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Université de La Rochelle
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Knut Aase, Bernt Øksendal, Nicolas Privault et Jan Ubøe

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KNUT AASE[1),2)], BERNT ØKSENDAL[2),1)], NICOLAS PRIVAULT[3)], AND JAN UBØE[4)]

[1)]Norwegian School of Economics and Business Administration
Helleveien 30, N-5035 Bergen - Sandviken, Norway. e-mail: knut.aase@nhh.no

Corresponding author: Bernt Øksendal

[2)]Department of Mathematics, University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway. e-mail: oksendal@math.uio.no

[3)]Department of Mathematics, Université de la Rochelle, Avenue Marillac
F-17042 La Rochelle Cedex 1, France. e-mail: nicolas.privault@univ-lr.fr

[4)]Stord/Haugesund College, Skåregaten 103, N-5 500, Haugesund, Norway.
e-mail: jan.uboe@hsh.no

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[1)]Norwegian School of Economics and Business Administration
Helleveien 30, N-5035 Bergen - Sandviken, Norway. e-mail: knut.aase@nhh.no

[2)]Department of Mathematics, University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway.
e-mail: oksendal@math.uio.no

[3)]Department of Mathematics, Université de la Rochelle, Avenue Marillac
F-17042 La Rochelle Cedex 1, France. e-mail: nicolas.privault@univ-lr.fr

[4)]Stord/Haugesund College, Skåregaten 103, N-5 500, Haugesund, Norway.
e-mail: jan.uboe@hsh.no

ABSTRACT. We use a white noise approach to Malliavin calculus to prove the following white noise generalization of the Clark-Haussmann-Ocone formula

$$F(\omega) = E[F] + \int_0^T E[D_t F|\mathcal{F}_t] \diamond W(t)dt$$

Here $E[F]$ denotes the generalized expectation, $D_t F(\omega) = \frac{dF}{d\omega}$ is the (generalized) Malliavin derivative, $\diamond$ is the Wick product and $W(t)$ is 1-dimensional Gaussian white noise. The formula holds for all $f \in \mathcal{G}^* \supset L^2(\mu)$, where $\mathcal{G}^*$ is a space of stochastic distributions and μ is the white noise probability measure. We also establish similar results for multidimensional Gaussian white noise, for multidimensional Poissonian white noise and for combined Gaussian and Poissonian noise. Finally we give an application to mathematical finance: We compute the replicating portfolio for a European call option in a Poissonian Black & Scholes type market.

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1. Introduction

Let $B_t(\omega) = B(t,\omega); t \geq 0, \omega \in \Omega$ be a 1-dimensional Wiener process (Brownian motion) on a probability space $(\Omega, \mathcal{F}, P)$ such that $B(0,\omega) = 0$ a.s. P. For $t \geq 0$ let $\mathcal{F}_t$ be the σ-algebra generated by $\{B(s,\cdot); s \leq t\}$. Fix $T > 0$. The *Clark-Haussmann-Ocone (CHO) theorem* states that if $F = F(\omega) \in L^2(P)$ is $\mathcal{F}_T$-measurable and $F \in \mathbb{D}_{1,2}$ (see definitions below), then

$$F(\omega) = E[F] + \int_0^T E[D_t F|\mathcal{F}_t](\omega) dB_t(\omega) \tag{1.1}$$

where $D_t F = \frac{dF}{d\omega}(t)$ denotes the Malliavin derivative of F at t. This result and its generalizations have important applications in economics, where (basically) $E[D_t F|\mathcal{F}_t]$ represents the replicating portfolio of a given T-claim F. (See, e.g., [K-O], [Ø])

Usually this result is presented and proved in the context of analysis on the Wiener space $\Omega = C_0([0,T])$, the space of all real continuous functions on $[0,T]$ starting at 0. Then one can identify each Wiener process path $B(\cdot,\omega)$ with one element $\omega(\cdot) \in \Omega$, which is a computational advantage. It is in this setting that the Malliavin derivative and its properties are usually studied. See, e.g., [N], [U]. However, the drawback with this setting is that the Malliavin derivative only exists for $F \in \mathbb{D}_{1,2}$. This excludes many interesting applications. For example, in mathematical finance one is interested in computing the replicating portfolios of a given T-claim F. If, say, the claim is a digital option of the form

$$F(\omega) = \mathcal{X}_{[K,\infty)}(B_T(\omega)) = \begin{cases} 1 & \text{if } B_T(\omega) \geq K \\ 0 & \text{if } B_T(\omega) < K \end{cases} \tag{1.2}$$

then D_tF does not exist and formula (1.1) cannot be applied. The purpose of this paper is to present a new proof of the Clark-Haussmann-Ocone formula in the setting of white noise analysis. One of the advantages with this approach is that it allows a generalization of the Clark-Haussmann-Ocone formula which is valid for all $\mathcal{F}_T$-measurable $F \in \mathcal{G}^*$, a space of stochastic distributions which contains $L^2(\mu)$, where μ is the white noise probability measure (μ corresponds to P in the Wiener space setting). The generalization has the form (See Theorem 3.15)

$$F(\omega) = E[F] + \int_0^T E[D_tF|\mathcal{F}_t] \diamond W_t dt \tag{1.3}$$

where $\diamond$ denotes the Wick product and $W_t \in (\mathcal{S})^*$ is white noise. $E[F]$ is the generalized expectation of $F \in \mathcal{G}^*$, $E[D_tF|\mathcal{F}_t]$ is the generalized expectation and the integral on the right hand side is an $(\mathcal{S})^*$-valued (Bochner) integral. In view of the identity

$$\int_0^T Y(t,\omega)\delta B(t) = \int_0^T Y(t,\omega) \diamond W_t dt \tag{1.4}$$

valid for all Skorohod integrable processes $Y(t,\omega)$, we see that (1.3) is indeed a generalization of (1.1). In fact, if $F \in L^2(\mu)$ then (1.3) simplifies to

$$F(\omega) = E[F] + \int_0^T E[D_tF|\mathcal{F}_t] dB(t) \tag{1.5}$$

where $D_tF \in \mathcal{G}^*, E[D_tF|\mathcal{F}_t] \in L^2(\mu)$ for a.a. t and

$$E\Big[\int_0^T E[D_tF|\mathcal{F}_t]^2 dt\Big] < \infty \qquad \text{(Theorem 3.11)} \tag{1.6}$$

We emphasize that in the Wiener space setting another generalization of (1.1) has been obtained by S. Ustunel [U, Theorem 1 p.44]. His generalization is valid for all $F \in \mathbb{D}_{-\infty}$, the *Meyer-Watanabe distributions*. Since $\mathbb{D}_{-\infty} \subsetneq \mathcal{G}^*$, our result is more general. Moreover, our approach is entirely different. Recently other approaches to the Malliavin calculus and the Clark-Haussmann-Ocone theorem have been given by F. E. Benth [B], M. de Faria, M. J. Oliveira & L. Streit [dO-S], and G. Våge [V].

Our white noise setup can be easily modified to cover more general situations. This is demonstrated in Sections 4-6. In Section 4 we prove the following multidimensional version of the generalized Clark-Haussmann-Ocone theorem:

Let $B(t,\omega) := (B_1(t,\omega_1),\ldots,B_m(t,\omega_m)); \omega = (\omega_1,\ldots,\omega_m) \in \Omega$ be m-dimensional Brownian motion with filtration $\mathcal{F}_T^{(m)}$. Then if $F \in (\mathcal{G}^*)^m$ is $\mathcal{F}_T^{(m)}$-measurable, we have

$$F(\omega) = E[F] + \int_0^T \left(\sum_{j=1}^m E\left[\frac{\partial F}{\partial \omega_j}(t,\omega) \,\middle|\, \mathcal{F}_T^{(m)} \right] \diamond W_j(t,\omega_j) \right) dt \tag{1.7}$$

where we have used the notation $\left(\frac{\partial F}{\partial \omega_1},\ldots,\frac{\partial F}{\partial \omega_m}\right)$ for the Malliavin gradient of F at t (Theorem 4.2).

If we replace the *Gaussian* white noise probability measure μ by the *Poissonian* white noise probability measure ν (on the same underlying space $S'(\mathbb{R})$), then we obtain a similar theory where Gaussian white noise $W(t)$ is replaced by Poissonian white noise $V(t)$ and Brownian motion $B(t)$ is replaced by compensated Poisson process $Q(t)$. The spaces $\mathcal{G}^* = \mathcal{G}^*(\nu)$ can be defined in a similar way as for the Gaussian case and the Malliavin gradient too. Thus we obtain the following generalized Clark-Haussmann-Ocone theorem for the compensated Poisson process:

If $F \in \mathcal{G}^*(\nu_m)$ is measurable with respect to the filtration $\mathcal{H}_t^{(m)}$ of the m-dimensional compensated Poisson process $Q(t)$, then

$$F(\omega) = E[F] + \int_0^T \left(\sum_{j=1}^m E\left[\frac{\partial F}{\partial \omega_j}(t,\omega) \,\Big|\, \mathcal{H}_T^{(m)} \right] \diamond V_j(t,\omega_j) \right) dt \tag{1.8}$$

This result is proved in Section 5 (See Theorem 5.3). We also point out how the above theory can be modified to cover the case with combinations of Gaussian and Poissonian noises. Finally, in Section 6 we apply our results to compute the replicating portfolios for the European call option in a Poisson Black and Scholes type market: Consider a market $X(t) = (A(t), S(t))$ consisting of two investment possibilities:

i) *a bank account*, where the price $A(t)$ at time t is given by

$$dA(t) = \rho(t)A(t)dt \;;\; A(0) = 1 \tag{1.9}$$

ii) *a stock*, where the price $S(t)$ at time t is given by

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dQ(t) \;;\; S(0) = x > 0 \tag{1.10}$$

where $\rho(t), \mu(t)$, and $\sigma(t)$ are *deterministic* functions in $L^2[0,T]$ ($T > 0$ constant), $\sigma(t) \geq \epsilon$ for some $\epsilon > 0$, and where $Q(t) = P(t) - t$ is the compensated Poisson process. Let $(\xi(t,\omega), \eta(t,\omega))$ denote the *portfolio*, i.e., $\xi(t), \eta(t)$ gives the number of units of investments #1, #2, respectively, held by an agent at time t.

Consider $u(t) = \frac{\mu(t)-\rho(t)}{\sigma(t)}$, and find a new measure $\tilde{\nu}$ such that $\tilde{Q}(t) := \int_0^t u(s)ds + Q(t)$ is a compensated Poisson process with respect to $\tilde{\nu}$. We will prove the following theorem.

THEOREM 6.1

The price $V(0)$ of a European call option with payoff $F(\omega) = (S(T) - K)^+$ in the Poissonian market defined by (1.9), (1.10) and satisfying $u(t) \leq 1 - \epsilon_1$ for some $\epsilon_1 > 0$, is given by $V(0) = e^{-\int_0^T \rho(s)ds} E_{\tilde{\nu}}[F]$. Moreover, the replicating portfolio for this claim is given by

$$\begin{aligned} \eta(t) = &\frac{1}{\sigma(t)S(t)} e^{-\int_t^T \rho(s)ds} E_{\tilde{\nu}}[(\sigma(t)Y^y(T-t) - (K - Y^y(T-t))^+) \\ &\cdot \mathcal{X}_{[K/(1+\sigma(t)),\infty)}(Y^y(T-t))]_{y=S(t)} \end{aligned} \tag{1.11}$$

where

$$Y^y(t) = y \exp\left[\int_0^t \ln[1+\sigma(s)]d\tilde{Q}(s) + \int_0^t (\mu(s) - \sigma(s) + \ln[1+\sigma(s)](1-u(s)))ds \right] \tag{1.12}$$

and

$$\xi(t) = \frac{V(t) - \eta(t)S(t)}{A(t)} \tag{1.13}$$

2. Background from white noise theory

In this section we briefly recall some of the basic concepts and results from Gaussian white noise theory. Our presentation and notation will follow that of [H-Ø-U-Z] closely. More information about white noise analysis can be found in [H-K-P-S].

Let $\Omega = S'(\mathbb{R})$ be the space of tempered distributions on the set $\mathbb{R}$ of real numbers and let μ be the *Gaussian white noise probability measure* on Ω defined (in virtue of the Bochner-Minlos theorem) by the property

$$\int_\Omega e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}||\phi||^2} \tag{2.1}$$

for all $\phi \in S(\mathbb{R})$ (the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$), where $||\phi||^2 = \int_\mathbb{R} \phi^2(x)dx$ and $< \omega, \phi >$ denotes the action of $\omega \in S'(\mathbb{R})$ on ϕ. ($S'(\mathbb{R})$ is the dual of $S(\mathbb{R})$). Note that if $\omega \in L^2(\mathbb{R})$, then $< \omega, \phi >= \int_\mathbb{R} \omega(x)\phi(x)dx$. From (2.1) we deduce that

$$E[< \cdot, \phi >] = 0 \text{ and } E[< \cdot, \phi >^2] = ||\phi||^2 \ ; \ \phi \in S(\mathbb{R}) \tag{2.2}$$

where $E = E_\mu$ denotes expectation with respect to μ. This isometry allows us to define a Brownian motion $B(t) = B(t,\omega)$ as the continuous version of $\tilde{B}(t) = \tilde{B}(t,\omega) = < \omega, \mathcal{X}_{[0,t]}(\cdot) >$ (which exist in $L^2(\mu)$) where

$$\mathcal{X}_{[0,t]}(s) = \begin{cases} 1 & \text{if } 0 \le s \le t \\ -1 & \text{if } -t \le s \le 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

Then $< \omega, \psi >= \int_\mathbb{R} \psi(t)dB(t)$ for all $\psi \in L^2(\mathbb{R})$. We let $\mathcal{F}_t$ denote the σ algebra generated by $\{B(s,\cdot)\}_{0\le s\le t}$. If $f(t_1,\dots,t_n) \in \hat{L}^2(\mathbb{R}^n)$, i.e., f_n is symmetric and $||f_n||^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f_n^2(t_1,\dots,t_n)dt_1\cdots dt_n < \infty$, then we can define the *iterated Itô integral*

$$\int_{\mathbb{R}^n} f_n dB^{\otimes n} := n! \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{t_n} \cdots \Big(\int_{-\infty}^{t_2} f(t_1,\dots,t_n)dB(t_1) \Big) \cdots \Big) dB(t_n) \tag{2.4}$$

In the following we let

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2/2} \right) \ ; \ n = 0,1,2,\dots \tag{2.5}$$

be the *Hermite polynomials* and we let $\{\xi_n\}_{n=1}^\infty$ be the basis of $L^2(\mathbb{R})$ consisting of the *Hermite functions*

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-x^2/2} h_{n-1}(\sqrt{2}\,x) \ ; \ n = 1,2,\dots \tag{2.6}$$

Several parts of the theory can be carried out without the explicit use of this particular basis. In some cases, however, the choice of an explicit basis is important. For example, explicit estimates of $\xi_n(t)$ are needed to deduce that the Gaussian white noise $W(t,\cdot)$ defined in (3.12) belongs to the space $(S)^*$ of Hida distributions.

If $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ are two functions, we let $f \otimes g : \mathbb{R}^{m+n} \to \mathbb{R}$, resp. $f \hat{\otimes} g : \mathbb{R}^{m+n} \to \mathbb{R}$, be the *tensor product*, resp. *symmetric tensor product* of f and g (see [H-Ø-U-Z, Section 2.2]). Similarly we let $f^{\otimes n} = f \otimes \cdots \otimes f$ (n times) be the tensor powers of f. The set of multi-indices $\alpha = (\alpha_1,\dots,\alpha_m)$ of nonnegative integers is denoted by $\mathcal{I} = (\mathbb{N}_0^\mathbb{N})_c$.

Here $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $z = (z_1, z_2, \ldots)$ is a sequence of numbers or functions, we use the multi-index notation

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \text{ if } \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathcal{I}$$

We list some of the basic results we need:

THEOREM 2.1

(Itô 1951 [I]) Suppose $\psi_1, \ldots, \psi_n$ are orthonormal functions in $L^2(\mathbb{R})$. Then for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{I}$, we have

$$\int_{\mathbb{R}^{|\alpha|}} \psi^{\hat{\otimes}\alpha}(x) dB^{\otimes|\alpha|}(x) = h_{\alpha_1}(< \omega, \psi_1 >) \cdots h_{\alpha_n}(< \omega, \psi_n >)$$

COROLLARY 2.2

$$\int_{\mathbb{R}^{|\alpha|}} \xi^{\hat{\otimes}\alpha}(x) dB^{\otimes|\alpha|}(x) = H_\alpha(\omega)$$

where $H_\alpha(\omega) = h_{\alpha_1}(< \omega, \xi_1 >) \cdots h_{\alpha_n}(< \omega, \xi_n >), \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{I}$.

COROLLARY 2.3

Let $\diamond$ denote the Wick product, defined by

$$(H_\alpha \diamond H_\beta)(\omega) = H_{\alpha+\beta}(\omega) \; ; \; \alpha, \beta \in \mathcal{I}$$

and extended linearly ([HØUZ, Def.2.4.1]). Then if $f_n \in \hat{L^2}(\mathbb{R}^n), g_n \in \hat{L^2}(\mathbb{R}^m)$, we have

$$\left(\sum_n \int_{\mathbb{R}^n} f_n dB^{\otimes n}\right) \diamond \left(\sum_n \int_{\mathbb{R}^n} g_n dB^{\otimes n}\right) = \sum_{m,n} \int_{\mathbb{R}^{m+n}} f_n \hat{\otimes} g_m dB^{\otimes(m+n)}$$

PROOF

$$\int_{\mathbb{R}^{|\alpha|}} \xi^{\hat{\otimes}\alpha} dB^{\otimes|\alpha|} \diamond \int_{\mathbb{R}^{|\beta|}} \xi^{\hat{\otimes}\beta} dB^{\otimes|\beta|} = H_\alpha \diamond H_\beta = H_{\alpha+\beta}$$
$$= \int_{\mathbb{R}^{|\alpha+\beta|}} \xi^{\hat{\otimes}(\alpha+\beta)} dB^{\otimes|\alpha+\beta|} = \int_{\mathbb{R}^{|\alpha+\beta|}} \xi^{\hat{\otimes}\alpha} \hat{\otimes} \xi^{\hat{\otimes}\beta} dB^{\otimes|\alpha+\beta|}$$

using Corollary 2.2.

□

COROLLARY 2.4

$< \omega, \phi >^{\diamond n} = ||\phi||^n h_n\left(\frac{<\omega,\phi>}{||\phi||}\right)$, where $< \omega, \phi >^{\diamond n} = < \omega, \phi > \diamond \cdots \diamond < \omega, \phi >$ (n times)

PROOF

$$< \omega, \phi >^{\diamond n} = \Big(\int_{\mathbb{R}} \phi dB\Big)^{\diamond n} = \int_{\mathbb{R}^n} \phi^{\otimes n} dB^{\otimes n}$$
$$= ||\phi||^n \int_{\mathbb{R}^n} \Big(\frac{\phi}{||\phi||}\Big)^{\otimes n} dB^{\otimes n} = ||\phi||^n h_n\Big(\frac{< \omega, \phi >}{||\phi||}\Big)$$

by Corollary 2.2.

□

Recall that

$$E\Big[\int_{\mathbb{R}} u(s, \omega) dB(s) | \mathcal{F}_t\Big] = \int_0^t u(s, \omega) dB(s)$$

From this we deduce that (see [HØUZ,p 47])

LEMMA 2.5

$$E\Big[\int_{\mathbb{R}^n} f_n dB^{\otimes n} | \mathcal{F}_t\Big] = \int_{\mathbb{R}^n} f_n \mathcal{X}_{[0,t]^n} dB^{\otimes n}$$

STOCHASTIC TEST FUNCTIONS AND STOCHASTIC DISTRIBUTIONS

The following spaces of stochastic test functions, $\mathcal{G} = \mathcal{G}(\mu)$, and stochastic distributions $\mathcal{G}^* = \mathcal{G}^*(\mu)$, have been studied by J. Potthoff and M. Timpel. See [P-T] and the references therein.

DEFINITION 2.6

(i) Let $k \in \mathbb{N}$. We say that $f(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n dB^{\otimes n}$ belongs to the space $\mathcal{G}_k(\mu)$ if

$$||f||^2_{\mathcal{G}_k} := \sum_{n=0}^{\infty} n! ||f_n||^2_{L^2(\mathbb{R}^n)} e^{2kn} < \infty \tag{2.7}$$

We define

$$\mathcal{G}(\mu) = \mathcal{G} = \bigcap_{k \in \mathbb{N}} \mathcal{G}_k$$

and equip $\mathcal{G}$ with the projective topology.

(ii) We say that a formal expansion

$$G = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} g_n dB^{\otimes n}$$

belongs to the space $\mathcal{G}_{-q}(\mu)$ ($q \in \mathbb{N}$) if

$$||G||^2_{\mathcal{G}_{-q}} := \sum_{n=0}^{\infty} n! ||f_n||^2_{L^2(\mathbb{R}^n)} e^{-2qn} < \infty \tag{2.8}$$

We define

$$\mathcal{G}^*(\mu) = \mathcal{G}^* = \bigcup_{q \in \mathbb{N}} \mathcal{G}_{-q}$$

and equip $\mathcal{G}^*$ with the inductive topology. Note that $\mathcal{G}^*$ is the dual of $\mathcal{G}$, with action

$$< G, f >= \sum_{n=0}^{\infty} n! \int_{\mathbb{R}^n} f_n(x) g_n(x) dx \tag{2.9}$$

if $G \in \mathcal{G}^*$ and $f \in \mathcal{G}$ is as above. Also note that the connection between the expansion

$$G(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} g_n dB^{\otimes n} \tag{2.10}$$

and the Hermite expansion

$$G(\omega) = \sum_{\alpha} c_\alpha H_\alpha(\omega) \tag{2.11}$$

is given by

$$g_n(x) = \sum_{|\alpha|=n} c_\alpha \xi^{\hat{\otimes}\alpha}(x) \; ; \; n = 0, 1, 2, \ldots \tag{2.12}$$

(see [HØUZ, (2,2.33)]). Since this gives

$$||\int_{\mathbb{R}^n} g_n dB^{\otimes n}||^2_{L^2(\mu)} = n!||g_n||^2_{L^2(\mathbb{R}^n)} = \sum_{|\alpha|=n} c_\alpha^2 \alpha! \tag{2.13}$$

it follows that we can express the $\mathcal{G}_r$-norm of G in terms of the Hermite expansion as follows:

$$||G||^2_{\mathcal{G}_r} = \sum_{n=0}^{\infty} \Big(\sum_{|\alpha|=n} c_\alpha^2 \alpha! \Big) e^{2rn} \; ; \; r \in \mathbb{Z} \tag{2.14}$$

Two important properties of these spaces are the following (see [P-T] for a proof):

$$f, g \in \mathcal{G} \;\Rightarrow\; f \diamond g \in \mathcal{G} \tag{2.15}$$

and

$$f, g \in \mathcal{G}^* \;\Rightarrow\; f \diamond g \in \mathcal{G}^* \tag{2.16}$$

We also recall the *Hida stochastic test function space* $(\mathcal{S})$ and the *Hida stochastic distribution space* $(\mathcal{S})^*$ as follows:

For a formal expansion $f(\omega) = \sum_{\alpha \in \mathcal{I}} c_\alpha H_\alpha(\omega)$ define the norm

$$||f||^2_{0,k} := \sum_{\alpha \in \mathcal{I}} c_\alpha^2 \alpha! (2\mathbb{N})^{k\alpha} \text{ for } k \in \mathbb{Z} \text{ (the integers)} \tag{2.17}$$

where $(2\mathbb{N})^{k\alpha} = (2 \cdot 1)^{k\alpha_1} (2 \cdot 2)^{k\alpha_2} \cdots (2m)^{k\alpha_m}$ if $\alpha = (\alpha_1, \ldots, \alpha_m)$. Let

$$(\mathcal{S})_{0,k} := \{f; \; ||f||_{0,k} < \infty\} \tag{2.18}$$

and define

$$(\mathcal{S}) := \bigcap_{k \in \mathbb{N}} (\mathcal{S})_{0,k} \tag{2.19}$$

with projective topology, and

$$(\mathcal{S})^* := \bigcup_{q \in \mathbb{N}} (\mathcal{S})_{0,-q} \tag{2.20}$$

with inductive topology. Note that

$$(\mathcal{S}) \subset \mathcal{G} \subset L^2(\mu) \subset \mathcal{G}^* \subset (\mathcal{S})^*$$

For more information about these and related spaces see [H-K-P-S] and [H-Ø-U-Z]. We now extend the concept of conditional expectation to $\mathcal{G}^*$. This has been done in a different context by Benth and Potthoff [B-P].

DEFINITION 2.7

Let $F = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n dB^{\otimes n} \in \mathcal{G}^*$. Then the conditional expectation of F with respect to $\mathcal{F}_t$ is defined by

$$E[F|\mathcal{F}_t] = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n \cdot \mathcal{X}_{[0,t]^n} dB^{\otimes n} \tag{2.21}$$

Note that this coincides with the usual conditional expectation if $F \in L^2(\mu)$. Also note that

$$||E[F|\mathcal{F}_t]||_{\mathcal{G}_r} \leq ||F||_{\mathcal{G}_r} \text{ for all } r \in \mathbb{Z} \tag{2.22}$$

In particular

$$E[F|\mathcal{F}_t] \in \mathcal{G}^* \text{ for all } t \tag{2.23}$$

LEMMA 2.8

Let $F, G \in \mathcal{G}^*$. Then

$$E[F \diamond G|\mathcal{F}_t] = E[F|\mathcal{F}_t] \diamond E[G|\mathcal{F}_t]$$

PROOF

We may assume, without loss of generality, that $F = \int_{\mathbb{R}^n} f_n dB^{\otimes n} = \sum_{|\alpha|=n} c_\alpha \int_{\mathbb{R}^n} \xi^{\hat{\otimes}\alpha} dB^{\otimes n}$ and similarly with G. Then Corollary 2.3 and Definition 2.7 give

$$\begin{aligned} E[F \diamond G|\mathcal{F}_t] &= E\Big[\int_{\mathbb{R}^{m+n}} f_n \hat{\otimes} g_m dB^{\otimes(m+n)} |\mathcal{F}_t\Big] \\ &= \int_{\mathbb{R}^{m+n}} f_n \hat{\otimes} g_m \cdot \mathcal{X}_{[0,t]^{m+n}} dB^{\otimes(m+n)} \\ &= \int_{\mathbb{R}^{m+n}} f_n \cdot \mathcal{X}_{[0,t]^n} \hat{\otimes} g_m \cdot \mathcal{X}_{[0,t]^m} dB^{\otimes(m+n)} \\ &= E[F|\mathcal{F}_t] \diamond E[G|\mathcal{F}_t] \end{aligned}$$

□

From now on we will use the following notation:

As before let $\xi_1, \xi_2, \ldots$ be the Hermite functions, and put

$$X_i = X_i(\omega) = <\omega, \xi_i> = \int_{\mathbb{R}} \xi_i(s) dB(s) \; ; \; i = 1, 2, \ldots \tag{2.24}$$

and

$$X_i^{(t)}(\omega) = \int_0^t \xi_i(s) dB(s) \; ; \; i = 1, 2, \ldots \tag{2.25}$$

and

$$X = (X_1, X_2, \ldots), \; X^{(t)} = (X_1^{(t)}, X_2^{(t)}, \ldots)$$

Note that with this notation we have

$$X^{\diamond\alpha}(\omega) = (X_1^{\diamond\alpha_1} \diamond \ldots \diamond X_m^{\diamond\alpha_m})(\omega) = H_\alpha(\omega) \qquad \text{if } \alpha = (\alpha_1, \ldots, \alpha_m) \tag{2.26}$$

for all multi-indices α.

COROLLARY 2.9

If $F = \sum_\alpha c_\alpha X^{\diamond\alpha} \in \mathcal{G}^*$, then

$$E[F|\mathcal{F}_t] = \sum_\alpha c_\alpha (X^{(t)})^{\diamond\alpha}$$

PROOF

We know that $E[X_i|\mathcal{F}_t] = X_i^{(t)}$. Now apply Lemma 2.8.

□

DEFINITION 2.10

We say that $F \in \mathcal{G}^*$ is $\mathcal{F}_T$-measurable if

$$E[F|\mathcal{F}_T] = F$$

COROLLARY 2.11

$F \in \mathcal{G}^*$ is $\mathcal{F}_T$-measurable if and only if F can be written

$$F = \sum_\alpha c_\alpha (X^{(T)})^{\diamond\alpha} \quad \textit{(convergence in } \mathcal{G}^*)$$

for some numbers c_α.

REMARK

A general discussion of $\mathcal{G}$ and $\mathcal{G}^*$ and the related spaces $\mathcal{G}^\beta$ and $\mathcal{G}^{-\beta}$ can be found in [G-K-S].

3. The 1-dimensional case

Let us first recall the definition of a directional derivative:

DEFINITION 3.1

Let $F : S'(\mathbb{R}) \to \mathbb{R}$ be a given function and let $\gamma \in S'$. Then we say that F has a *directional derivative in the direction* γ if

$$D_\gamma F(\omega) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(F(\omega + \epsilon\gamma) - F(\omega) \right) \tag{3.1}$$

exists in$(S)^*$. If this is the case, we call $D_\gamma F$ the *directional* (or *Gateaux*) derivative of F in the direction γ.

EXAMPLE 3.2

Let $F(\omega) = \int_\mathbb{R} \phi(t) dB(t) = < \omega, \phi >$ for some $\phi \in L^2(\mathbb{R})$. Then, if $\gamma \in L^2(\mathbb{R})$, we have

$$D_\gamma F(\omega) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (< \omega + \epsilon\gamma, \phi > - < \omega, \phi >) = < \gamma, \phi > = \int_\mathbb{R} \gamma(t)\phi(t)dt \tag{3.2}$$

PROOF

By linearity we have $< \omega + \epsilon\gamma, \phi > = < \omega, \phi > + \epsilon < \gamma, \phi >$. Hence

$$\frac{1}{\epsilon} (< \omega + \epsilon\gamma, \phi > - < \omega, \phi >) = < \gamma, \phi >$$

□

Motivated by this example we make the following definitions:

DEFINITION 3.3

A function $Z(t) : \mathbb{R} \to (S)^*$ is $(S)^*$-integrable if

$$< Z(t), f > \in L^1(\mathbb{R}) \quad \text{for all } f \in (S) \tag{3.3}$$

Then the $(S)^*$-integral of $Z(t)$, denoted by $\int_\mathbb{R} Z(t)dt$, is the unique $(S)^*$-element such that

$$< \int_\mathbb{R} Z(t)dt, f > = \int_\mathbb{R} < Z(t), f > dt \; ; \; f \in (S) \tag{3.4}$$

It is a consequence of Proposition 8.1 in [H-K-P-S] that (3.4) defines $\int_{\mathbb{R}} Z(t)dt$ as an element of $(S)^*$.

DEFINITION 3.4

We say that $F : S'(\mathbb{R}) \to \mathbb{R}$ is *differentiable* if there exists a map $K : \mathbb{R} \to (S)^*$ such that

$$K(t,\omega)\gamma(t) \text{ is } (S)^*\text{-integrable}$$

and

$$D_\gamma F(\omega) = \int_{\mathbb{R}} K(t,\omega)\gamma(t)dt \quad \text{for all } \gamma \in L^2(\mathbb{R}) \tag{3.5}$$

In this case we put

$$D_t F(\omega) := \frac{dF}{d\omega}(t,\omega) := K(t,\omega)\ ;\ t \in \mathbb{R} \tag{3.6}$$

The set of all differentiable $F : S' \to \mathbb{R}$ is denoted by $\mathbb{D}$.

EXAMPLE 3.5

If $F(\omega) =< \omega, \phi >$ as in Example 3.2, then we see by (3.2) that F is differentiable and

$$D_t F(\omega) = \phi(t) \tag{3.7}$$

If

$$P(x) = \sum_\alpha c_\alpha x^\alpha\ ;\ x \in \mathbb{R}^{\mathbb{N}}, c_\alpha \in \mathbb{R} \tag{3.8}$$

is a polynomial, we define its Wick version at $X = (X_1, \ldots, X_m)$ by

$$P^\diamond(X) = \sum_\alpha c_\alpha X^{\diamond\alpha} \tag{3.9}$$

Such (finite) sums are called *(stochastic) polynomials.*

LEMMA 3.6 (THE CHAIN RULE I)

Let $P(x) = \sum_\alpha c_\alpha x^\alpha$ *be a polynomial in* n *variables* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. *Then* $P(X) \in \mathbb{D}, P^\diamond(X) \in \mathbb{D}$ *and*

$$D_t P(X) = \sum_{i=1}^n \frac{\partial P}{\partial x_i}(X_1, \ldots, X_n)\xi_i(t) = \sum_\alpha c_\alpha \sum_i \alpha_i X^{\alpha-\epsilon^{(i)}}\xi_i(t) \tag{3.10}$$

and

$$D_t P^\diamond(X) = \sum_{i=1}^n \frac{\partial P}{\partial x_i}^\diamond (X_1, \ldots, X_n)\xi_i(t) = \sum_\alpha c_\alpha \sum_i \alpha_i X^{\diamond(\alpha-\epsilon^{(i)})}\xi_i(t) \tag{3.11}$$

Here and in the following, $\epsilon^{(i)} = (0,0,\ldots,1)$ *is the i'th unit vector (with 1 on entry i).*

PROOF

Put $F(\omega) = P(X(\omega))$ and choose $\gamma \in L^2(\mathbb{R})$. Then by the classical chain rule we have

$$\begin{aligned} &\frac{1}{\epsilon}\left(F(\omega+\epsilon\gamma) - F(\omega)\right) = \frac{1}{\epsilon}\left(P(X(\omega+\epsilon\gamma)) - P(X(\omega))\right) \\ =&\frac{1}{\epsilon}\left(P(X(\omega) + \epsilon < \gamma, \xi_i >) - P(X(\omega))\right) \\ \to& \sum_{i=1}^n \frac{\partial P}{\partial x_i}(X(\omega)) < \gamma, \xi_i > \quad \text{in } L^2(\mu),\ \text{as } \epsilon \to 0 \end{aligned}$$

since

$$E_\mu[|X(\omega)|^N] < \infty \ \text{ for all } N \in \mathbb{N}$$

We conclude that

$$D_\gamma F(\omega) = \int_{\mathbb{R}} \left(\sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(X(\omega)) \cdot \xi_i(t) \right) \gamma(t)dt$$

Similarly, let $G(\omega) = P^\diamond(X) = X^{\diamond\alpha}(\omega)$ for some $\alpha \in \mathcal{I}$, and choose $\gamma \in L^2(\mathbb{R})$. Then

$$\begin{aligned}
&\frac{1}{\epsilon}(G(\omega+\epsilon\gamma) - G(\omega)) = \frac{1}{\epsilon}(X^{\diamond\alpha}(\omega+\epsilon\gamma) - X^{\diamond\alpha}(\omega)) \\
&= \frac{1}{\epsilon}(H_\alpha(\omega+\epsilon\gamma) - H_\alpha(\omega)) = \frac{1}{\epsilon}\left(\prod_i h_{\alpha_i}(<\omega+\epsilon\gamma, \xi_i>) - \prod_i h_{\alpha_i}(<\omega, \xi_i>)\right) \\
&\overset{\epsilon\to 0}{\longrightarrow} \sum_i h'_{\alpha_i}(<\omega,\xi_i>) \prod_{j:j\neq i} h_{\alpha_j}(<\omega,\xi_j>)\cdot <\gamma,\xi_i> \\
&= \sum_i \alpha_i h_{\alpha_i-1}(<\omega,\xi_i>) \prod_{j:j\neq i} h_{\alpha_j}(<\omega,\xi_j>)\cdot <\gamma,\xi_i> \\
&= \sum_i \alpha_i X^{\diamond(\alpha-\epsilon^{(i)})}(\omega) <\gamma,\xi_i>,
\end{aligned}$$

where we have used the well known property

$$h'_m(x) = m h_{m-1}(x); \ m = 1, 2, \ldots$$

for Hermite polynomials.

□

Let

$$W(t,\omega) = \sum_{i=1}^{\infty} \xi_i(t) H_{\epsilon^{(i)}}(\omega) \tag{3.12}$$

be *Gaussian white noise.* We have $W(t,\cdot) \in (\mathcal{S})^*$ for all t and

$$\int_0^T X(t,\omega)\delta B(t) = \int_0^T X(t,\omega) \diamond W(t,\omega)dt \text{ (integration in } (\mathcal{S})^*) \tag{3.13}$$

for all Skorohod integrable processes $X(t)$. See [HØUZ, Theorem 2.5.9] and the references therein. Note that if X_j is as in (2.24), then

$$t \mapsto X_j(t)$$

is differentiable in $(\mathcal{S})^*$ and

$$\frac{d}{dt}\Big(\int_0^t \xi_j(s)dB(s)\Big) = \frac{d}{dt}\Big(\int_0^t \xi_j(s)W(s)ds\Big) = \xi_j(t)W(t) \in (\mathcal{S})^*$$

By induction this gives

LEMMA 3.7 (THE CHAIN RULE II)
Let

$$P(x) = \sum_\alpha c_\alpha x^\alpha$$

be a polynomial in $\mathbb{R}^n$*. Let*

$$X_j = \int_{\mathbb{R}} \xi_j(s)dB(s)$$

be as in (2.24). Then

$$\frac{d}{dt}P^{\diamond}(X^{(t)}) = \sum_{j=1}^{n} \left(\frac{\partial P}{\partial x_j}\right)^{\diamond} (X^{(t)}) \diamond \xi_j(t)W(t)$$

LEMMA 3.8 (THE CLARK-OCONE FORMULA FOR POLYNOMIALS)

Let $F(\omega)$ *be* $\mathcal{F}_T$*-measurable and suppose* $F(\omega) = P^{\diamond}(X)$ *for some polynomial* $P(x) = \sum_\alpha c_\alpha x^\alpha, X = (X_1, \dots, X_n)$ *with* $X_j =< \omega, \xi_j >$ *as in (3.9). Then*

(i) $F(\omega) = P^{\diamond}(X^{(T)})$ *where* $X_j^{(T)} =< \omega, \xi_j \cdot \mathcal{X}_{[0,T]} >$

and

(ii) $F(\omega) = E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dB(t)$

PROOF

We have $F(\omega) = E[F|\mathcal{F}_T] = P^{\diamond}(X^{(T)})$ by Corollary 2.9. Hence by Lemma 3.6 and Lemma 3.7

$$\int_0^T E[D_tF|\mathcal{F}_t]dB(t) = \int_0^T E\Big[\sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} (X)\xi_i(t)|\mathcal{F}_t\Big]dB(t)$$

$$= \int_0^T \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} (X^{(t)})\xi_i(t) \diamond W(t)dt$$

$$= \int_0^T \frac{d}{dt}P^{\diamond}(X^{(t)})dt = \Big|_0^T P^{\diamond}(X^{(t)}) = P^{\diamond}(X^{(T)}) - P^{\diamond}(X^{(0)})$$

$$= F - P^{\diamond}(0) = F - E[F]$$

□

We proceed to consider a Clark-Haussmann-Ocone formula for the space $L^2(\mu)$. Suppose that $F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in \mathcal{G}^*$. Then as noted in (2.26), we may write

$$F(\omega) = \sum_\alpha c_\alpha X^{\diamond\alpha}(\omega)$$

Hence F is a limit in $\mathcal{G}^*$ of stochastic polynomials. So in view of Lemma 3.6, it is natural to make the following definition:

DEFINITION 3.9

Let $F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in \mathcal{G}^*$. Then we define the *stochastic derivative* of F at t by

$$\begin{aligned} D_tF(\omega) &:= \frac{dF}{d\omega}(t,\omega) \\ &:= \sum_\alpha c_\alpha \sum_i \alpha_i H_{\alpha-\epsilon^{(i)}}(\omega) \cdot \xi_i(t) \\ &= \sum_\beta \Big(\sum_i c_{\beta+\epsilon^{(i)}}(\beta_i+1)\xi_i(t)\Big)H_\beta(\omega) \end{aligned} \tag{3.14}$$

REMARK

The stochastic derivative is also called the *Hida derivative* or - in the context of the Wiener space - the *Malliavin derivative*.

The following result is crucial:

LEMMA 3.10

a) *Suppose* $F \in \mathcal{G}^*$. *Then* $D_t F \in \mathcal{G}^*$ *for a.a.* $t \in \mathbb{R}$.

b) *Suppose* $F, F_m \in \mathcal{G}^*$ *for all* $m \in \mathbb{N}$ *and*

$$F_m \to F \text{ in } \mathcal{G}^*$$

Then there exists a subsequence $\{F_{m_k}\}_{k=1}^{\infty}$ *such that*

$$D_t F_{m_k} \to D_t F \text{ in } \mathcal{G}^*, \text{ for a.a. } t > 0$$

PROOF

a) Suppose $F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in \mathcal{G}^*$. Then

$$D_t F(\omega) = \sum_\alpha c_\alpha \sum_i \alpha_i H_{\alpha - \epsilon^{(i)}}(\omega) \cdot \xi_i(t)$$
$$= \sum_\beta \Big(\sum_i c_{\beta+\epsilon^{(i)}} (\beta_i + 1) \xi_i(t) \Big) H_\beta(\omega) =: \sum_\beta g_\beta(t) H_\beta(\omega)$$

where

$$g_\beta(t) = \sum_i c_{\beta+\epsilon^{(i)}} (\beta_i + 1) \xi_i(t)$$

Choose $q < \infty$ s.t. $||F||^2_{\mathcal{G}_{-q}} := \sum_m \sum_{|\alpha|=m} c_\alpha^2 \alpha! e^{-2qm} < \infty$ (see (2.8)). We will prove that

$$||D_t F||^2_{\mathcal{G}_{-q-1}} := \sum_n \Big(\sum_{|\beta|=n} g_\beta^2 \beta! \Big) e^{-(q+1)n} < \infty \text{ for a.a. } t$$

Note that

$$\int_{\mathbb{R}} g_\beta^2(t) dt = \int_{\mathbb{R}} \Big(\sum_i c_{\beta+\epsilon^{(i)}} (\beta_i + 1) \xi_i(t) \Big)^2 dt = \sum_i c^2_{\beta+\epsilon^{(i)}} (\beta_i + 1)^2$$

So

$$\sum_{|\beta|=n} \Big(\int_{\mathbb{R}} g_\beta^2 dt \Big) \beta! = \sum_{|\beta|=n} \Big(\sum_i c^2_{\beta+\epsilon^{(i)}} (\beta_i + 1)(\beta + \epsilon^{(i)})! \Big)$$
$$\leq \sum_{|\beta|=n} (n+1) \sum_i c^2_{\beta+\epsilon^{(i)}} (\beta_i + 1)(\beta + \epsilon^{(i)})! \leq (n+1) \sum_{|\alpha|=|\beta|+1} c_\alpha^2 \alpha!$$

Hence using the fact that $(n+1)e^{-n} \leq 1$ for all n, we get

$$\int_{\mathbb{R}} ||D_t F||^2_{\mathcal{G}_{-q-1}} dt = \int_{\mathbb{R}} \sum_n \Big(\sum_{|\beta|=n} g_\beta^2 \beta! \Big) e^{-2(q+1)n} dt$$
$$\leq \sum_n (n+1) \Big(\sum_{|\alpha|=|\beta|+1} c_\alpha^2 \alpha! \Big) e^{-2(q+1)n} \tag{3.15}$$
$$\leq \sum_n \Big(\sum_{|\alpha|=|\beta|+1} c_\alpha^2 \alpha! \Big) e^{-2qn} \leq ||F||^2_{\mathcal{G}_{-q}} < \infty$$

Therefore

$$||D_tF||^2_{\mathcal{G}_{-q-1}} < \infty \text{ for a.a. } t$$

So

$$D_tF \in \mathcal{G}_{-q-1} \subset \mathcal{G}^* \text{ for a.a. } t$$

b) It suffices to prove that if $G_m \to 0$ in $\mathcal{G}_{-q}$, then there exist a subsequence $\{G_{m_k}\}_{k=1}^{\infty}$ such that $D_tG_{m_k} \to 0$ in $\mathcal{G}^*$ as $k \to \infty$, for a.a. t. By (3.15) we see that $||D_tG_m||_{\mathcal{G}_{-q-1}} \to 0$ in $L^2(\mathbb{R})$. So there exist a subsequence $\{||D_tG_{m_k}||_{\mathcal{G}_{-q-1}}\}_{k=1}^{\infty}$ such that

$$||D_tG_{m_k}||_{\mathcal{G}_{-q-1}} \to 0 \text{ for a.a. } t \text{ as } k \to \infty \tag{3.16}$$

Hence $D_tG_{m_k} \to 0$ in $\mathcal{G}^*$ for a.a. t as $k \to \infty$. The last part follows from (2.22).

□

THEOREM 3.11(THE CLARK-HAUSSMANN-OCONE THEOREM FOR $L^2(\mu)$)
Let λ denote Lebesgue measure on $\mathbb{R}$. Let $F(\omega) \in L^2(\mu)$ be $\mathcal{F}_T$-measurable. Then

$$(t,\omega) \mapsto E[D_tF|\mathcal{F}_t](\omega) \in L^2(\lambda \times \mu)$$

and

$$F(\omega) = E[F] + \int_0^T E[D_tF|\mathcal{F}_t]dB(t)$$

PROOF
Let $F(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha(\omega)$ be the chaos expansion of F and put

$$F_n(\omega) = \sum_{\alpha\in\mathcal{I}_n} c_\alpha H_\alpha(\omega) = \sum_{\alpha\in\mathcal{I}_n} C_\alpha X^{\diamond\alpha}$$

where $\mathcal{I}_n = \{\alpha \in \mathcal{I}\,;\ |\alpha| \le n \text{ \& length } (\alpha) \le n\}$. Then by Lemma 3.8, we have

$$F_n(\omega) = E[F_n] + \int_0^T E[D_tF_n|\mathcal{F}_t]dB(t) \text{ for all } n \tag{3.17}$$

By the Itô representation theorem we know that there is a unique $u(t,\omega)$ which is $\mathcal{F}_t$-adapted and such that

$$E\Big[\int_0^T u^2(t,\omega)dt\Big] < \infty$$

and such that

$$F(\omega) = E[F] + \int_0^T u(t,\omega)dB(t) \tag{3.18}$$

Since $F_n \to F$ in $L^2(\mu)$, we conclude that

$$E\Big[\int_0^T (E[D_tF_n|\mathcal{F}_t] - u(t,\omega))^2\,dt\Big]$$
$$= E[(F_n - F - E[F_n] + E[F])^2] \to 0 \text{ as } n \to \infty$$

So

$$E[D_tF_n|\mathcal{F}_t] \to u(t,\omega) \text{ in } L^2(\lambda \times \mu)$$

On the other hand, by Lemma 3.10 b), we know that, by taking a subsequence,

$$E[D_tF_n|\mathcal{F}_t] \to E[D_tF|\mathcal{F}_t] \text{ in } \mathcal{G}^* \text{ for a.a. } t$$

By taking another subsequence, we obtain that

$$E[D_tF_n|\mathcal{F}_t] \to u(t,\omega) \text{ in } L^2(\mu) \text{ for a.a. } t$$

We conclude that

$$u(t,\omega) = E[D_tF|\mathcal{F}_t] \text{ for a.a. } t$$

and the proof is complete.

□

We proceed to prove a Clark-Haussmann-Ocone theorem for $\mathcal{G}^*$. First we establish some auxiliary results:

LEMMA 3.12
Let $F \in \mathcal{G}_{-q} \subset (\mathcal{S})^*$ and $f \in (\mathcal{S})$. Then, with $||f||_{0,\hat{q}}$ as in (2.17),

$$| < F, f > | \leq ||F||_{\mathcal{G}_{-q}} \cdot ||f||_{0,\hat{q}} \,;\; \hat{q} = \frac{2q}{\ln 2}$$

PROOF
Suppose $F(\omega = \sum_\alpha a_\alpha H_\alpha(\omega)$, $f(\omega = \sum_\beta a_\beta H_\beta(\omega)$. Then

$$\begin{aligned} | < F, f > | &= | \sum_\alpha a_\alpha b_\alpha \alpha! | \\ =| \sum_m \Big(\sum_{|\alpha|=m} a_\alpha b_\alpha \alpha! \Big) | &\leq \sum_m \Big(\sum_{|\alpha|=m} a_\alpha^2 \alpha! \Big)^{\frac{1}{2}} \Big(\sum_{|\alpha|=m} b_\alpha^2 \alpha! \Big)^{\frac{1}{2}} \\ \leq \Big(\sum_m \Big(\sum_{|\alpha|=m} a_\alpha^2 \alpha! \Big) e^{-2qm} \Big)^{\frac{1}{2}} &\Big(\sum_m \Big(\sum_{|\alpha|=m} b_\alpha^2 \alpha! \Big) e^{2qm} \Big)^{\frac{1}{2}} \\ \leq ||F||_{\mathcal{G}_{-q}} \Big(\sum_\alpha b_\alpha \alpha! (2\mathbb{N})^{\hat{q}\alpha} \Big)^{\frac{1}{2}} &= ||F||_{\mathcal{G}_{-q}} \cdot ||f||_{0,\hat{q}} \end{aligned}$$

□

LEMMA 3.13
Let $F \in \mathcal{G}, f \in (\mathcal{S})$. Then

$$\int_{\mathbb{R}} < E[D_tF|\mathcal{F}_t], f >^2 dt < \infty$$

PROOF
By Lemma 3.12 and (3.15), we have

$$\begin{aligned} \int_{\mathbb{R}} < E[D_tF|\mathcal{F}_t], f >^2 dt &\leq \int_{\mathbb{R}} ||E[D_tF|\mathcal{F}_t]||^2_{\mathcal{G}_{-p}} ||f||^2_{0,\hat{p}} dt \\ \leq ||f||^2_{0,\hat{p}} \int_{\mathbb{R}} ||D_tF||^2_{\mathcal{G}_{-p}} dt &< \infty \text{ for some } p \in \mathbb{N} \end{aligned} \tag{3.19}$$

□

LEMMA 3.14

Suppose $F_n, F \in \mathcal{G}^*$ and $F_n \to F$ in $(S)^*$ as $n \to \infty$. Then, as $n \to \infty$

$$\int_0^T E[D_t F_n|\mathcal{F}_t] \diamond W(t)dt \to \int_0^T E[D_t F|\mathcal{F}_t] \diamond W(t)dt \quad in\ (S)^* \tag{3.20}$$

PROOF

Note that both integrals in (3.20) exist by Lemma 3.13. Moreover, by (3.19) and (3.15) we also get for $f \in (S)$, for some $p \in \mathbb{N}$

$$\begin{aligned}
&| < \int_0^T E[D_t F_n|\mathcal{F}_t] \diamond W(t)dt - \int_0^T E[D_t F|\mathcal{F}_t] \diamond W(t)dt, f > | \\
&= \int_0^T | < E[D_t(F_n - F)|\mathcal{F}_t], f > |dt \le \sqrt{T}||f||_{0,\hat{p}} \Big(\int_0^T ||D_t(F_n - F)||^2_{\mathcal{G}_{-p}} dt \Big)^{\frac{1}{2}} \\
&\le \sqrt{T}||f||_{0,\hat{p}} ||F_n - F||_{\mathcal{G}_{-p+1}} \to 0 \ \text{ as } n \to \infty
\end{aligned}$$

Since this holds for all $f \in (S)$, (3.20) follows.

□

THEOREM 3.15 (THE CLARK-HAUSSMANN-OCONE THEOREM FOR $\mathcal{G}^*$)

Let $F(\omega) \in \mathcal{G}^*$ be $\mathcal{F}_T$-measurable. Then

$$D_t F \in \mathcal{G}^* \quad and \quad E[D_t F|\mathcal{F}_t] \in \mathcal{G}^* \quad for\ a.a.\ t$$

$E[D_t F|\mathcal{F}_t] \diamond W(t)$ is integrable in $(S)^*$ and

$$F(\omega) = E[F] + \int_0^T E[D_t F|\mathcal{F}_t] \diamond W(t)dt \tag{3.21}$$

where $E[F]$ denotes the generalized expectation of F.

PROOF

Let $F_n(\omega) = \sum_{\alpha \in \mathcal{I}_n} c_\alpha H_\alpha(\omega)$ be as in the proof of Theorem 3.11. Then by Lemma 3.8, we have

$$F_n(\omega) = E[F_n] + \int_0^T E[D_t F_n|\mathcal{F}_t] \diamond W(t)dt$$

for all n. Therefore

$$F(\omega) = E[F] + \lim_{n\to\infty} \int_0^T E[D_t F_n|\mathcal{F}_t] \diamond W(t)dt$$

The limit must exist in $\mathcal{G}^*$ and hence in $(S)^*$. The result then follows from Lemma 3.14.

□

EXAMPLE 3.16

Let $\phi : [0,T] \to \mathbb{R}$ be a deterministic function such that $||\phi||^2_{[0,T]} := \int_0^T \phi^2(s)ds < \infty$. Define

$$Y(t) = \int_0^t \phi(s)dB_s, \ 0 \le t \le T \tag{3.22}$$

Then the Donsker delta function $\delta_{Y(T)}(\cdot) : \mathbb{R} \to \mathcal{G}^*$, see [A-Ø-U], is given by the expression

$$\delta_{Y(T)}(y) = \frac{1}{\sqrt{2\pi ||\phi||^2_{[0,T]}}} \cdot \exp^{\diamond}\left[-\frac{(y - Y(T))^{\diamond 2}}{2||\phi||^2_{[0,T]}} \right] \tag{3.23}$$

In this case the generalized expectation is given by

$$E[\delta_{Y(T)}(y)] = \frac{1}{\sqrt{2\pi ||\phi||^2_{[0,T]}}} \cdot \exp\left[-\frac{y^2}{2||\phi||^2_{[0,T]}} \right] \tag{3.24}$$

and from the chain rule we get

$$\begin{aligned} &E[D_t \delta_{Y(T)}(y)|\mathcal{F}_t] \\ =&\frac{1}{\sqrt{2\pi ||\phi||^2_{[0,T]}}} \cdot \exp^{\diamond}\left[-\frac{(y - Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}} \right] \diamond \frac{y - Y(t)}{||\phi||^2_{[0,T]}} \diamond \phi(t) \end{aligned} \tag{3.25}$$

Using (3.24) and (3.25) in (3.21), we get a Clark-Haussmann-Ocone formula for this particular Donsker delta function. Moreover, the formula can be integrated to get explicit formulas for T-claims of the form $f(Y(T))$ where $f : \mathbb{R} \to \mathbb{R}$ is bounded and measurable, see the results in [A-Ø-U].

4. The multidimensional Gaussian case

The framework for multidimensional Gaussian white noise theory is based on the following construction:

Let μ be the Gaussian white noise probability measure on $S'(\mathbb{R})$, as defined in Section 2. Fix a natural number m and put

$$\Omega := S'(\mathbb{R}) \times \cdots \times S'(\mathbb{R}) \;\; (m \text{ factors})$$

and

$$\mu_m = \mu \times \cdots \times \mu \;\; (m \text{ factors}) \tag{4.1}$$

For $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$ and $\phi = (\phi_1, \ldots, \phi_m) \in \mathcal{S} := \mathcal{S}(\mathbb{R}) \times \cdots \times \mathcal{S}(\mathbb{R})$ we put

$$< \omega, \phi >= \sum_{i=1}^{m} < \omega_i, \phi_i > \tag{4.2}$$

Then we see that

$$\int_{\Omega} e^{i<\omega,\phi>} d\mu_m(\omega) = e^{-\frac{1}{2}||\phi||^2} \tag{4.3}$$

where $||\phi||^2 = \sum_{i=1}^{m} ||\phi_i||^2_{L^2(\mathbb{R})}$ if $\phi = (\phi_1, \ldots, \phi_m) \in \mathcal{S}$. As in the case $m = 1$, we get that

$$\tilde{B}(t,\omega) = (\tilde{B}_1(t,\omega_1), \ldots, \tilde{B}_m(t,\omega_m)) = (< \omega_1, \mathcal{X}_{[0,t]} >, \ldots, < \omega_m, \mathcal{X}_{[0,t]} >)$$

has a continuous version $B(t,\omega)$, which will be an m-dimensional Brownian motion on $\mathbb{R}$. Let $\mathcal{I}^m = \mathcal{I} \times \cdots \times \mathcal{I}$ (m factors) be the set of all m-tuples $\Gamma = (\gamma^{(1)}, \ldots, \gamma^{(m)})$ of multi-indices $\gamma^{(j)} \in \mathcal{I}$. For $\Gamma \in \mathcal{I}^m$ define

$$\mathbb{H}_{\Gamma} = \mathbb{H}^{(m)}_{\Gamma}(\omega) = \prod_{i=1}^{m} H_{\gamma^{(i)}} \tag{4.4}$$

Then the family $\{\mathbb{H}_\Gamma\}_{\Gamma\in\mathcal{J}^m}$ constitutes an orthogonal basis for $L^2(\mu_m)$ and

$$||\mathbb{H}_\Gamma||^2_{L^2(\mu_m)} = \Gamma! = \gamma^{(1)}!\cdots\gamma^{(m)}! \tag{4.5}$$

(see [HØUZ, Theorem 2.2.3]). We now proceed to define the spaces $\mathcal{G}$ and $\mathcal{G}^*$ using $\mathbb{H}_\Gamma$: If $G(\omega) = \sum_{\Gamma\in\mathcal{J}^m} c_\Gamma \mathbb{H}_\Gamma(\omega)$ and $r \in \mathbb{Z}$, define

$$||G||^2_{\mathcal{G}_r} = ||G||^2_{\mathcal{G}_r(\mu_m)} := \sum_{n=0}^{\infty}\Big(\sum_{|\Gamma|=n} c_\Gamma^2\Gamma!\Big)e^{2rn} \tag{4.6}$$

and

$$\mathcal{G}_r = \mathcal{G}_r(\mu_m) = \{G\,;\ ||G||_{\mathcal{G}_r} < \infty\} \tag{4.7}$$

Then put

$$\mathcal{G} = \mathcal{G}(\mu_m) = \bigcap_{r\in\mathbb{N}} \mathcal{G}_r \text{ with projective topology} \tag{4.8}$$

and

$$\mathcal{G}^* = \mathcal{G}^*(\mu_m) = \bigcup_{r\in\mathbb{N}} \mathcal{G}_{-r} \text{ with inductive topology} \tag{4.9}$$

The Hida spaces $(\mathcal{S}) = (\mathcal{S})_m$ and $(\mathcal{S})^* = (\mathcal{S})^*_m$ are defined similarly. The Wick product is defined by

$$\Big(\sum_{\Gamma\in\mathcal{J}^m} a_\Gamma\mathbb{H}_\Gamma\Big)\diamond\Big(\sum_{\Lambda\in\mathcal{J}^m} b_\Lambda\mathbb{H}_\Lambda\Big) = \sum_{\Gamma,\Lambda} a_\Gamma b_\Lambda \mathbb{H}_{\Gamma+\Lambda} \tag{4.10}$$

One verifies that (2.15) and (2.16) still holds in this setting. The *multidimensional Gaussian white noise* is defined by

$$W(t,\omega) = (W_1(t,\omega_1),\ldots,W_m(t,\omega_m)) \tag{4.11}$$

where

$$W_j(t,\omega_j) = \sum_{k=0}^{\infty}\xi_k(t)H_{\epsilon^{(k)}}(\omega_j) \tag{4.12}$$

(similar to (3.12). The *directional derivative* of $F:\Omega\to\mathbb{R}$ in the direction $\gamma = (\gamma_1,\ldots,\gamma_m) \in (L^2(\mathbb{R}))^m$ is defined by

$$D_\gamma F(\omega) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(F(\omega+\epsilon\gamma) - F(\omega)) \text{ (limit in } (\mathcal{S})^*_m) \tag{4.13}$$

We say that $F:\Omega\to\mathbb{R}$ is *differentiable* if there exists a map $K = (K_1,\ldots,K_m):\mathbb{R}\to((\mathcal{S})^*)^m$ such that

$$D_\gamma F(\omega) = \int_{\mathbb{R}} K(t,\omega)\cdot\gamma(t)dt \text{ for all } \gamma\in(L^2(\mathbb{R}))^m \tag{4.14}$$

where $K(t,\omega)\cdot\gamma(t) = \sum_{j=1}^m K_j(t,\omega)\gamma_j(t)$. If this is the case, we put

$$\frac{\partial F}{\partial\omega_i}(t,\omega) = K_i(t,\omega)\,;\ 1\le i\le m \tag{4.15}$$

and we call the vector

$$K(t,\omega) := \Big(\frac{\partial F}{\partial\omega_1}(t,\omega),\ldots,\frac{\partial F}{\partial\omega_m}(t,\omega)\Big) \tag{4.16}$$

the *stochastic gradient* of F (at t). The reader can easily verify that in this modified setting the proofs in the 1-dimensional case carries over to the multidimensional case with only minor modifications. Thus we obtain (compare with Theorem 3.11 and Theorem 3.15):

THEOREM 4.1 (THE CLARK-HAUSSMANN-OCONE FORMULA FOR $L^2(\mu_m)$)
Let $F(\omega) \in L^2(\mu_m)$ be measurable with respect to the σ-algebra $\mathcal{F}_T^{(m)}$ which is generated by $\{B_i(s,\omega)\}_{0\le s\le T, 1\le i\le m}$. Then

$$\frac{\partial F}{\partial \omega_j}(t,\omega) \in \mathcal{G}^*(\mu_m) \quad \textit{and} \quad E[\frac{\partial F}{\partial \omega_i}(t,\omega)|\mathcal{F}_t^{(m)}] \in L^2(\mu_m)$$

for a.a. t, $1 \le j \le m$. Moreover,

$$E\Big[\int_0^T \Big(\sum_{j=1}^m E[\frac{\partial F}{\partial \omega_i}(t,\omega)|\mathcal{F}_t^{(m)}]^2\Big)dt\Big] < \infty$$

and we have

$$F(\omega) = E_{\mu_m}[F] + \int_0^T \sum_{j=1}^m E[\frac{\partial F}{\partial \omega_i}(t,\omega)|\mathcal{F}_t^{(m)}]dB_j(t,\omega_j)$$

THEOREM 4.2 (THE CLARK-HAUSSMANN-OCONE FORMULA FOR $\mathcal{G}^*(\mu_m)$)
Let $F(\omega) \in \mathcal{G}^*(\mu_m)$ be $\mathcal{F}_T^{(m)}$-measurable. Then

$$\frac{\partial F}{\partial \omega_j}(t,\omega) \in \mathcal{G}^*(\mu_m) \quad \textit{and} \quad E[\frac{\partial F}{\partial \omega_i}(t,\omega)|\mathcal{F}_t^{(m)}] \in \mathcal{G}^*(\mu_m)$$

for a.a. t, $1 \le j \le m$. $E[\frac{\partial F}{\partial \omega_i}(t,\omega)|\mathcal{F}_t^{(m)}]$ is integrable in $(S)^*(\mu_m)$ and

$$F(\omega) = E_{\mu_m}[F] + \int_0^T \Big(\sum_{j=1}^m E[\frac{\partial F}{\partial \omega_i}(t,\omega)|\mathcal{F}_t^{(m)}] \diamond W_j(t,\omega_j)\Big)dt$$

5. The Poissonian case

The white noise machinery can also be adapted to the Poissonian case. To achieve this we replace the measure μ defined by (2.1) by the *Poissonian* white noise probability measure ν defined on $S'(\mathbb{R})$ by

$$\int_{S'(\mathbb{R})} e^{i<\omega,\phi>}d\nu(\omega) = \exp\Big[\int_{\mathbb{R}}\Big(e^{i\phi(x)}-1\Big)dx\Big]\,;\phi \in S(\mathbb{R}) \tag{5.1}$$

Then we obtain a *Poisson process* $P(t,\omega)$ as the right-continuous version of

$$\tilde{P}(t,\omega) := <\omega, \mathcal{X}_{[0,t]}(\cdot)> \tag{5.2}$$

This gives

$$<\omega,\phi> = \int_{\mathbb{R}} \phi(x)dP(x)\ ;\ S(\mathbb{R})$$

If we define

$$Q(t,\omega) = P(t,\omega) - t\ ;\ t \in \mathbb{R}$$

(the compensated Poisson process), then we obtain that any $F \in L^2(\nu)$ can be represented as a sum of iterated integrals with respect to Q:

$$F(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} g_n(x) dQ^{\otimes n}(x) \tag{5.3}$$

where $g_n \in \hat{L}^2(\mathbb{R}^n)$ for all n. Moreover,

$$||F||^2_{L^2(\nu)} = \sum_{n=0}^{\infty} n!||g_n||^2_{L^2(\mathbb{R}^n)} \tag{5.4}$$

There is also a basis of polynomials analogous to the Hermite functionals $H_\alpha(\omega)$ in the Gaussian case. These are called the *Charlier polynomials* and denoted by $C_\alpha(\omega)$; $\alpha \in \mathcal{I}$. The first Charlier polynomials are

$$C_0(\omega) = 1, \; C_{\epsilon^{(j)}}(\omega) = <\omega, \xi_j> - \overline{\xi}_j \tag{5.5}$$

(where $\overline{\xi}_j = \int_{\mathbb{R}} \xi_j(t)dt$),

$$C_{\epsilon^{(i)}+\epsilon^{(j)}}(\omega) = <\omega, \xi_i><\omega, \xi_j> - <\omega, \xi_i\xi_j> - <\omega, \xi_i> \overline{\xi}_j - <\omega, \xi_j> \overline{\xi}_i - \overline{\xi}_i\overline{\xi}_j \tag{5.6}$$

We have

$$E_\nu[C_\alpha C_\beta] = \alpha!\delta_{\alpha,\beta}$$

We refer to [B-G], [H-Ø-U-Z], [H-Ø], [IY] and the references therein for more information. We now proceed as in the Gaussian case to define the Poissonian versions $(\mathcal{S})_\nu, \mathcal{G}(\nu), \mathcal{G}^*(\nu)$, and $(\mathcal{S})^*_\nu$ of the stochastic test function and distribution spaces $(\mathcal{S}), \mathcal{G}, \mathcal{G}^*$, and $(\mathcal{S})^*$ we defined in (2.19), (2.20), (2.7), and (2.8). For example, the Poissonian Hida test function space $(\mathcal{S})_\nu$ consists of all expansions

$$F(\omega) = \sum_{\alpha \in \mathcal{I}} a_\alpha C_\alpha(\omega)$$

such that

$$||F||^2_{\nu;0,k} := \sum_{\alpha \in \mathcal{I}} a^2_\alpha \alpha!(2\mathbb{N})^{k\alpha} < \infty \tag{5.7}$$

for all $k \in \mathbb{N}$. The Poissonian Wick product $\hat{\diamond}$ is defined on the Charlier polynomials by

$$(C_\alpha \hat{\diamond} C_\beta)(\omega)) = C_{\alpha+\beta}(\omega) \; ; \; \alpha, \beta \in \mathcal{I} \tag{5.8}$$

and extended linearly to $(\mathcal{S})^*_\nu$. As in the Gaussian case, we get that all the four spaces $(\mathcal{S})_\nu, \mathcal{G}(\nu), \mathcal{G}^*(\nu)$, and $(\mathcal{S})^*_\nu$ are closed under Wick products and that the Wick product is a commutative, associative and distributive (over addition) binary operation on these spaces. In spite of the many similarities, there are important distinctions between the Gaussian and the Poissonian case. For example, in the Gaussian case we have seen that with our definition (3.1), (3.5) of directional derivative D_γ and Malliavin derivative D_t, we have both the ordinary chain rule (5.9) and the Wick chain rule (5.10)/(5.11):

$$D_t(<\omega, \xi_i>^n) = n <\omega, \xi_i>^{n-1} \xi_i(t) \tag{5.9}$$

and

$$D_t(<\omega, \xi_i>^{\diamond n}) = n <\omega, \xi_i>^{\diamond(n-1)} \xi_i(t) \tag{5.10}$$

i.e.,

$$D_t(H_{n\epsilon^{(i)}}(\omega)) = nH_{(n-1)\epsilon^{(i)}}(\omega)\xi_i(t) \tag{5.11}$$

See Lemma 3.6.

If we maintain our definitions (3.1),(3.5) in the Poissonian case, then as before we get that the ordinary chain rule (5.9) holds. However, *the Wick chain rule will fail.* To see this, consider the following example:

EXAMPLE 5.1

Let $F(\omega) = <\omega, \xi_i>^{\hat{\diamond}2}$. Then by (5.5) and (5.6)

$$F(\omega) = (C_{\epsilon^{(i)}}(\omega) + \overline{\xi}_i)^{\hat{\diamond}2} = C_{2\epsilon^{(i)}}(\omega) + 2C_{\epsilon^{(i)}}(\omega)\overline{\xi}_i + \overline{\xi}_i^2 = <\omega, \xi_i>^2 - <\omega, \xi_i^2> \tag{5.12}$$

Therefore, by (5.9),

$$D_t(<\omega, \xi_i>^{\hat{\diamond}2}) = 2<\omega, \xi_i>\xi_i(t) - \xi_i(t)^2 \neq 2<\omega, \xi_i>\xi_i(t) \tag{5.13}$$

Equivalently, using (5.5),

$$D_t\left(C_{2\epsilon^{(i)}}(\omega)\right) = 2C_{\epsilon^{(i)}}(\omega)\xi_i(t) - \xi_i^2(t) \neq 2C_{\epsilon^{(i)}}(\omega)\xi_i(t) \tag{5.14}$$

The reader may find these results contradictory in view of the following "argument"

$$\begin{aligned} &\text{“}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(<\omega+\epsilon\gamma, \xi_i>^{\hat{\diamond}2} - <\omega, \xi_i>^{\hat{\diamond}2}\right) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left((<\omega, \xi_i> + \epsilon<\gamma, \xi_i>)^{\hat{\diamond}2} - <\omega, \xi_i>^{\hat{\diamond}2}\right) \\ &= \lim_{\epsilon\to 0} 2<\omega, \xi_i><\gamma, \xi_i> + \epsilon<\gamma, \xi_i>^2 = 2<\omega, \xi_i><\gamma, \xi_i>\text{”} \end{aligned} \tag{5.15}$$

which - if it were correct - would imply that

$$\text{“}D_t\left(<\omega, \xi_i>^{\hat{\diamond}2}\right) = 2<\omega, \xi_i><\gamma, \xi_i>\text{”} \tag{5.16}$$

which seemingly contradicts (5.13). However, there is a flaw in the argument leading to (5.16), because if $F(\omega) = <\omega, \xi_i>^{\hat{\diamond}2}$, then by (5.12)

$$\begin{aligned} &<\omega+\epsilon\gamma, \xi_i>^{\hat{\diamond}2} = F(\omega+\epsilon\gamma) = <\omega+\epsilon\gamma, \xi_i>^2 - <\omega+\epsilon\gamma, \xi_i^2> \\ &= <\omega, \xi_i>^2 + 2\epsilon<\omega, \xi_i><\gamma, \xi_i> + \epsilon^2<\gamma, \xi_i>^2 - <\omega, \xi_i^2> - \epsilon<\gamma, \xi_i^2> \\ &= <\omega, \xi_i>^{\hat{\diamond}2} + 2\epsilon<\omega, \xi_i><\gamma, \xi_i> + \epsilon^2<\gamma, \xi_i>^2 - \epsilon<\gamma, \xi_i^2> \\ &= (<\omega, \xi_i> + \epsilon<\gamma, \xi_i>)^{\hat{\diamond}2} - \epsilon<\gamma, \xi_i^2> \neq (<\omega, \xi_i> + \epsilon<\gamma, \xi_i>)^{\hat{\diamond}2} \end{aligned} \tag{5.17}$$

which shows that the equality (5.15) is false.

Since it is the *Wick chain rule* and not the ordinary chain rule that is needed in our proof of the Clark-Haussmann-Ocone formula, we must in the Poissonian case abandon the Malliavin derivative D_t based on the directional derivative D_γ in the definitions (3.1), (3.5) and replace D_t by the stochastic derivative $\hat{D}_t$ defined analogously to (3.14):

DEFINITION 5.2

Let $F(\omega) = \sum_{\alpha\in\mathcal{I}} a_\alpha C_\alpha(\omega) \in \mathcal{G}^*(\nu)$. Then we define the *stochastic derivative* of F at t by

$$\hat{D}_t F(\omega) = \frac{\hat{dF}}{d\omega}(t, \omega) = \sum_\alpha c_\alpha \sum_i a_{\alpha_i} C_{\alpha-\epsilon^{(i)}}(\omega)\xi_i(t) \tag{5.18}$$

Note that with this definition the ordinary chain rule does *not* hold. For example, from (5.12) and the Wick chain rule, we get

$$\hat{D}_t \left(< \omega, \xi_i >^2 \right) = \hat{D}_t \left(< \omega, \xi_i >^{\hat{\diamond} 2} + < \omega, \xi_i^2 > \right) = 2 < \omega, \xi_i > \xi_i(t) + \xi_i^2(t) \qquad (5.19)$$

More precisely, $\hat{D}$ is a *finite difference operator*, cf. Th. 6.5 of [IY], and (I.12) in [D-K-W]. In particular, from Prop. 1 of [N-V] we have

$$\hat{D}_t F(\omega) = F(\omega + \delta_t) - F(\omega), \quad a.a.\ t, \omega$$

if F is in the L^2 domain of $\hat{D}$, where $\delta_t \in S'(\mathbb{R})$ is the Dirac measure at t.

The *Poissonian white noise* is defined by

$$V(t,\omega) = \sum_{j=1}^{\infty} \xi_j(t) C_{\epsilon^{(j)}}(\omega)$$

We see that $V(t,\cdot) \in (S)_\nu^*$ for all t and as in the Gaussian case we get

$$\int_0^T Y(t,\omega)\delta Q(t,\omega) = \int_0^T Y(t,\omega) \hat{\diamond} V(t,\omega) dt \qquad (5.20)$$

for all Q-Skorohod integrable $Y(t,\omega)$. With these definitions the proofs of Section 3 carry over to the Poissonian case. We omit the details.

In the multidimensional Poissonian case we put

$$\Omega = S'(\mathbb{R}) \times \cdots \times S'(\mathbb{R}) \ \ (k \text{ factors})$$

and

$$\nu_k = \nu \times \cdots \times \nu \ \ (k \text{ factors})$$

and proceed as in Section 4. Again one can verify that the proofs of Section 3 carry over also to the Poissonian case. Again we omit the details. The results are:

THEOREM 5.3 (THE CLARK-HAUSSMANN-OCONE FORMULA FOR $L^2(\nu_k)$ and $\mathcal{G}^*(\nu_k)$)
Interpreted within the framework of $\hat{D}_t$ and $\hat{\diamond}$, the Clark-Haussmann-Ocone formula applies for $L^2(\nu_k)$ and $\mathcal{G}^*(\nu_k)$.

REMARK
Other approaches to stochastic calculus/Malliavin calculus for jump processes can be found in [B-C], [B-G-J], and [E-T]. In [P], the CHO formula has been expressed using both the finite difference operator and the derivation with respect to jump times of [C-P].

Similarly to obtain a white noise theory for the case with m Gaussian noises and k Poissonian noises, we consider the measure

$$\Theta = \Theta_{m,k} = \mu_m \times \nu_k$$

on

$$\Omega = S'(\mathbb{R}) \times \cdots \times S'(\mathbb{R}) \ \ (m + k \text{ factors})$$

By considering appropriate tensor products of the Hermite polynomials H_α and the Charlier polynomials C_β, we obtain an orthonormal basis for $L^2(\Theta_{m,k})$ just as explained in the beginning of Section 4. See [H-Ø] for more information about this construction. We encourage the reader to (once again) verify that the proofs of Sections 3 and 4 carry over.

THEOREM 5.4 (THE CLARK-HAUSSMANN-OCONE FORMULA FOR THE SPACES $L^2(\mu_m \times \nu_k)$ and $\mathcal{G}^*(\mu_m \times \nu_k)$)
Interpreted within the framework of $\hat{D}_t$ and $\hat{\diamond}$, the Clark-Haussmann-Ocone formula applies for $L^2(\mu_m \times \nu_k)$ and $\mathcal{G}^*(\mu_m \times \nu_k)$.

6. Application to mathematical finance: Hedging in a Poissonian market

As an application of the results above, consider a market $X(t) = (A(t), S(t))$ consisting of two investment possibilities:

(i) *a bank account*, where the price $A(t)$ at time t is given by

$$dA(t) = \rho(t)A(t)dt \; ; \; A(0) = 1 \tag{6.1}$$

(ii) *a stock*, where the price $S(t)$ at time t is given by

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dQ(t) \; ; \; S(0) = x > 0 \tag{6.2}$$

where $\rho(t), \mu(t)$, and $\sigma(t)$ are *deterministic* functions in $L^2[0,T]$ ($T > 0$ constant), $\sigma(t) \geq \epsilon$ for some $\epsilon > 0$. As before $Q(t) = P(t) - t$ is the compensated Poisson process. It is well known (see, e.g., [HØ, Example 2.2]) that the solution of (6.2) is given by

$$S(t) = x \exp\left[\int_0^t \ln[1+\sigma(s)]dQ(s) + \int_0^t (\mu(s) - \sigma(s) + \ln[1+\sigma(s)])ds\right] \tag{6.3}$$

Let $(\xi(t,\omega), \eta(t,\omega))$ be a *portfolio*, i.e., $\xi(t), \eta(t)$ gives the number of units of investments #1, #2, respectively, held by an agent at time t. The total value $V(t)$ at time t of such a portfolio is then given by

$$V(t) = \xi(t)A(t) + \eta(t)S(t) \tag{6.4}$$

Assume that the portfolio is *self-financing*, in the sense that

$$dV(t) = \xi(t)dA(t) + \eta(t)dS(t) \tag{6.5}$$

From (6.4) we get

$$\xi(t) = \frac{V(t) - \eta(t)S(t)}{A(t)} \tag{6.6}$$

which substituted in (6.5) gives

$$dV(t) = \rho(t)V(t)dt + \sigma(t)\eta(t)S(t)\left(\frac{\mu(t)-\rho(t)}{\sigma(t)}dt + dQ(t)\right) \tag{6.7}$$

Define

$$u(t) = \frac{\mu(t) - \rho(t)}{\sigma(t)} \tag{6.8}$$

Suppose

$$u(t) \leq 1 - \epsilon_1 \qquad \text{for some } \epsilon_1 > 0 \tag{6.9}$$

Then by the Girsanov theorem for the compensated Poisson process (see [B-V-W]), we get: If we define the measure $\tilde{\nu}$ on $\mathcal{H}_T$ by

$$d\tilde{\nu}(\omega) = Z_T(\omega)d\nu(\omega),$$

where

$$Z_T(\omega) = \exp^{\diamond}\left[-\int_0^T u(t)dQ(t)\right] = \exp\left[\int_0^T \ln[1-u(t)]dP(s) + \int_0^T u(t)dt\right] \tag{6.10}$$

(see, e.g., [HØ,(2.43)]), then the process

$$\tilde{Q}(t,\omega) := \int_0^t u(s)ds + Q(t,\omega) \tag{6.11}$$

is a compensated Poisson process with respect to the measure $\tilde{\nu}$. Substituting (6.11) into (6.7), we get

$$dV(t) = \rho(t)V(t)dt + \sigma(t)\eta(t)S(t)d\tilde{Q}(t)$$

or

$$e^{-\int_0^T \rho(s)ds}V(T) = V(0) + \int_0^T e^{-\int_0^t \rho(s)ds}\sigma(t)\eta(t)S(t)d\tilde{Q}(t) \tag{6.12}$$

Suppose we want to hedge a given $\mathcal{H}_T$-measurable claim $F(\omega) \geq 0$. Then we seek $V(0)$ and $\eta(t)$ such that $V(T) = F$ a.s., i.e.,

$$e^{-\int_0^T \rho(s)ds}F(\omega) = V(0) + \int_0^T e^{-\int_0^t \rho(s)ds}\sigma(t)\eta(t)S(t)d\tilde{Q}(t) \tag{6.13}$$

Define

$$G(\omega) = e^{-\int_0^T \rho(s)ds}F(\omega)$$

If $G \in L^2(\tilde{\nu})$, then by Theorem 5.1 applied to $\tilde{\nu}$ we get (with $\tilde{D}_t$ denoting the stochastic derivative w.r.t. $\tilde{\nu}$)

$$G(\omega) = E_{\tilde{\nu}}[G] + \int_0^T E_{\tilde{\nu}}[\tilde{D}_t G|\mathcal{H}_t]d\tilde{Q}(t) \tag{6.14}$$

(Observe that $Q(t)$ and $\tilde{Q}(t)$ generate the same filtration $\mathcal{H}_t = \tilde{\mathcal{H}_t}$). Comparing (6.13) with (6.14), we get, by uniqueness,

$$V(0) = e^{-\int_0^T \rho(s)ds}E_{\tilde{\nu}}[F] \qquad \text{(the price of the claim } F) \tag{6.15}$$

and

$$\eta(t) = e^{-\int_t^T \rho(s)ds}\sigma(t)^{-1}S(t)^{-1}E_{\tilde{\nu}}[\tilde{D}_t F|\mathcal{H}_t] \tag{6.16}$$

As an example, consider the *European call option*, i.e.,

$$F(\omega) = (S(T) - K)^+ \tag{6.17}$$

where $K > 0$ is some constant (the exercise price). We may write

$$F(\omega) = f(S(T)) \qquad \text{where } f(x) = (x-K)^+ \; ; \; x > 0$$

From Prop. 1 of [N-V] we have

$$\tilde{D}_t F(\omega) = F(\omega + \delta_t) - F(\omega) = (S(T)(\omega + \delta_t) - K)^+ - (S(T)(\omega) - K)^+ \tag{6.18}$$

By (6.3) and (6.11) we have

$$S(T) = x \exp\left[\int_0^T \ln[1+\sigma(t)]d\tilde{Q}(t) + \int_0^T (\mu(t) - \sigma(t) + \ln[1+\sigma(t)](1-u(t)))dt\right] \tag{6.19}$$

hence

$$S(T)(\omega + \delta_t) = (1+\sigma(t))S(T)(\omega) \tag{6.20}$$

Combining this with (6.18), we get for $0 \le t \le T$

$$\begin{aligned}\tilde{D}_t F =& ((1+\sigma(t))S(T)-K)\mathcal{X}_{[K/(1+\sigma(t)),K]}(S(T)) + \sigma(t)S(T)\mathcal{X}_{[K,\infty)}(S(T))\\ =&\sigma(t)S(T)\mathcal{X}_{[K/(1+\sigma(t)),\infty)}(S(T)) + (S(T)-K)\mathcal{X}_{[K/(1+\sigma(t)),K]}(S(T))\\ =&\sigma(t)S(T)\mathcal{X}_{[K/(1+\sigma(t)),\infty)}(S(T)) - (K-S(T))^+\mathcal{X}_{[K/(1+\sigma(t)),\infty)}(S(T))\\ =&(\sigma(t)S(T)-(K-S(T))^+)\mathcal{X}_{[K/(1+\sigma(t)),\infty)}(S(T)),\end{aligned} \tag{6.21}$$

and

$$E_{\tilde{\nu}}[\tilde{D}_t F|\mathcal{H}_t] = E_{\tilde{\nu}}[(\sigma(t)S(T)-(K-S(T))^+)\mathcal{X}_{[K/(1+\sigma(t)),\infty)}(S(T))|\mathcal{H}_t]. \tag{6.22}$$

By the Markov property of the process $S(t)$ with respect to $\tilde{\nu}$ (see (6.19)), we see that

$$\begin{aligned}&E_{\tilde{\nu}}[(\sigma(t)S(T)-(K-S(T))^+)\mathcal{X}_{[K/(1+\sigma(t)),\infty)}(S(T))|\mathcal{H}_t]\\ =&E_{\tilde{\nu}}[(\sigma(t)Y^y(T-t)-(K-Y^y(T-t))^+)\mathcal{X}_{[K/(1+\sigma(t)),\infty)}(Y^y(T-t))]_{y=S(t)},\end{aligned} \tag{6.23}$$

where $Y^y(t)$ is the process defined by

$$dY^y(t) = Y^y(t)((\mu(t)-u(t)\ln[1+\sigma(t)])dt + \sigma(t)d\tilde{Q}(t))\ ;\ Y^y(0)=y \tag{6.24}$$

i.e.

$$Y^y(t) = y\exp\left[\int_0^t \ln[1+\sigma(s)]d\tilde{Q}(s) + \int_0^t (\mu(s)-\sigma(s)+\ln[1+\sigma(s)](1-u(s)))ds\right] \tag{6.25}$$

Since the law of $\tilde{Q}(t)$ is known, we can also write down an explicit formula for the expression (6.22). We summarize what we have found in the following

THEOREM 6.1

The price $V(0)$ of a European call option with payoff

$$F(\omega) = (S(T)-K)^+$$

in the Poissonian market defined by (6.1), (6.2) and satisfying (6.9), is given by (6.15). Moreover, the replicating portfolio $\xi(t),\eta(t)$ for this claim is given by (6.6) and

$$\begin{aligned}\eta(t) =& \frac{1}{\sigma(t)S(t)}e^{-\int_t^T \rho(s)ds}E_{\tilde{\nu}}[(\sigma(t)Y^y(T-t)-(K-Y^y(T-t))^+)\\ &\cdot\mathcal{X}_{[K/(1+\sigma(t)),\infty)}(Y^y(T-t))]_{y=S(t)}\end{aligned} \tag{6.26}$$

with $Y^y(t)$ given by (6.25).

REMARK

The formula (6.15) for the price $V(0)$ is well-known. However, the hedging formula (6.25) appears to be new. Note that the alternative approach often used to compute hedging strategies (the PDE approach) seems difficult to apply here because it involves the calculation of

$$\frac{\partial f}{\partial x}(T-t,x)$$

where $f(T-t,x)$ is the price at time $T-t$ if $S(t)=x$. One can express f in terms of an expectation with respect to ν and this leads to a series expansion for f. This series, however, cannot be differentiated term by term. Pricing in models described by diffusions plus jumps is treated in [C-G]. However, that paper does not study the question of finding replicating portfolios.

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