left[\exp(-TR_T)(S_T-K)^+\right] \\
&= \sum_{k=0}^\infty E_Q\left[\exp(-TR_T)(S_T-K)^+ \mid N_T = k\right] Q(N_T = k),
\end{aligned}$$

with $Q(N_T = k) = \exp(-\Gamma_T)(\Gamma_T)^k/k!$, $k\in\mathbb{N}$. Conditionally to $\{N_T = k\}$, the jump times $(T_1,\ldots,T_n)$ have the law

$$\frac{1}{(\Gamma_T)^k}1_{\{0<t_1<\cdots<t_k<T\}}\gamma_{t_1}\cdots\gamma_{t_k}dt_1\cdots dt_k,$$

since the process $(N_{\Gamma_t^{-1}})_{t\in\mathbb{R}_+}$ is a standard Poisson process. Hence, conditionally to $\{N(\Gamma^{-1}(\Gamma_T)) = k\} = \{N_T = k\}$, its jump times $(\Gamma_{T_1},\ldots,\Gamma_{T_k})$ have a uniform law on $[0,\Gamma_T]^k$. We then use the fact that $(\tilde{B}_t)_{t\in\mathbb{R}_+}$ and $(N_t)_{t\in\mathbb{R}_+}$ are also independent under Q since $(\mu_t)_{t\in\mathbb{R}_+}$ is deterministic, and the identity in law

$$S_T \overset{law}{=} S_0 X_T\exp\left(-\int_0^T \phi_s\lambda_s(1+\phi_s\psi_s)\sigma_s ds\right)\prod_{k=1}^{k=N_T}\left(1+\sigma_{T_k}\phi_{T_k}\right),$$

where

$$X_T = \exp\left(TR_T - \Gamma_T^\sigma/2 + \left(\frac{\Gamma_T^\sigma}{T}\right)^{1/2} W_T\right),$$

and W_T is independent of N.

$\square$

6.3 The hedging strategy

An important problem is to determine the hedging strategy, i.e. to compute the processes (θ_t^0, θ_t^1) such that

$$C(t, S_t) = \theta_t^0 \exp\left(\int_0^t r_s ds\right) + \theta_t^1 S_t,$$

and

$$dC(t, S_t) = \theta_t^0 \exp\left(\int_0^t r_s ds\right) r_t dt + \theta_t^1 dS_t$$

(self-financing condition). It suffices to compute $(\theta_t^1)_{t\in\mathbb{R}_+}$ such that

$$d\left[\exp\left(-\int_0^t r_s ds\right) C(t, S_t)\right] = \theta_t^1 d\left[\exp\left(-\int_0^t r_s ds\right) S_t\right] \tag{15}$$

We shall note $\tilde{S}_t = \exp\left(-\int_0^t r_s ds\right) S_t$, $t \in \mathbb{R}_+$, the discounted price process.
In the following proposition we still denote by D the gradient operator defined relatively to the multiple stochastic integrals with respect to $(Z_t)_{t\in\mathbb{R}_+}$ instead of $(M_t)_{t\in\mathbb{R}_+}$, ($(Z_t)_{t\in\mathbb{R}_+}$ also has the chaos representation property because $(\alpha_t^2(1+\phi_t\psi_t))_{t\in\mathbb{R}_+}$ is deterministic). The hedging strategy is computed using the gradient D from the representation formula (9).

Proposition 9 *If $(r_s)_{s\in\mathbb{R}_+} = 0$, the hedging strategy is given by*

$$\theta_t^1 = \frac{1}{\sigma_t S_t} E[D_t(S_T - K)^+ \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+.$$

Proof. Let $F = (S_T - K)^+$. We have $dZ_t = (\sigma_t S_{t^-})^{-1} dS_t$, hence

$$\begin{aligned} F &= E[F] + \int_0^\infty E[D_t F \mid \mathcal{F}_t] dZ_t \\ &= E[F] + \int_0^T \frac{1}{\sigma_t S_{t^-}} E[D_t F \mid \mathcal{F}_t] dS_t, \end{aligned}$$

since F is $\mathcal{F}_T$-measurable.

$\square$

From Remark 1, the only regularity property that should be assumed on $F = (S_T - K)^+$ is its square-integrability, and this method can be applied to any square integrable pay-off F.

As in the standard Black-Scholes model, it is possible to determine the hedging strategy in terms of the "delta" of the price in the case $(r_t)t \in \mathbb{R}_+$ is deterministic.
An application of Itô's lemma leads to

$$\begin{aligned} dC(t,S_t) &= \left[\frac{\partial C}{\partial t} + r_t S_t \frac{\partial C}{\partial x} + \frac{1}{2}\frac{\partial^2 C}{\partial x^2} i_t \alpha_t^2 S_t^2 \sigma_t^2 + \lambda_t(1+\phi_t\psi_t)\Theta C\right](t,S_t)dt \\ &\quad + S_t\sigma_t \frac{\partial C}{\partial x}(t,S_t)dZ_t + \Theta C(t,S_t)\left[dN_t - \lambda_t(1+\phi_t\psi_t)dt\right] \end{aligned} \tag{16}$$

where

$$\Theta C(t,S_t) = C(t,S_t(1+\sigma_t\phi_t)) - C(t,S_t) - \frac{\partial C}{\partial x}(t,S_t)S_t\sigma_t\phi_t.$$

The process $\widetilde{C}_t = C(t,S_t)\exp\left(-\int_0^t r_s ds\right)$ is a Q-martingale, and $d\widetilde{C}_t = \theta_t^1 d\widetilde{S}_t$; therefore, by identification of (15) and (16)

$$\begin{aligned} r_t C(t,S_t) &= \left[\frac{\partial C}{\partial t} + r_t S_t \frac{\partial C}{\partial x} + \frac{1}{2} i_t \alpha_t^2 S_t^2 \sigma_t^2 \frac{\partial^2 C}{\partial x^2} + \lambda_t(1+\phi_t\psi_t)\Theta C\right](t,S_t), \\ \theta_t^1 S_t \sigma_t dZ_t &= S_t\sigma_t \frac{\partial C}{\partial x}(t,S_t)dZ_t + \Theta C(t,S_t)\left[dN_t - \lambda_t(1+\phi_t\psi_t)dt\right]. \end{aligned}$$

Therefore, by identification of the Brownian and Poisson parts,

$$\begin{cases} \theta_t^1 S_t \sigma_t i_t = S_t \sigma_t \dfrac{\partial C}{\partial x}(t,S_t) i_t \\[2ex] \theta_t^1 S_t \sigma_t \phi_t = S_t \sigma_t \dfrac{\partial C}{\partial x}(t,S_t)\phi_t + \Theta C(t,S_t). \end{cases} \tag{17}$$

On $\{t \in \mathbb{R}_+ \ : \ \phi_t = 0\} = \{t \ : \ i(t) = 1\}$, the term $\Theta C(t,S_t)$ vanishes. Therefore, (17) reduces to

$$\theta_t^1 = \frac{\partial C}{\partial x}(t,S_t) + \frac{\Theta C(t,S_t)}{S_t\sigma_t\phi_t}.$$

The process $(\theta_t^1)_{t\in\mathbb{R}_+}$ is equal to the usual "delta" on the set $\{t \in \mathbb{R}_+ \ : \ i_t = 1\}$ and to $\dfrac{C(t,S_t(1+\phi_t\sigma_t)) - C(t,S_t)}{S_t\phi_t\sigma_t}$ on the set $\{t \in \mathbb{R}_+ \ : \ i_t = 0\}$.
The usual PDE for the price of a call splits into two parts. On $\{t : \phi(t) = 0\}$, we obtain the usual PDE

$$\frac{\partial C}{\partial t}(t,x) + r_t x \frac{\partial C}{\partial x}(t,x) + \frac{1}{2}\alpha_t^2 x^2 \sigma_t^2 \frac{\partial^2 C}{\partial x^2}(t,x) = r_t C(t,x),$$

whereas on the set $\{t \in \mathbb{R}_+ \ : \ \phi_t \neq 0\}$ we have

$$\frac{\partial C}{\partial t}(t,x) + r_t x \frac{\partial C}{\partial x}(t,x) + \lambda_t(1+\phi_t\psi_t)\Theta C(t,x) = r_t C(t,x).$$

Each equation has to be solved using a terminal condition, which is computed by backward induction from the condition at maturity $C(T,x) = (x-K)^+$.

6.4 The greeks

The price of the European call is still convex and increasing with respect to the value of the underlying asset. Here a new "greek" parameter has to be defined to make precise the dependence w.r.t. the drift of the asset, under the historical probability, i.e. $\Upsilon = \frac{\partial C}{\partial \mu}(t,x)$ in the case where this drift is constant.

However, it turns out that the sign of this term depends on all the parameters of the model.

7 Conclusion

The predictable representation property and the existence of a EMM give to this model the main properties for mahematical finance's purpose. For example, it is possible to solve a consumption/investment problem, and to use the backward stochastic differential equation tools. It remains to find a method to fit the parameters to some data. The continuous part of the driving martingale can have an arbitrary (deterministic) quadratic variation. Concerning the discontinuous part, its jumps as well as its intensity can be independently chosen as deterministic functions of time. This allows to model different types of transitions from continuous to discontinuous paths, for example starting with small jumps and great intensity.

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