Prépublications du Département de Mathématiques

Université de La Rochelle
Avenue Marillac
17042 La Rochelle Cedex 1
http://www.univ-lr.fr/Labo/MATH

Generalized Cahn-Hilliard equations with a logarithmic free energy

Laurence Cherfils et Alain Miranville

Avril 1999

Classification: 35B40, 35B45, 35A05.
Mots clés: Cahn-Hilliard equation, microforces, logarithmic free energy, global attractor, exponential attractor.

1999/02

Generalized Cahn-Hilliard equations with a logarithmic free energy

L. Cherfils [1], *A. Miranville* [2]

Abstract: Our aim in this article is to study some models of Generalized Cahn-Hilliard equations with a logarithmic free energy. We obtain the existence and uniqueness of solutions and then study the existence of finite dimensional attractors.

Key words: Cahn-Hilliard equation, microforces, logarithmic free energy, global attractor, exponential attractor.

AMS classification scheme numbers: 35B40, 35B45, 35A05.

1 Introduction

The Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} = \kappa_0 \Delta (f'(u) - \alpha \Delta u), \tag{1}$$

where κ_0, $\alpha > 0$, is very central to material sciences. This equation, which is a conservation law, describes the transport of atoms between unit cells (the order parameter u corresponds to a density of atoms). The function f, which corresponds here to a free energy, is generally a double well potential, whose wells define the phases of the material. We recall that the equation that was originally called the Cahn-Hilliard equation (by Cahn and Hilliard [2]) was based on a polynomial (of degree 4) free energy. Elliot and Luckhauss [7], and then Debussche and Dettori [4], considered later a logarithmic free energy.

In particular, for both a polynomial and a logarithmic free energy, the different authors obtained the existence and uniqueness of solutions and also the existence (and estimates on the dimension) of attractors.

In [8], M. Gurtin considers more general forms of the Cahn-Hilliard equation, namely

$$\begin{cases} \dfrac{\partial}{\partial t}(u - \beta\, div(B_0 \nabla u)) + \alpha\, div(B_0 \nabla \Delta u) - div(B_0 \nabla f'(u)) + div(B_0 \nabla \gamma) = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{2}$$

where $\alpha > 0$, $\beta > 0$. We shall assume here that the mobility tensor $B_0 \in \mathcal{M}_n(\mathbb{R})$ is symmetric, strictly positive and has constant coefficients. The equation is posed in $\Omega \times \mathbb{R}^+$, where $\Omega = \Pi_{i=1}^n]0, L_i[$, $L_i > 0$, $n = 1, 2$, or 3. The function $\gamma = \gamma(x)$ is assumed to be

[1]Université de La Rochelle, LMCA, Avenue Marillac, 17042 La Rochelle cedex 1, France

[2]Université de Poitiers, Mathématiques, SP2MI, Boulevard 3, Téléport 2, 86960 Futuroscope cedex, France

regular enough (say, in $H^3(\Omega)$), and $\Omega-$periodic. Throughout the paper, we thus consider periodic boundary conditions, i.e. u is Ω-periodic. However, we shall comment at the end of the article on the case of Neumann-like boundary conditions.

The models we consider are derived by assuming that there should be microforces whose working accompanies changes in the order parameter u. Indeed, according to M. Gurtin, even though the derivation (and thus the mathematical study) of (1) is useful and important, this equation should not be regarded as basic, but rather as precursor of more complete theories (e.g. the taking of internal microforces into account). For instance, it seems reasonable that interaction forces may be characterized macroscopically by fields that perform work when u undergoes changes ; hence the presence of microforces.

In [9], the authors considered some of the models of Cahn-Hilliard equations derived in [8] for a polynomial free energy. They obtained in particular the existence and uniqueness of weak and strong solutions. They also obtained the existence of the global attractor, which is a compact invariant set that attracts the trajectories as time goes to infinity and thus characterizes the long time behavior of the system. Furthermore, they could prove that the global attractor has finite dimension (in the sense of the fractal or the Hausdorff dimension).

Our aim in this article is to make a similar study in the case of a logarithmic free energy, i.e.

$$f'(u) \;=\; -\theta_c\, u \;+\; \frac{\theta}{2}\, \ln\big(\frac{1+u}{1-u}\big) \;; \quad 0<\theta\,<\theta_c. \tag{3}$$

The idea is, following in particular Dettori-Debussche [4], to approximate f' by a polynomial expansion

$$f_N'(u) \;=\; -\theta_c\, u \;+\; \theta \sum_{k=0}^{N} \frac{u^{2k+1}}{2k+1}, \tag{4}$$

and to pass to the limit as the degree of the polynomial goes to infinity. It is thus important to derive *a priori* estimates that are independent of the degree of the polynomial by a careful treatment of the nonlinear term in particular (note that most of the estimates derived in [9] depend strongly on the degree of the free energy). We thus obtain the existence (and uniqueness) of solutions and the existence of the global attractor. We finally study the dimension of the global attractor. We can note here that the regularizing effect introduced in the generalized Cahn-Hilliard equation (corresponding to the term $\beta\,\frac{\partial}{\partial t}\, div(B_0\,\nabla\, u)$) allows us to obtain a more satisfactory result than in [4] for the dimension of the global attractor, in the sense that we are not restricted as in [4] to sets with constant average. Indeed, in that case, we are able to prove the existence of exponential attractors (which contain the global attractor, have finite fractal dimension and attract exponentially the trajectories).

2 Preliminaries

2.1 Notations and preliminary results

Throughout the article, we denote by $|.|$ and $(.,.)$ the norm and the scalar product in $L^2(\Omega)$, and we set $m(u) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u(x)\,dx$ for u in $L^1(\Omega)$. We also set $\varphi(u) = \ln\left(\dfrac{1+u}{1-u}\right)$, $\phi(u) = \displaystyle\int_0^u \varphi(s)\,ds$, $\varphi_N(u) = 2\displaystyle\sum_{k=0}^{N}\frac{u^{2k+1}}{2k+1}$ and $\phi_N(u) = 2\displaystyle\sum_{k=0}^{N}\frac{u^{2k+2}}{(2k+1)(2k+2)}$. Furthermore, for $u \in H^2_{per}(\Omega)$, we set $\Delta_0\, u = div(B_0\,\nabla u)$, $A_0\, u = -\Delta_0\, u$ and we have $H^2_{per}(\Omega) = \{v \in H^1_{per}(\Omega)\,;\, A_0^{\frac{1}{2}}\,\nabla\, v \in H^1_{per}(\Omega)\,\}$. Hereafter, C_0, C_1, C_2, ... will denote fixed constants, whereas c, c', c''.... will denote generic positive constants which may vary from line to line. In any case, these constants will be independent of the polynomial degree N.

Proposition 1 *For u, v smooth enough and Ω-periodic and B_0 symmetric, strictly positive, with constant coefficients, we have:*

$$(\,-\Delta_0 u,\, v\,) \;=\; (\,B_0\,\nabla u,\, \nabla v\,) = (\,B_0^{\frac{1}{2}}\,\nabla u,\, B_0^{\frac{1}{2}}\,\nabla v\,), \tag{5}$$

$$(\,\Delta_0(\Delta u),\, v\,) \;=\; (\,\Delta u,\, \Delta_0 v\,) = (\,\nabla\,(B_0^{\frac{1}{2}}\,\nabla u),\, \nabla\,(B_0^{\frac{1}{2}}\,\nabla v)\,). \tag{6}$$

Proof :
Identity (5) results from elementary calculations. Furthermore, the operator Δ_0 is selfadjoint and, for u regular enough, we have

$$\begin{aligned}(\Delta_0\,(\Delta u),\, v) &= (div\, B_0 \nabla\,(\Delta u),\, v)\\ &= -(B_0^{\frac{1}{2}}\,\nabla\,(\Delta u),\, B_0^{\frac{1}{2}}\,\nabla v).\end{aligned}$$

Since $B_0^{\frac{1}{2}}\,\nabla\,(\Delta u) = \Delta\,(B_0^{\frac{1}{2}}\,\nabla\, u)$, we integrate by parts and find (6).

$\blacksquare$

Concerning the operator A_0, we have the following well-known results:

Proposition 2 *The operator A_0 is linear, positive, selfadjoint and possesses a basis of eigenvectors $(w_j)_{j\in\mathbb{N}}$ which is orthonormal in $L^2(\Omega)$ and is associated with the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$,*

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_j \to +\infty.$$

Furthermore, A_0 is a strictly positive, selfadjoint operator with compact inverse when restricted to subspaces of functions with vanishing average.

We can thus give the following definition:

Definition 1

For $s \in I\!R$, we set

$$A_0^s u = \sum_{j=1}^{\infty} \lambda_j^s u_j w_j, \qquad where \quad u = \sum_{j=0}^{\infty} u_j w_j,$$

$$V_s = D(A_0^{\frac{s}{2}}) = \{ u = \sum_{j=0}^{\infty} u_j w_j, \quad \sum_{j=1}^{\infty} \lambda_j^s u_j^2 < +\infty \},$$

which we endow with the seminorm and the semiscalar product

$$|u|_s = |A_0^{\frac{s}{2}} u| \quad ; \quad (u,v)_s = (A_0^{\frac{s}{2}} u, A_0^{\frac{s}{2}} v),$$

and the norm

$$\|u\|_s = (|u|_s^2 + m(u)^2)^{\frac{1}{2}}.$$

Let $\tilde{u} = u - m(u)$. We then have

$$|u|_s = \|\tilde{u}\|_s = \left(\sum_{j=1}^{\infty} \lambda_j^s u_j^2 \right)^{\frac{1}{2}}.$$

We refer the reader to [4] and [10] for the properties of the spaces V_s. Nevertheless, we point out that

Proposition 3

The norms $\|.\|_2$ and $(|\nabla B_0^{\frac{1}{2}} \nabla u|^2 + |u|^2)^{\frac{1}{2}}$ are equivalent to the usual H^2-norm in V_2. Moreover, the constants in these equivalences can be chosen independently of Ω.

Proof: We infer from classical regularity results (see [1]) for the second order elliptic problem

$$\begin{cases} -\,div\,(B_0^s \nabla u) + u = f \quad \text{in } \Omega, \\ \qquad u \text{ is } \Omega - \text{periodic}, \end{cases} \tag{7}$$

$s = 1$, that $\|.\|_2$ and the usual H^2-norm are equivalent on V_2. In the same way, taking $s = \frac{1}{2}$ in (7), we deduce the equivalence of the norms $(|div(B_0^{\frac{1}{2}} \nabla u)|^2 + |u|^2)^{\frac{1}{2}}$ and $\|.\|_{H^2(\Omega)}$. In the latter case, we also have:

$$\begin{aligned} |\nabla(B_0^{\frac{1}{2}} \nabla u)|^2 &= \sum_{i,j=1}^{n} |\frac{\partial}{\partial x_i} (B_0^{\frac{1}{2}} \nabla u)_j|^2 \\ &\geq \sum_{i=1}^{n} |\frac{\partial}{\partial x_i} (B_0^{\frac{1}{2}} \nabla u)_i|^2, \end{aligned}$$

where $(B_0^{\frac{1}{2}}\,\nabla u)_i$ denotes the ith component of the vector $B_0^{\frac{1}{2}}\,\nabla u$. Furthermore, it can easily be shown that, for $n\,=1,2$ or 3,

$$\sum_{i=1}^{n} a_i^2 \,\geq\, \varepsilon\,(\sum_{i=1}^{n} a_i)^2, \quad \forall\,\varepsilon < \frac{1}{3}.$$

Thus, for $\varepsilon\,<\frac{1}{3}$,

$$\begin{aligned} |\nabla(B_0^{\frac{1}{2}}\nabla u)|^2 \;&\geq\; \varepsilon\left|\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\,(B_0^{\frac{1}{2}}\nabla u)_i\right|^2 \\ &\geq\; \varepsilon\,|\,div\,(B_0^{\frac{1}{2}}\nabla u)\,|^2. \end{aligned}$$

Finally,

$$|\nabla(B_0^{\frac{1}{2}}\,\nabla u)|^2 \,+\, |u|^2 \,\geq\, \varepsilon\,(|divB_0^{\frac{1}{2}}\nabla u|^2 \,+\, |u|^2) \,\geq\, c\,\|u\|^2_{H^2(\Omega)},$$

hence the equivalence of norms (the second inequality is straightforward).

There remains to prove that the constants occuring in the equivalence of norms can be chosen independently of Ω. Let us for instance prove this result for the norms $(|\nabla(B_0^{\frac{1}{2}}\,\nabla u)|^2\,+\,|u|^2)^{\frac{1}{2}}$ and $\|u\|_{H^2(\Omega)}$, i.e. actually, thanks to the result obtained above, for the norms $(|div(B_0^{\frac{1}{2}}\,\nabla u)|^2\,+\,|u|^2)^{\frac{1}{2}}$ and $\|u\|_{H^2(\Omega)}$. We would proceed similarly for the norms $\|u\|_2$ and $\|u\|_{H^2(\Omega)}$.

For simplicity, we take $n\,=\,3$. We thus have $\Omega\;=\;\Pi_{i=1}^3\,]0,L_i[$ and we assume that $L_i\;\leq\;l$ (say, $l\,=\,1$ in the sequel). We thus cannot consider elongated domains of the form $\Omega\,=\,]0,\frac{1}{\delta}[\times]0,\delta[\times]0,\delta[,\delta\to 0^+$ (for which $|\Omega|=\delta\to 0$).

We set $K_i\,=\,E(\frac{1}{L_i})+1,\;i\,=\,1,2,3$. We set $\tilde{\Omega}\;=\;\prod_{i=1}^n\,]0,\,K_i\,L_i[$ and we extend u to $\tilde{\Omega}$ by periodicity in the three directions ; let $\tilde{u}$ denote the extension of u to $\tilde{\Omega}$. We note that $K_iL_i\,\in\,[1,2]$ so that $]0,1[^3\subset\,\tilde{\Omega}\,\subset]0,2[^3$. Furthermore, we have

$$\left\{\begin{aligned} &|div(\,B_0^{\frac{1}{2}}\nabla\tilde{u})|_{L^2(\tilde{\Omega})} \;=\; \sqrt{\prod_{i=1}^{3}K_i}\;|div(\,B_0^{\frac{1}{2}}\nabla u)|, \\ &|\tilde{u}|_{L^2(\tilde{\Omega})} \;=\; \sqrt{\prod_{i=1}^{3}K_i}\;|u|, \\ &\|\tilde{u}\|_{H^2(\tilde{\Omega})} \;=\; \sqrt{\prod_{i=1}^{3}K_i}\;\|u\|_{H^2(\Omega)}. \end{aligned}\right.$$

It thus suffices to prove the equivalence of norms with u and Ω replaced by $\tilde{u}$ and $\tilde{\Omega}$ respectively (and with constants that are independent of the L_i.) So, we now assume that $\Omega=\prod_{i=1}^3\,]0,L_i[$, where $L_i\in\,[1,2],\;i=1,2,3$. We set $\Omega_0=]0,1[^3$ and we consider the change of variables $x_i^{'}=\frac{x_i}{L_i}$ that maps Ω onto Ω_0. We set $\bar{u}(x_1^{'},x_2^{'},x_3^{'})\;=\;u(x_1,x_2,x_3)$ and we denote by $\overline{div},\overline{\nabla}$ the divergence and gradient operators with respect to the variables $x_i^{'},i=1,2,3$. We then have

$$div(B_0^{\frac{1}{2}}\nabla u)\;=\;\overline{div}\,\overline{B}\overline{\nabla}\bar{u},$$

where $\overline{B} = (\bar{b}_{ij}),\ \bar{b}_{ij} = \frac{b_{ij}}{L_i L_j}$, if $B_0^{\frac{1}{2}} = (b_{ij})$.

Since $\bar{B} = \mathrm{diag}(\dfrac{1}{L_1}, \dfrac{1}{L_2}, \dfrac{1}{L_3})\, B_0^{\frac{1}{2}}\, \mathrm{diag}(\dfrac{1}{L_1}, \dfrac{1}{L_2}, \dfrac{1}{L_3})$, where

$$\mathrm{diag}(\frac{1}{L_1}, \frac{1}{L_2}, \frac{1}{L_3}) = \begin{pmatrix} \frac{1}{L_1} & 0 & 0 \\ 0 & \frac{1}{L_2} & 0 \\ 0 & 0 & \frac{1}{L_3} \end{pmatrix},$$

we deduce that $\bar{B}$ is symmetric, positive definite. We then have

$$\begin{aligned} |div(B_0^{\frac{1}{2}} \nabla u)|^2 + c|u|^2 &= |\Omega|\, (|\overline{div}(\overline{B}\, \overline{\nabla}\, \bar{u}|^2_{L^2(\Omega_0)} + c\, |\bar{u}|^2_{L^2(\Omega_0)}) \\ &\geq c'\, |\Omega|\, \|\bar{u}\|^2_{H^2(\Omega_0)}, \end{aligned}$$

where c' depends on Ω_0, on the coercivity constant of $\overline{B}$ and on $\max(|\bar{b}_{ij}|)$ (see the details of the proofs in [1]). This constant can be chosen independently of Ω. Furthermore, returning to the x variable

$$\begin{aligned} |\Omega|\, \|\bar{u}\|^2_{H^2(\Omega_0)} &= \sum_{i=1}^{3} L_i^4\, |\frac{\partial^2 u}{\partial x_i^2}|^2 + \sum_{i \neq j} L_i^2 L_j^2\, |\frac{\partial^2 u}{\partial x_i \partial x_j}|^2 + \sum_{i=1}^{3} L_i^2\, |\frac{\partial u}{\partial x_i}|^2 + |u|^2 \\ &\geq c\, \|u\|^2_{H^2(\Omega)}, \end{aligned}$$

where c can be chosen independently of Ω. Therefore

$$|div(B_0^{\frac{1}{2}} \nabla u)|^2 + c|u|^2 \geq c'\, \|u\|^2_{H^2(\Omega)},$$

where c' does not depend on Ω. This finishes the proof (the other inequality is straightforward).

■

We next establish a Lemma that will be useful in the sequel.

Lemma 1

(i) For every u, v in $L^2(\Omega)$, we have

$$(u, v)_0 = (u, v) - m(u)\, m(v)\, |\Omega|, \tag{8}$$

$$(\tilde{u}, \tilde{v}) = (\tilde{u}, v) = (u, v)_0. \tag{9}$$

(ii) There exists $C_0 \in I\!R^{+*}$ such that

$$|v|_1^2 \leq C_0\, |\nabla v|^2, \quad \forall v \in V_1. \tag{10}$$

Proof

(i) Let $u = \sum_{i=0}^{\infty} u_i\, w_i$ and $v = \sum_{j=0}^{\infty} v_j\, w_j$. Then

$$
\begin{aligned}
(u,\, v)_0 &= \Big(\sum_{i=1}^{\infty} u_i\, w_i,\, \sum_{j=1}^{\infty} v_j\, w_j\Big)\\
&= \sum_{i=1}^{\infty} u_i\, v_i\\
&= \sum_{i=0}^{\infty} u_i\, v_i - u_0\, v_0\\
&= (u,\, v) - u_0\, v_0.
\end{aligned}
$$

It can easily be shown that

$$
w_0 = |\Omega|^{-\frac{1}{2}} \quad \text{and} \quad u_0 = (u,\, w_0) = |\Omega|^{\frac{1}{2}}\, m(u),
$$

which ends the proof of (8). Identity (9) is straightforward.

(ii) Let v belong to V_2. We have

$$
\begin{aligned}
|v|_1^2 = |A_0^{\frac{1}{2}}\, v|^2 &= (A_0^{\frac{1}{2}}\, v,\, A_0^{\frac{1}{2}}\, v)\\
&= (A_0\, v,\, v)\\
&= -\,(div(B_0\, \nabla\, v),\, v)\\
&= |B_0^{\frac{1}{2}}\, \nabla v|^2.
\end{aligned}
$$

Then, we infer the existence of a real constant C_0 satisfying

$$
|v|_1^2 \le C_0\, |\nabla\, v|^2,
$$

and we obtain (10) by density.

■

2.2 Preliminary estimates

Equations (2)-(4) can be written as (taking $f' = f_N'$)

$$
\begin{cases}
\dfrac{\partial u_N}{\partial t} + A_0\, K_N(u_N) = 0,\\
K_N(u_N) = \beta\, \dfrac{\partial u_N}{\partial t} - \alpha\Delta u_N - \theta_c\, u_N + \dfrac{\theta}{2}\, \varphi_N(u_N) - \gamma,\\
u_{N_{|t=0}} = u_0,\\
u_N \text{ is } \Omega - \text{ periodic.}
\end{cases}
\tag{11}
$$

The solutions of (11) formally satisfy the conservation property (obtained by integrating the equation over Ω)

$$
m(u_N(t)) = m(u_0) \quad \forall t \ge 0. \tag{12}
$$

We shall derive here only formal *a priori* estimates. These estimates can be justified by making Galerkin approximations and by passing to the limit (see [4] and [10] for more details).

We set $\mu = m(u_0)$ and

$$J_N(u_N) \,=\, \frac{\alpha}{2}\,|\nabla u_N|^2 \,-\, \frac{\theta_c}{2}\int_\Omega u_N^2\,dx \,+\, \frac{\theta}{2}\int_\Omega \phi_N(u_N)\,dx \,-\, \int_\Omega \gamma\,u_N\,dx. \tag{13}$$

Then,

$$\begin{aligned}
\frac{d}{dt}\,J_N(u_N) \quad &= \quad -\alpha\,(\Delta u_N,\,\frac{\partial u_N}{\partial t}) - \theta_c\,(u_N,\,\frac{\partial u_N}{\partial t}) \,+\, \frac{\theta}{2}\int_\Omega \varphi_N(u_N)\,\frac{\partial u_N}{\partial t}\,dx \,-\, (\gamma,\,\frac{\partial u_N}{\partial t}) \\
&= \quad (K_N(u_N),\,\frac{\partial u_N}{\partial t}) \,-\, \beta\,|\frac{\partial u_N}{\partial t}|^2 \\
&= \quad -\,\beta\,|\frac{\partial u_N}{\partial t}|^2 \,-\, (K_N(u_N),\,A_0\,(K_N(u_N)).
\end{aligned}$$

Thus

$$\frac{d}{dt}\,J_N(u_N) \,=\, -\beta\,|\frac{\partial u_N}{\partial t}|^2 \,-\, |K_N(u_N)|_1^2 \,\le\, 0, \tag{14}$$

and we deduce that J_N is a Lyapounov function.

Since $\displaystyle\int_\Omega \gamma\,u_N\,dx \,\le\, \frac{1}{2}\,|\gamma|^2 \,+\, \frac{1}{2}\,|u_N|^2$, we obtain, using (13)

$$J_N(u_N) \,-\, \frac{\theta_c}{2}\,|u_N|^2 \,\ge\, \frac{\alpha}{2}|\nabla u_N|^2 \,-\, \frac{(2\theta_c+1)}{2}|u_N|^2 \,+\, \frac{\theta}{2}\int_\Omega \phi_N(u_N)\,dx \,-\, \frac{1}{2}\,|\gamma|^2.$$

Since

$$\begin{aligned}
\frac{\theta}{4}\int_\Omega \phi_N(u_N)\,dx \quad &- \quad \frac{(2\theta_c+1)}{2}\int_\Omega u_N^2\;dx \\
&= \quad \int_\Omega \left\{\theta\,(\frac{u_N^2}{4} + \frac{u_N^4}{24} + \frac{u_N^6}{60} + ...) \,-\, (\frac{2\theta_c+1}{2})\,u_N^2\right\}dx \\
&\ge \quad \int_\Omega \left\{\frac{\theta\,u_N^4}{24} \,-\, (\frac{2\theta_c+1}{2})\,u_N^2\right\}dx,
\end{aligned}$$

and since $\displaystyle\frac{\theta\,y^2}{24} \,-\, \theta'\,y \,+\, \frac{6\theta'^2}{\theta} \,\ge\, 0, \quad \forall y \in \mathbb{R}$, it follows that

$$\frac{\theta}{4}\int_\Omega \phi_N(u_N)\,dx \,-\, \frac{(2\theta_c+1)}{2}\int_\Omega u_N^2\,dx \,\ge\, -\frac{3}{2}\,\frac{(2\theta_c+1)^2}{\theta}\,|\Omega|.$$

Consequently, for every $t > 0$, we have

$$J_N(u_N) - \frac{\theta_c}{2}|u_N|^2 \geq \frac{\alpha}{2}|\nabla u_N|^2 + \frac{\theta}{4}\int_\Omega \phi_N(u_N)\,dx - \frac{3}{2}\frac{(2\theta_c+1)^2}{\theta}|\Omega| - \frac{1}{2}|\gamma|^2. \tag{15}$$

Taking the scalar product of (11) with $\frac{\partial}{\partial t}K_N(u_N)$, we obtain

$$(\frac{\partial u_N}{\partial t}, \frac{\partial}{\partial t}K_N(u_N)) + \frac{1}{2}\frac{d}{dt}|K_N(u_N)|_1^2 \leq 0. \tag{16}$$

Remark 1 The solutions we shall obtain are not regular enough to prove, when passing to the limit in the Galerkin method, the equality in (16). As it will sometimes be the case hereafter, we can only justify, by lower semicontinuity, the inequality (see for instance [4]).

Since the function $\varphi_N'(u_N)$ is positive, we have

$$(\frac{\partial u_N}{\partial t}, \frac{\partial}{\partial t}K_N(u_N)) \geq -\theta_c|\frac{\partial u_N}{\partial t}|^2 + \beta\int_\Omega \frac{\partial u_N}{\partial t}\frac{\partial^2 u_N}{\partial t^2}\,dx + \alpha|\nabla\frac{\partial u_N}{\partial t}|^2.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}|K_N(u_N)|_1^2 + \frac{\beta}{2}\frac{d}{dt}|\frac{\partial u_N}{\partial t}|^2 + \alpha|\nabla\frac{\partial u_N}{\partial t}|^2 \leq \theta_c|\frac{\partial u_N}{\partial t}|^2.$$

We have the following interpolation type inequality ([3]):

$$|\frac{\partial u_N}{\partial t}|_0^2 = |\frac{\partial u_N}{\partial t}|^2 \leq |\frac{\partial u_N}{\partial t}|_{-1}|\frac{\partial u_N}{\partial t}|_1.$$

Indeed, we have, for $u \in V_1$ such that $m(u) = 0$,

$$|u|^2 = (A_0^{\frac{1}{2}}A_0^{-\frac{1}{2}}u, u) = (A_0^{\frac{1}{2}}u, A_0^{-\frac{1}{2}}u) \leq |A_0^{\frac{1}{2}}u|\,|A_0^{-\frac{1}{2}}u| \leq |u|_{-1}|u|_1.$$

Consequently, we infer from (10) and Young's inequality ($ab \leq \frac{\alpha a^2}{2C_0} + \frac{C_0 b^2}{2\alpha}$) that

$$\frac{1}{2}\frac{d}{dt}\left(|K_N(u_N)|_1^2 + \beta|\frac{\partial u_N}{\partial t}|^2\right) + \frac{\alpha}{C_0}|\frac{\partial u_N}{\partial t}|_1^2$$

$$\leq \frac{\alpha}{2C_0}|\frac{\partial u_N}{\partial t}|_1^2 + \frac{C_0\theta_c^2}{2\alpha}|\frac{\partial u_N}{\partial t}|_{-1}^2.$$

Moreover, we have (with (11))

$$\begin{aligned} |\frac{\partial u_N}{\partial t}|_{-1}^2 &= (A_0^{-\frac{1}{2}}\frac{\partial u_N}{\partial t}, A_0^{-\frac{1}{2}}\frac{\partial u_N}{\partial t}) \\ &= (A_0^{\frac{1}{2}}K_N(u_N), A_0^{\frac{1}{2}}K_N(u_N)) \\ &= |K_N(u_N)|_1^2. \end{aligned}$$

Thus, we finally conclude that, for every $t > 0$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(|K_N(u_N)|_1^2 + \beta\,|\frac{\partial u_N}{\partial t}|^2\right) &\leq c'\,|K_N(u_N)|_1^2 \\ &\leq c'\left(|K_N(u_N)|_1^2 + \beta\,|\frac{\partial u_N}{\partial t}|^2\right),\end{aligned} \tag{17}$$

where $c' = \dfrac{C_0\theta_c^2}{2\alpha}$ does not depend on N.

3 Existence and uniqueness of solutions

In [9], the authors proved the existence of a unique solution belonging to $L^\infty(0,T;\,V_1) \cap L^2(0,T;\,V_2)$ for problem (2)-(4) when the initial condition u_0 belongs to V_1. However, this earlier result relies on a priori estimates which strongly depend on the polynomial degree N. Therefore, it does not allow to pass to the limit as N goes to infinity to prove the existence of a solution for problem (2)-(3). Thus, as in [4], we first have to derive a priori estimates independent of N for the solutions of problem (2)-(4).

3.1 A priori estimates independent of the polynomial degree

We establish here a priori estimates (independent on N) for the proof of the existence of solutions when the initial value u_0 is assumed to satisfy

$$u_0 \in V_1\,; \quad |u_0|_{L^\infty} \leq 1\,; \quad m(u_0) = \mu \in]-1,1[. \tag{18}$$

The function J_N being a Lyapounov function, we deduce from (18) that, $\forall t \geq 0$,

$$J_N(u_N(t)) \leq J_N(u_0) \leq J(u_0) < +\infty, \tag{19}$$

where $J(u_0) = \dfrac{\alpha}{2}|\nabla u_0|^2 - \dfrac{\theta_c}{2}\displaystyle\int_\Omega u_0^2\,dx + \dfrac{\theta}{2}\int_\Omega \phi(u_0)\,dx - \int_\Omega \gamma\,u_0\,dx$. Inequalities (15) and (19) then yield

$$\frac{\alpha}{2}|\nabla u_N|^2 \leq J(u_0) + \frac{3(2\theta_c+1)^2}{2\theta}|\Omega| + \frac{1}{2}|\gamma|^2,$$

and using (10), we have

$$|u_N(t)|_1 \leq c, \tag{20}$$

$$\|u_N(t)\|_1 \leq (c^2+\mu^2)^{\frac{1}{2}} \quad \forall\, t > 0. \tag{21}$$

In conclusion, u_N is bounded (uniformly with respect to N) in $L^\infty(\mathbb{R}^+;\,V_1)$.

Integrating (14) between 0 and T and applying (15) and (19), we find

$$\beta \int_0^T |\frac{\partial u_N(t)}{\partial t}|^2\, dt \,+\, \int_0^T |K_N(u_N(t))|_1^2\, dt \,\leq\, J(u_0) \,+\, \frac{3(2\theta_c+1)^2}{2\theta}\,|\Omega| \,+\, \frac{1}{2}\,|\gamma|^2.$$

Thus, we have

$$\int_0^T |\frac{\partial u_N(t)}{\partial t}|^2\, dt \quad\leq\quad c, \tag{22}$$

$$\int_0^T |K_N(u_N(t))|_1^2\, dt \quad=\quad \int_0^T \|\tilde{K}_N(u_N(t))\|_1^2\, dt \,\leq\, c, \tag{23}$$

where c is independent of N. Multiplying (17) by t, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(t\,|K_N(u_N)|_1^2 \,+\, \beta\, t\,|\frac{\partial u_N}{\partial t}|^2\right)\\ &\leq\, c\,t\,\left(|K_N(u_N)|_1^2 \,+\, \beta\,|\frac{\partial u_N}{\partial t}|^2\right) \,+\, |K_N(u_N)|_1^2 \,+\, \beta\,|\frac{\partial u_N}{\partial t}|^2,\end{aligned}$$

which yields, using Gronwall's Lemma (for $0\,\leq t\,\leq T$)

$$t\,|K_N(u_N)|_1^2 \,+\, \beta\, t\,|\frac{\partial u_N}{\partial t}|^2 \,\leq\, e^{cT}\,\left(\int_0^T |K_N(u_N(t))|_1^2\, dt \,+\, \beta\,\int_0^T |\frac{\partial u_N(t)}{\partial t}|^2\, dt\right).$$

Applying (22) and (23), we conclude that for every $t\,\in[0,T]$

$$\sqrt{t}\,|K_N(u_N(t))|_1 \,=\, \sqrt{t}\,\|\tilde{K}_N(u_N(t))\|_1 \,\leq\, c, \tag{24}$$

$$\sqrt{t}\,|\frac{\partial u_N(t)}{\partial t}| \,\leq\, c. \tag{25}$$

We now take the semiscalar product in V_0 of $K_N(u_N)$ by $\varphi_N(u_N)$. We then have, thanks to (11)

$$\begin{aligned}&\frac{\theta}{2}\,|\varphi_N(u_N)|_0^2 \,-\, \alpha\,(\Delta u_N,\, \varphi_N(u_N))_0 \,=\\ &(K_N(u_N),\, \varphi_N(u_N))_0 \,-\, \beta\,(\frac{\partial u_N}{\partial t},\, \varphi_N(u_N))_0 \,+\, \theta_c\,(u_N,\, \varphi_N(u_N))_0 \,+\, (\gamma,\, \varphi_N(u_N))_0.\end{aligned}$$

Since $m(\Delta\, u_N) \,= 0$, we note that

$$
\begin{aligned}
(\Delta u_N,\, \varphi_N(u_N))_0 &= (\Delta u_N,\, \varphi_N(u_N)) \\
&= -\int_\Omega \nabla u_N\, \varphi_N'(u_N)\, \nabla u_N\, dx \;\leq\; 0,
\end{aligned}
$$

and since $ab \leq \dfrac{\theta a^2}{16} + \dfrac{4b^2}{\theta}$ when $\theta > 0$, we have

$$
\begin{aligned}
\frac{\theta}{2}\,|\varphi_N(u_N)|_0^2 &\leq \frac{\theta}{4}\,|\varphi_N(u_N)|_0^2 + \frac{4}{\theta}\,|K_N(u_N)|_0^2 + \frac{4\beta^2}{\theta}\,|\frac{\partial u_N}{\partial t}|_0^2 \\
&\quad + \frac{4\theta_c^2}{\theta}\,|u_N|_0^2 + \frac{4}{\theta}\,|\gamma|_0^2 \\
&\leq c\,|K_N(u_N)|_1^2 + \frac{4\beta^2}{\theta}\,|\frac{\partial u_N}{\partial t}|^2 + \frac{4\theta_c^2}{\theta}\,|u_N|_0^2 + \frac{4}{\theta}\,|\gamma|_0^2.
\end{aligned}
$$

Therefore,

$$
t\,|\varphi_N(u_N)|_0^2 \leq c\,t\,|K_N(u_N)|_1^2 + c'\,t\,|\frac{\partial u_N}{\partial t}|^2 + c''\,t\,|u_N|_0^2 + c'''\,t\,|\gamma|_0^2.
$$

Thanks to (24), (25), (20), we finally find, for every $t \in [0,T]$

$$
\sqrt{t}\,\|\tilde{\varphi}_N(u_N(t))\|_0 = \sqrt{t}\,|\varphi_N(u_N(t))|_0 \leq c, \quad c \text{ independent of } N. \tag{26}
$$

Multiplying $K_N(u_N)$ by $A_0\,u_N$, we obtain

$$
\begin{aligned}
(K_N(u_N),\, A_0\,u_N) &= \beta\,(u_N,\, \frac{\partial\, u_N}{\partial t})_1 - \theta_c\,|u_N|_1^2 + \frac{\theta}{2}\,(\varphi_N(u_N),\, A_0\,u_N) \\
&\quad - (\gamma, u_N)_1 + \alpha\,(\Delta\, u_N,\, \Delta_0\, u_N).
\end{aligned}
$$

We also have (cf. Proposition 1)

$$
\begin{aligned}
(\varphi_N(u_N),\, A_0\,u_N) &= -(\Delta_0\,u_N,\, \varphi_N(u_N)) \\
&= (B_0^{\frac{1}{2}}\,\nabla\,u_N,\, B_0^{\frac{1}{2}}\,\varphi_N'(u_N)\,\nabla u_N) \\
&= \int_\Omega \varphi_N'(u_N)\,(B_0^{\frac{1}{2}}\,\nabla\,u_N)^2\,dx \;\geq\; 0,
\end{aligned}
$$

and

$$
\alpha\,(\Delta u_N,\, \Delta_0\,u_N) = \alpha\,|\nabla\,(B_0^{\frac{1}{2}}\,\nabla u_N)|^2.
$$

We thus obtain

$$
(K_N(u_N),\, A_0 u_N) \geq \beta\,(u_N,\, \frac{\partial u_N}{\partial t})_1 - \theta_c\,|u_N|_1^2 - (\gamma, u_N)_1 + \alpha\,|\nabla\,(B_0^{\frac{1}{2}}\,\nabla u_N)|^2,
$$

which can be rewritten

$$\begin{aligned}\alpha\,|\nabla\,(B_0^{\frac{1}{2}}\,\nabla u_N)|^2\,+\,\beta\,(u_N,\,\frac{\partial u_N}{\partial t})_1 \quad &\leq \quad (K_N(u_N),\,A_0\,u_N)\,+\,\theta_c\,|u_N|_1^2\,+\,(\gamma,u_N)_1 \\ &\leq \quad (K_N(u_N),u_N)_1\,+\,\theta_c\,|u_N|_1^2\,+\,|\gamma|_1\,|u_N|_1.\end{aligned}$$

We finally have

$$\alpha\,|\nabla\,(B_0^{\frac{1}{2}}\,\nabla u_N)|^2\,+\,\frac{\beta}{2}\,\frac{d}{dt}\,|u_N|_1^2 \leq \frac{1}{2}\,|K_N(u_N)|_1^2\,+\,(\theta_c\,+1)\,|u_N|_1^2\,+\,\frac{1}{2}\,|\gamma|_1^2. \tag{27}$$

Then, integrating the last identity between 0 and T and applying (18), (20) and (23), we obtain

$$\begin{aligned}&\alpha\int_0^T\,|\nabla\,(B_0^{\frac{1}{2}}\,\nabla u_N)|^2\,dt \\ &\leq\,\frac{1}{2}\int_0^T\,|K_N(u_N)|_1^2\,dt\,+\,(\theta_c\,+1)\int_0^T\,|u_N|_1^2\,dt\,+\,\frac{\beta}{2}\,|u_0|_1^2\,+\,\frac{1}{2}\,|\gamma|_1^2\,T \\ &\leq c, \qquad c \text{ independent of } N,\end{aligned}$$

which yields (cf. Proposition 3)

$$\int_0^T\,\|u_N(t)\|_2^2\,dt\,\leq\,c, \quad c\,\text{ independent of } N. \tag{28}$$

We also infer from (27), (20), (24), that $\forall\,t\,\in\,[0,T]$

$$\begin{aligned}\alpha\,t\,|\nabla\,(B_0^{\frac{1}{2}}\,\nabla u_N)|^2 \quad &\leq \quad \frac{t}{2}\,|K_N(u_N)|_1^2\,+\,(\theta_c\,+1)\,t\,\,|u_N|_1^2\,+\,\frac{t}{2}\,|\gamma|_1^2\,+\,\beta\,t\,|(u_N,\,\frac{\partial u_N}{\partial t})_1| \\ &\leq \quad c\,+\,\beta\,t\,|(A_0\,u_N,\,\frac{\partial u_N}{\partial t})| \\ &\leq \quad c\,+\,\beta\,t\,|\frac{\partial u_N}{\partial t}|\,|u_N|_2,\end{aligned}$$

where c depends on T, but not on N.

Furthermore, we have

$$\alpha\,t\,(|\nabla\,(B_0^{\frac{1}{2}}\,\nabla u_N)|^2\,+\,|u_N|^2)\,\leq\,c'\,+\,\beta\,t\,|\frac{\partial u_N}{\partial t}|\,|u_N|_2,$$

and Proposition 3 yields that there exists C_1 independent of N such that

$$\begin{aligned} \alpha\, t\, C_1\, \|u_N\|_2^2 &\leq c' + \beta\, t\, |\frac{\partial u_N}{\partial t}|\, |u_N|_2 \\ &\leq c' + \frac{\beta^2\, t}{2\alpha C_1}\, |\frac{\partial u_N}{\partial t}|^2 + \frac{\alpha t C_1}{2}\, \|u_N\|_2^2. \end{aligned}$$

Thanks to (25), we finally deduce that

$$\sqrt{t}\, \|u_N(t)\|_2 \leq c'' \quad \forall\, t \in [0, T], \tag{29}$$

where c'' is independent of N.

3.2 Existence and uniqueness of the solution

Now that a priori estimates independent of N have been established, we can have existence results for problem (2)-(3). Note that this problem can also be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} + A_0\, K(u) = 0, \\ K(u) = \beta\, \dfrac{\partial u}{\partial t} - \alpha \Delta u - \theta_c\, u + \dfrac{\theta}{2}\, \varphi(u) - \gamma, \\ u_{|t=0} = u_0, \\ u \text{ is } \Omega - \text{ periodic.} \end{cases} \tag{30}$$

Theorem 1 *Let u_0 satisfy (18). Then, equation (30) possesses a unique solution u which belongs to*

$$\mathcal{C}^0([0, T]\,;\, V_1) \cap L^2(]0, T[\,;\, V_2) \quad \forall\, T > 0.$$

Moreover, $K(u) \in L^2(]0, T[\,;\, V_1)$, and

$$|u(t)|_{L^\infty} \leq 1 \quad \forall t \geq 0,$$

$$\{\, x \in \Omega\,;\, |u(x, t)| = 1\,\} \text{ has measure } 0 \text{ for } t > 0.$$

Proof :

(i) We first prove the uniqueness of the solution.

Let u and v be 2 solutions of problem (30) corresponding to the initial data u_0 and v_0 respectively. Then, $w = u - v$ satisfies

$$\frac{\partial}{\partial t}\, (I + \beta\, A_0)\, w - \alpha\, A_0(\Delta w) - \theta_c\, A_0\, w + \frac{\theta}{2}\, A_0\, (\varphi(u) - \varphi(v)) = 0. \tag{31}$$

Taking the scalar product of (31) by $A_0^{-1} w$, we obtain

$$\frac{1}{2}\, \frac{d}{dt} |w|_{-1}^2 + \frac{\beta}{2} \frac{d}{dt} |w|_0^2 - \alpha\, (\Delta w,\, w)_0 - \theta_c\, |w|_0^2 + \frac{\theta}{2}\, (\varphi(u) - \varphi(v),\, u - v)_0 \leq 0.$$

Since we consider periodic boundary conditions, we have

$$\begin{aligned}(\Delta w,\, w)_0 &= (\Delta w,\, w)\\ &= -\int_\Omega (\nabla w)^2\, dx \,\leq 0.\end{aligned}$$

We also have

$$\begin{aligned}(\varphi(u)-\varphi(v),\, u-v)_0 &= (\varphi(u)-\varphi(v),\, u-v) \,-\, (\varphi(u)-\varphi(v),\, m(u)-m(v))\\ &\geq -\,(\varphi(u)-\varphi(v),\, m(u)-m(v)),\end{aligned}$$

thanks to (9) and to the monotonicity of the function φ. We thus find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|_{-1}^2 \,+\, \frac{\beta}{2}\frac{d}{dt}|w|_0^2 &\leq \theta_c\,|w|_0^2 \,+\, \frac{\theta}{2}\,(\varphi(u)-\varphi(v),\, m(u)-m(v))\\ &\leq c\,(|w|_{-1}^2 \,+\, \beta\,|w|_0^2) \,+\, \frac{\theta}{2}\,(\varphi(u)-\varphi(v),\, m(u)-m(v)).\end{aligned}$$

If $u_0 = v_0$, then $m(u(t)) = m(v(t)),\ \forall t > 0$, and consequently

$$\frac{1}{2}\,\frac{d}{dt}\,(|w|_{-1}^2 \,+\, \beta|w|_0^2) \,\leq\, c\,(|w|_{-1}^2 \,+\, \beta|w|_0^2),$$

hence the uniqueness, using Gronwall's Lemma.

(ii) Concerning the existence of the solution, we proceed as in [4]. However, for the sake of completeness, we give the details of the proof. First, we show that $\sqrt{t}\,m(\varphi_N(u_N(t)))$ is bounded, $\forall t\, \in [0,T]$, in order to deduce, according to (26), that $\sqrt{t}\,\varphi_N(u_N(x,t))$ is bounded in $L^\infty(0,T\,;\,V_0)$.

We assume that $\sqrt{t}\,m(\varphi_N(u_N(t)))$ is unbounded, and consequently that there exist 2 sequences $(t_k)_{k\in\mathbb{N}}$ and $(N_k)_{k\in\mathbb{N}}$ such that

$$\begin{cases} t_k \underset{k\to+\infty}{\longrightarrow} t^* \quad \in [0,T], \\ \\ \sqrt{t_k}\,m(\varphi_{N_k}(u_{N_k}(t_k))) \underset{k\to+\infty}{\longrightarrow} +\infty. \end{cases} \tag{32}$$

Let $k_1\, \in \mathbb{N}$ be such that $m(\varphi_{N_k}(u_{N_k}(t_k))) > 0, \quad \forall k \geq k_1$. For $k \geq k_1$, we set

$$F_k = \{\, x\in\Omega\,;\, \varphi_{N_k}(u_{N_k}(x,t_k)) \,<\, \frac{m(\varphi_{N_k}(u_{N_k}(t_k)))}{2}\,\} \quad \text{and} \quad E_k \,= \Omega\setminus F_k.$$

From (26) we have

$$|F_k|\,\frac{m(\varphi_{N_k}(u_{N_k}(t_k)))^2}{4} \,\leq\, \int_{F_k} \tilde{\varphi}_{N_k}(u_{N_k}(t_k))^2\, dx \,\leq\, \frac{C\,|\Omega|}{t_k}.$$

Then, thanks to (32) and (20) we see that

$$|F_k| \underset{k\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad \int_{F_k} u_{N_k}(x,t_k)\,dx \underset{k\to+\infty}{\longrightarrow} 0.$$

Furthermore,

$$\int_{E_k} u_{N_k}(x,t_k)\,dx \;\geq\; \varphi_{N_k}^{-1}\left(\frac{m(\varphi_{N_k}(u_{N_k}(t_k)))}{2}\right)|E_k|.$$

Then, letting $k \to +\infty$, we obtain

$$|\Omega| \left(\limsup_{k\to+\infty} m(u_{N_k}(t_k))\right) \;\geq\; |\Omega|,$$

which is in contradiction with

$$m(u_{N_k}(t_k)) \;=\; m(u_0) \;\in]-1,1[, \quad \forall\, t_k \;\in\; [0,T].$$

The case $\sqrt{t_k}\, m(\varphi_{N_k}(u_{N_k}(t_k))) \underset{k\to+\infty}{\longrightarrow} -\infty$ is analogous and is thus omitted.

In conclusion, $\sqrt{t}\, m(\varphi_N(u_N(t)))$ is bounded $\forall t \in [0,T]$ and

$$\sqrt{t}\, \|\varphi_N(u_N(t))\|_0 \;\leq\; c, \quad \forall t \;\in [0,T], \tag{33}$$

where c is independent of N.

Next, we infer from (22) and (28) that u_N is bounded in $L^2(]0,T[\,;V_2)$ and that $\dfrac{\partial u_N}{\partial t}$ is bounded in $L^2(]0,T[\,;V_0)$. Then, we deduce from classical compactness theorems the existence of a function $u \;\in\; \mathcal{C}^0([0,T];V_0)$ and a subsequence $(u_m)_{m\in\mathbb{N}}$ converging to u in $L^2(]0,T[\,;V_2)$ weak and in $L^2(]0,T[\,;V_1)$ strong. Moreover, by (33), there exists a function φ^* such that

$$\sqrt{t}\, \varphi_m(u_m(x,t)) \underset{m\to+\infty}{\longrightarrow} \sqrt{t}\, \varphi^* \quad \text{in } L^\infty(0,T;V_0) \quad \text{weak } *\,.$$

We now have to prove that $\varphi^* \;=\; \varphi(u)$. To this aim, we need auxiliary results. For an arbitrary small $\varepsilon \;\in]0,1[$ and for every $t \;\in]0,T[$, we denote by $E_\varepsilon^N(t)$ (resp. $\chi_\varepsilon^N(t)$, resp. $meas(E_\varepsilon^N(t))$) the set $E_\varepsilon^N(t) \;=\; \{\, x\in\Omega\,;\, |u_N(x,t)| > 1-\varepsilon\,\}$ (resp. its caracteristic function, resp. its measure). We have in fact $meas(E_\varepsilon^N(t)) \;=\; \int_{E_\varepsilon^N(t)} dx \;=\; \int_\Omega \chi_\varepsilon(t)\,dx$.

We also denote by $E_\varepsilon(t)$ (resp. $\chi_\varepsilon(t)$, resp. $meas(E_\varepsilon(t))$) the set $\{\, x\in\Omega\,;\, |u(x,t)| \;>\; 1-\varepsilon\,\}$ (resp. its caracteristic function, resp. its measure). We then have

$$\begin{aligned}
\left\{\int_{E_\varepsilon^N(t)} \varphi_N(u_N(x,t))\,dx\right\}^{\frac12} \quad &\geq \quad |E_\varepsilon^N(t)|^{\frac12} \left\{\inf_{x\in E_\varepsilon^N(t)} \left(\sum_{k=0}^{N} \frac{u_N(x,t)^{2k+1}}{2k+1}\right)^2\right\}^{\frac12} \\
&\geq \quad |E_\varepsilon^N(t)|^{\frac12} \inf_{x\in E_\varepsilon^N(t)} \left(\sum_{k=0}^{N} \frac{u_N(x,t)^{2k+1}}{2k+1}\right) \\
&\geq \quad |E_\varepsilon^N(t)|^{\frac12} \sum_{k=0}^{N} \frac{(1-\varepsilon)^{2k+1}}{2k+1}.
\end{aligned}$$

Thus, (33) implies that

$$\begin{aligned} |E_\varepsilon^N(t)|^{\frac{1}{2}} &\leq \frac{c}{\sqrt{t}\,\displaystyle\sum_{k=0}^{N}\frac{(1-\varepsilon)^{2(k+1)}}{2k+1}} \\ &\leq \frac{c'}{\sqrt{t}\;\ln(\dfrac{2-\varepsilon}{\varepsilon})}. \end{aligned} \tag{34}$$

Letting $N \to +\infty$, we deduce from Fatou's Lemma that

$$\begin{aligned} |E_\varepsilon(t)| &= \int_\Omega \chi_\varepsilon(t)\,dt \\ &\leq \int_\Omega \liminf_{N\to+\infty} \chi_\varepsilon^N(t)\,dt \\ &\leq \liminf_{N\to+\infty} |E_\varepsilon^N(t)|. \end{aligned}$$

Letting $\varepsilon \to 0$, it follows from (34) that

$$\text{Meas}\,\{\, x \in \Omega,\ |u(x,t)| \,\geq\, 1 \,\} \,=0. \tag{35}$$

Let us now study the term $|\varphi_m(u_m(x,t)) - \varphi(u_m(x,t))|$. Thanks to (35), we know that there exist $m_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$|u_m(x,t)| \,\leq\, 1-\delta \quad \text{a.e. in } \Omega,\ \forall m \geq m_0.$$

Thus,

$$\begin{aligned} |\varphi_m(u_m(x,t)) - \varphi(u_m(x,t))| &\leq \sum_{k=m+1}^{+\infty} \frac{|u_m(x,t)|^{2k+1}}{2k+1} \\ &\leq \frac{1}{2m+3}\sum_{k=m+1}^{+\infty}(1-\delta)^{2k+1} \\ &\leq \frac{c\,(1-\delta)^{2m+3}}{2m+3}, \end{aligned}$$

where c is independent of m. This implies

$$|\varphi_m(u_m(x,t)) - \varphi(u_m(x,t))| \underset{m\to+\infty}{\longrightarrow} 0,$$

and therefore

$$\begin{aligned} |\varphi_m(u_m(x,t)) - \varphi(u(x,t))| &\leq |\varphi_m(u_m(x,t)) - \varphi(u_m(x,t))| \,+\, |\varphi(u_m(x,t)) - \varphi(u(x,t))| \\ &\underset{m\to+\infty}{\longrightarrow} 0. \end{aligned}$$

We thus have proved that

$$\varphi_m(u_m(x,t)) \to \varphi(u(x,t)) \quad \text{a.e. in } \Omega.$$

Using Lebesgue's Theorem, we finally conclude that

$$\varphi_m(u_m) \to \varphi(u) \quad in\ \mathcal{D}'((0,T)\times\Omega).$$

Thus, $\varphi^* = \varphi(u)$ and u is a solution of (30). There remains to prove that u belongs to $\mathcal{C}^0([0,T]\,;\,V_1)$.
According to (29), we already know that u belongs to $\mathcal{C}^0([\tau,T]\,;\,V_1)$, $\tau > 0$. We refer the reader to [4] and [5] for the study of the continuity at $t = 0$.

■

4 Existence and dimension of the global attractor

For $s = 0,1,2$ and $\sigma \geq 0$, we denote by V_s^σ the space

$$V_s^\sigma = \{u \in V_s\,;\,|m(u)| \leq \sigma\},$$

which we endow with the norm of V_s. We derive from (12) and Theorem 1 the existence of the semigroup:

$$\begin{array}{rcll} S(t)\,:\,V_1^\sigma & \to & V_1^\sigma, & \sigma < 1, \\ u_0 & \mapsto & u(t). & \end{array}$$

We easily prove that this semigroup is continuous in the space $V_1^\sigma \cap \{u \in L^\infty\,;\,|u|_{L^\infty} \leq 1\}$ endowed with the norm of V_1. We first establish a priori estimates in order to prove the existence of bounded absorbing sets in V_s^σ, $s = 0,1,2$, $\sigma < 1$, for equation (30). We then derive the existence of the global attractor in V_0^σ and V_1^σ. Finally, we show that equation (30) possesses an exponential attractor in V_0^σ ; hence we conclude that the global attractor has finite fractal dimension.

4.1 Time uniform estimates independent of the polynomial degree

We multiply equation (11) by $A_0^{-1}u_N$ to obtain

$$\frac{1}{2}\frac{d}{dt}|u_N|_{-1}^2 + (K_N(u_N),\,u_N)_0 \leq 0,$$

where

$$(K_N(u_N),\,u_N)_0 = \frac{\beta}{2}\frac{d}{dt}|u_N|_0^2 - \theta_c\,|u_N|_0^2 - \alpha\,(\Delta u_N,\,u_N)_0 + \frac{\theta}{2}(\varphi_N(u_N),\,u_N)_0 - (\gamma, u_N)_0.$$

Moreover,

$$
\begin{aligned}
(\Delta u_N,\, u_N)_0 &= (\Delta u_N, u_N),\\
|u_N|_0 &\le |u_N|,
\end{aligned}
$$

and (cf. Lemma 1)

$$
\begin{aligned}
(\varphi_N(u_N), u_N)_0 &= (\varphi_N(u_N), u_N - \mu)\\
&= 2\sum_{k=0}^{N}\left(\int_\Omega \frac{u_N^{2k+2}}{2k+1}\,dx - \int_\Omega \frac{u_N^{2k+1}}{2k+1}\,\mu\,dx\right).
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_N|_{-1}^2 + \frac{\beta}{2}\frac{d}{dt}|u_N|_0^2 - \theta_c\,|u_N|^2 + \alpha\,|\nabla u_N|^2 + \theta\sum_{k=0}^{N}\int_\Omega \frac{u_N^{2k+2}}{2k+1}\,dx\\
&\quad\le \theta\sum_{k=0}^{N}\int_\Omega \frac{u_N^{2k+1}}{2k+1}\,\mu\,dx + (\gamma, u_N) - m(\gamma)\,\mu\,|\Omega|.
\end{aligned} \tag{36}
$$

Applying successively Hölder's and Young's inequalities, we find

$$
\begin{aligned}
\sum_{k=0}^{N}\int_\Omega \frac{u_N^{2k+1}}{2k+1}\,\mu\,dx &\le \sum_{k=0}^{N}\frac{\mu}{2k+1}\left(\int_\Omega u_N^{2k+2}\,dx\right)^{\frac{2k+1}{2k+2}}|\Omega|^{\frac{1}{2k+2}}\\
&\le \sum_{k=0}^{N}\frac{1}{2k+1}\left(\frac{2k+1}{2k+2}\int_\Omega u_N^{2k+2}\,dx + \frac{1}{2k+2}\,\mu^{2k+2}\,|\Omega|\right)\\
&\le \sum_{k=0}^{N}\frac{1}{2k+2}\int_\Omega u_N^{2k+2}\,dx + \sum_{k=0}^{N}\frac{\mu^{2k+2}}{(2k+1)(2k+2)}\,|\Omega|.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\sum_{k=0}^{N}\int_\Omega \frac{u_N^{2k+2}}{2k+1}\,dx - \sum_{k=0}^{N}\int_\Omega \frac{u_N^{2k+2}}{2k+2}\,dx &= \sum_{k=0}^{N}\int_\Omega \frac{u_N^{2k+2}}{(2k+1)(2k+2)}\,dx\\
&= \frac{1}{2}\int_\Omega \phi_N(u_N)\,dx.
\end{aligned}
$$

Equation (36) then reduces to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|u_N|_{-1}^2 + \beta\,|u_N|_0^2\right) - \theta_c\,|u_N|^2 + \alpha\,|\nabla u_N|^2 - (\gamma, u_N) + \frac{\theta}{2}\int_\Omega \phi_N(u_N)\,dx\\
&\quad\le \sum_{k=0}^{N}\frac{\mu^{2k+2}}{(2k+1)(2k+2)}\,\theta\,|\Omega| - m(\gamma)\,\mu\,|\Omega|,
\end{aligned}
$$

which can be rewritten equivalently as

$$\frac{1}{2}\frac{d}{dt}\left(|u_N|_{-1}^2 + \beta\,|u_N|_0^2\right) + \frac{\alpha}{2}|\nabla u_N|^2 - \frac{\theta_c}{2}|u_N|^2 + J_N(u_N)$$
$$\leq \sum_{k=0}^{N} \frac{\mu^{2k+2}}{(2k+1)(2k+2)}\,\theta\,|\Omega| - m(\gamma)\,\mu\,|\Omega|. \tag{37}$$

Thanks to (15) and to the positivity of ϕ_N, we deduce that

$$\frac{1}{2}\frac{d}{dt}\left(|u_N|_{-1}^2 + \beta\,|u_N|_0^2\right) + \alpha\,|\nabla u_N|^2 - \frac{3(2\theta_c+1)^2}{2\theta}|\Omega| - \frac{1}{2}|\gamma|^2$$
$$\leq \sum_{k=0}^{N} \frac{\mu^{2k+2}}{(2k+1)(2k+2)}\,\theta\,|\Omega| - m(\gamma)\,\mu\,|\Omega|.$$

It follows from Lemma 1 and from Poincaré-type inequalities (see [10]) that

$$\frac{1}{2}\frac{d}{dt}\left(|u_N|_{-1}^2 + \beta\,|u_N|_0^2\right) + \frac{\alpha}{2C_0C_2}|u_N|_{-1}^2 + \frac{\alpha}{2C_0C_3}|u_N|_0^2$$
$$\leq \sum_{k=0}^{N} \frac{\mu^{2k+2}}{(2k+1)(2k+2)}\,\theta\,|\Omega| - m(\gamma)\,\mu\,|\Omega| + \frac{1}{2}|\gamma|^2 + \frac{3(2\theta_c+1)^2}{2\theta}|\Omega|,$$

which yields (since $|\mu| \leq \sigma$)

$$\frac{1}{2}\frac{d}{dt}\left(|u_N|_{-1}^2 + \beta\,|u_N|_0^2\right) + C_4\left(|u_N|_{-1}^2 + \beta\,|u_N|_0^2\right) \leq D_{\sigma,|\gamma|}^N, \tag{38}$$

where

$$D_{\sigma,|\gamma|}^N = \sum_{k=0}^{N} \frac{\sigma^{2k+2}}{(2k+1)(2k+2)}\,\theta\,|\Omega| + \frac{1}{2}|\gamma|^2 + \frac{3(2\theta_c+1)^2}{2\theta}|\Omega| + \sigma\,|\gamma|\,|\Omega|^{\frac{1}{2}}, \tag{39}$$

and

$$C_4 = \min\left(\frac{\alpha}{2C_0C_2}, \frac{\alpha}{2\beta C_0C_3}\right) > 0.$$

Remark 2 For $\sigma < 1$, we denote by $D_{\sigma,|\gamma|}$ the limit, as $N \to +\infty$, of $D_{\sigma,|\gamma|}^N$. We note that $D_{\sigma,|\gamma|}$ is bounded for every $0 < \sigma < 1$ and that

$$D_{\sigma,|\gamma|} \longrightarrow 0 \quad \text{when} \quad |\Omega| \to 0.$$

Thanks to (38) and Gronwall's Lemma, we finally obtain, for every $t > 0$

$$|u_N(t)|_{-1}^2 + \beta\,|u_N(t)|_0^2 \leq \left(|u_0|_{-1}^2 + \beta\,|u_0|_0^2\right)e^{-2C_4t} + \frac{D_{\sigma,|\gamma|}^N}{C_4}(1 - e^{-2C_4t}).$$

We thus deduce the

Proposition 4 *For every $\rho > 0$ and for every $u_0 \in V_0$ satisfying*

$$|u_0|_0 \leq \rho \quad \text{and} \quad |m(u_0)| \leq \sigma, \tag{40}$$

there exists $T_1(\rho) > 0$ such that the solution of (11) satisfies

$$|u_N(t)|_0 \leq \sqrt{\frac{2}{\beta\, C_4}\, D^N_{\sigma,|\gamma|}}\,, \quad \forall t \geq T_1(\rho). \tag{41}$$

Furthermore, if $\sigma < 1$, the solution of (30) satisfies

$$|u(t)|_0 \leq \sqrt{\frac{2}{\beta\, C_4}\, D_{\sigma,|\gamma|}}\,, \quad \forall t \geq T_1(\rho). \tag{42}$$

Remark 3 A consequence of Proposition 4 is the existence of a bounded absorbing set in $V_0^\sigma = \{\, u \in V_0\, ;\, |m(u)| \leq \sigma \,\}$ for problem (11) and in V_0^σ, $\sigma < 1$, in the case of equation (30).

Next, we integrate (37) from t to $t+1$. We obtain

$$\int_t^{t+1} J_N(u_N(s))\, ds \leq \frac{\theta_c}{2} \int_t^{t+1} |u_N(s)|^2\, ds - m(\gamma)\, \mu\, |\Omega|$$
$$+ \sum_{k=0}^{N} \frac{\mu^{2k+2}}{(2k+1)(2k+2)}\, \theta\, |\Omega| + \frac{1}{2}\, |u_N(t)|^2_{-1} + \frac{\beta}{2}\, |u_N(t)|^2_0.$$

Since J_N is decreasing along the trajectories and $|u_N(s)|^2 = |u_N(s)|_0^2 + \mu^2\, |\Omega|$, Proposition 4 yields

$$J_N(u_N(t+1)) \leq C_5\, D^N_{\sigma,|\gamma|} + \frac{\theta_c}{2}\, \sigma^2\, |\Omega|, \qquad \forall\, t > T_1(\rho). \tag{43}$$

Thanks to (15) and (43), we deduce that, for every $t > T_1(\rho) + 1$

$$\frac{\alpha}{2}\, |\nabla u_N(t)|^2 \leq C_5\, D^N_{\sigma,|\gamma|} + \frac{3\,(2\theta_c + 1)^2}{2\theta}\, |\Omega| + \frac{1}{2}\, |\gamma|^2 + \frac{\theta_c}{2}\, \sigma^2\, |\Omega|,$$

and we have the

Proposition 5 *Let u_0 satisfy assumption (40). The solution of problem (11) satisfies*

$$|u_N(t)|_1 \leq \sqrt{C_6\, D^N_{\sigma,|\gamma|}}\,, \qquad \forall\, t \geq T_1(\rho) + 1. \tag{44}$$

Moreover, if $\sigma < 1$, the solution of (30) satisfies

$$|u(t)|_1 \leq \sqrt{C_6\, D_{\sigma,|\gamma|}}\,, \qquad \forall\, t \geq T_1(\rho) + 1. \tag{45}$$

Now, we integrate (14) from t to $t+1$ and use (15) and (43). We obtain, $\forall t \geq T_1(\rho) + 1$,

$$\int_t^{t+1} \left(|K_N(u_N(s))|_1^2 + \beta \, |\frac{\partial u_N(s)}{\partial s}|^2 \right) ds \leq (C_5 + 1) \, D^N_{\sigma,|\gamma|} + \frac{\theta_c}{2} \sigma^2 \, |\Omega|.$$

Thus, the uniform Gronwall's Lemma applied to (17) yields

$$|K_N(u_N(t))|_1^2 + \beta \, |\frac{\partial u_N(t)}{\partial t}|^2 \leq C_7 \, D^N_{\sigma,|\gamma|}, \qquad \forall t \geq T_1(\rho) + 2. \tag{46}$$

Then, we infer from (27), (46) and Proposition 5 that

$$\alpha \, |\nabla \, (B_0^{\frac{1}{2}} \, \nabla u_N)|^2 + \beta \, (\frac{\partial u_N}{\partial t}, \, u_N)_1 \leq C_8 \, D^N_{\sigma,|\gamma|} + \frac{1}{2} \, |\gamma|_1^2.$$

With arguments similar to those used for the derivation of (29), we find the

Proposition 6
If u_0 satisfies assumption (40) and u_N is a solution of (11), then

$$\|u_N(t)\|_2^2 \leq C_9 \, (D^N_{\sigma,|\gamma|} + \frac{1}{2} \, |\gamma|_1^2), \quad \forall t > T_1(\rho) + 2. \tag{47}$$

Moreover, if $\sigma < 1$ and u is the solution of (30),

$$\|u(t)\|_2^2 \leq C_9 \, (D_{\sigma,|\gamma|} + \frac{1}{2} \, |\gamma|_1^2), \quad \forall t > T_1(\rho) + 2. \tag{48}$$

Remark 4 It can be noticed that the constant C_0 introduced in Lemma 1, as well as C_1 introduced in Proposition 3, are independent of $|\Omega|$. Moreover, it is well known that C_2 and C_3 (Poincaré-like constants) remain bounded (and can be chosen greater than a strictly positive constant) whenever $|\Omega|$ is. Consequently, it can be shown that it is also the case for the constants C_4, ..., C_9.

4.2 Existence and dimension of the global attractor

Propositions 4, 5, 6 lead to the existence of bounded absorbing sets in $V_s^\sigma, \sigma < 1$, and $s = 0, 1, 2$ for Equation (30). Thanks to the compact injections $V_2 \hookrightarrow V_1 \hookrightarrow V_0$, we thus have (cf. [10]) the

Theorem 2 *The semigroup $S(t)$ possesses the global attractor $\mathcal{A}_\sigma$ in V_0^σ and in V_1^σ, $\sigma < 1$. Furthermore, this attractor is bounded in V_2^σ.*

In the last part of this section, we prove that, when Ω is small, the global attractor $\mathcal{A}_\sigma$ has finite fractal dimension by proving the existence of an exponential attractor. To do so, we proceed as in [6], but we first need some preliminary results. We denote by $\mathrm{B}_{2,\sigma}$ a bounded absorbing set in V_2^σ, and set

$$X_\sigma = \overline{\bigcup_{t \geq T_2} S(t)\,\mathrm{B}_{2,\sigma}},$$

where T_2 is such that $S(t)\,\mathrm{B}_{2,\sigma} \subset \mathrm{B}_{2,\sigma}, \quad \forall t \geq T_2$. By construction, X_σ is clearly positively invariant by $S(t)$ and compact in $L^2(\Omega)$. Moreover, we have the

Proposition 7 *Let σ belong to $]-1,1[$. If $|\Omega|$ is small, there exists a constant $0 < \delta < 1$ such that*

$$\|u(t)\|_{L^\infty} \leq \delta < 1, \quad \forall\, u \in X_\sigma.$$

Proof: Thanks to Proposition 6, Remark 2 and Remark 4, we know that the right hand side of (48) tends to 0 when $|\Omega|$ tends to 0. Thus, we deduce that if $|\Omega|$ is small enough, there exists $0 < \delta < 1$ such that

$$\|u(t)\|_{L^\infty} \leq C\,\|u(t)\|_2 \leq \delta < 1, \quad \forall\, t > T_1(\rho) + 2,$$

where C depends on Ω and is bounded if Ω is bounded.

■

Next, we set $L = I - \beta\,div\,(B_0\,\nabla u)$ and note that

$$L\,(u - \Delta u) = \beta\,div(\,B_0\,\nabla\,\Delta u) + u - \Delta u - \beta\,div(B_0\,\nabla u).$$

Thus, equation (2) can be rewritten as

$$\frac{\partial L\,u}{\partial t} + \frac{\alpha}{\beta}\,L\,(u - \Delta u) + \frac{\alpha}{\beta}\,(\Delta u - u) + \alpha\,div(B_0\,\nabla u)$$
$$- div(B_0\,\nabla\,f^{'}(u)) + div(B_0\,\nabla\gamma) = 0,$$

or, equivalently

$$\frac{\partial\,u}{\partial t} + \frac{\alpha}{\beta}\,(u - \Delta u) + \frac{\alpha}{\beta}\,L^{-1}\,(\Delta u - u) + \alpha\,L^{-1}\,div(B_0\,\nabla u)$$
$$- L^{-1}\,div(B_0\,\nabla\,f^{'}(u)) + L^{-1}\,div(B_0\,\nabla\gamma) = 0.$$

The operators L and $\dfrac{\partial}{\partial x_i}$ clearly commute, so do L^{-1} and $\dfrac{\partial}{\partial x_i}$. Thus, equation (2) can be rewritten in the form

$$\begin{cases} \dfrac{\partial u}{\partial t} + A\,u + R(u) = 0, \\ \\ A = \dfrac{\alpha}{\beta}\,(I - \Delta), \\ \\ R(u) = \dfrac{\alpha}{\beta}\,(\,\Delta\, L^{-1}u - L^{-1}u) + \alpha\, div(B_0\, \nabla\, (L^{-1}u)) \\ \qquad\qquad - div(B_0\, \nabla\, (L^{-1} f^{'}(u)) + div(B_0\, \nabla\, (L^{-1}\gamma)). \end{cases} \tag{49}$$

We have the

Proposition 8 *If $|\Omega|$ is small enough, we have, for every $u,\, v \in X_\sigma$*

$$|R(u) - R(v)| \le c(\sigma)\, |A^{\frac{1}{2}}\,(u - v)|.$$

<ins>Proof:</ins>
For $u,\, v \in X_\sigma$, we have

$$\begin{aligned} |R(u) - R(v)| &\le \frac{\alpha}{\beta} |\Delta\, L^{-1}(u - v) - L^{-1}(u - v)| \\ &\quad + \alpha\, |div(B_0 \nabla (L^{-1}u)) - div(B_0 \nabla (L^{-1}v))| \\ &\quad + |div(B_0 \nabla (L^{-1} f^{'}(u))) - div(B_0\, \nabla (L^{-1} f^{'}(v)))| \\ &\le c\, \|L^{-1}(u - v)\|_{H^2} + c^{'}\, \|L^{-1}\, (f^{'}(u) - f^{'}(v))\|_{H^2}. \end{aligned}$$

Since

$$\|v\|_{H^2} \le c\, |L\, v|, \quad \forall v \in H^2_{per}(\Omega),$$

which yields

$$\|L^{-1}\, v\|_{H^2} \le c\, |v|, \quad \forall v \in L^2(\Omega),$$

we find

$$|R(u) - R(v)| \le c^{'}\, |u - v| + c^{''}\, |f^{'}(u) - f^{'}(v)|.$$

Since u and v are in X_σ, Proposition 7 holds as soon as $|\Omega|$ is small enough. We then have $|u|_{L^\infty} \le \delta < 1$ and $|v|_{L^\infty} \le \delta < 1$. Thus, we infer

$$|f^{'}(u) - f^{'}(v)| \le c(\sigma)\, |u - v|,$$

and consequently

$$\begin{aligned} |R(u) - R(v)| &\le c^{'}(\sigma)\, |u - v| \\ &\le c^{''}(\sigma)\, |A^{\frac{1}{2}}\,(u - v)|. \end{aligned}$$

■

We then have the

Theorem 3 *We assume that $|\Omega|$ is small enough. Then, the semigroup associated with (30) possesses an exponential attractor $\mathcal{M}_\sigma$ in X_σ. Consequently, the global attractor $\mathcal{A}_\sigma$, $\sigma < 1$, has finite fractal dimension.*

Proof: The existence of an exponential attractor $\mathcal{M}_\sigma \subset X_\sigma$ follows from Proposition 8 and the arguments of [6]. Moreover, since an exponential attractor $\mathcal{M}_\sigma$ contains $\mathcal{A}_\sigma$ and, by definition, has finite fractal dimension, we see that $\mathcal{A}_\sigma$ has finite fractal dimension.

■

Remark 5 We can obtain similar results for the polynomial nonlinearity f_N as in [4]. However, we are not able to obtain here, as in [4], an upper bound on the dimension of the global attractor that is uniform with respect to the degree of the polynomial.

Remark 6 We note that the result of Theorem 3 cannot be extended to the classical Cahn-Hilliard equation (for which $\beta = 0$) studied in [4]. In that case, we can only study the finite dimensionality of the global attractor on spaces of the form $\tilde{V}_s^\sigma = \{u \in V_s^\sigma;\, m(u) = \sigma\}$, which is less interesting from the physical point of view.

Remark 7 Proceeding as in [4], we can prove the upper semicontinuity of the attractor $\mathcal{A}_\sigma^N$ (corresponding to the polynomial free energy f_N) to $\mathcal{A}_\sigma$ as $N \to +\infty$.

Remark 8 We note that since $\theta < \theta_c$, $f''(0) < 0$. Therefore, $D_\sigma = \{x \in]-\sigma, \sigma[,\, f''(x) < 0\}$, $\sigma < 1$, is nonempty, and, proceeding as in [9], we can prove that the dimension of the global attractor is bounded from below by $\sup\limits_{x \in D_\sigma} \operatorname{card}\{k,\, \lambda_k < -\dfrac{f''(x)}{\alpha}\}$. We thus deduce that the dimension of the global attractor $\mathcal{A}_\sigma$, $\sigma < 1$, tends to $+\infty$ as α tends to 0 (for a fixed domain Ω).

5 Comments on *Neumann-like* boundary conditions

Another classical set of boundary conditions for the Cahn-Hilliard equation (see for instance [10]), of Neumann type, reads

$$(B_0\,\nu).\nabla u = (B_0\,\nu).\nabla\Delta u = 0 \quad \text{on } \partial\Omega, \tag{50}$$

where ν is the unit outer normal vector to $\partial\Omega$. We assume here that Ω is a bounded regular domain of $\mathbb{R}^n$, $n = 1, 2$ or 3.

The main difference with the previous case concerns the norm on V_2. Indeed, we only have, for the boundary conditions (50), assuming that u and v are regular enough

$$(div(B_0\, \nabla \Delta u),\, v\,) \; = \; \Big(\Delta u,\, div(B_0\, \nabla v) \Big). \tag{51}$$

If we wanted to proceed as in the periodic case, we would find

$$(div(B_0\, \nabla \Delta u),\, v\,) \; = \; (\nabla B_0^{\frac{1}{2}} \nabla u,\, \nabla B_0^{\frac{1}{2}} \nabla v) \; - \; \int_{\partial\Omega} \frac{\partial (B_0^{\frac{1}{2}} \nabla u)}{\partial \nu} \,.\, (B_0^{\frac{1}{2}} \nabla v)\, d\sigma,$$

and we would not be able to get rid of nor absorb the boundary term.

Now, the difficulty is that the term $(\Delta u,\, div(B_0 \nabla u))$ is not positive in general (it is positive however when $B_0 \, = \, \kappa_0\, I,\; \kappa_0 > 0$). We shall write instead $B_0 \, = \, \kappa_0\, I + \, B_0^{'},\; \kappa_0 > 0$ (for the sake of simplicity, we take $\kappa_0 = 1$) and

$$(\Delta\, u,\, div(B_0 \nabla u)) \; = \; |\Delta u|^2 \, - \, (\Delta u,\, div(B_0^{'} \nabla u)), \tag{52}$$

and we shall absorb, assuming that $B_0^{'}$ is small and that its diagonal terms vanish, the second term in the right-hand-side of (52). To do so, we need to prove that $(|\Delta u|^2 \, + \, c\, |u|^2)^{\frac{1}{2}},\; c > 0,$ is a norm on $V_2 \; = \; \{ u \, \in \, H^2(\Omega),\, (B_0 \nu).\nabla u \; = \, 0 \text{ on } \partial\Omega \, \}$ that is equivalent to the usual H^2-norm, with constants that are independent on B_0 (the functions belonging to V_2 depend on B_0 through the boundary conditions). We set

$$b_0 \; = \; \max_{i \neq j} \, |b_{ij}|,$$

where $B_0 \; = \; (b_{ij})$. Thanks to classical regularity results on second order elliptic problems, we see that the norm $(|div(B_0 \nabla .)|^2 \, + \, c\, |.|^2)^{\frac{1}{2}}$ is a norm on V_2 that is equivalent to the usual H^2-norm. A careful look at the proofs (see [1]) shows that all the constants depend either on the coercivity constant of B_0 or on max $|b_{ij}|$. Therefore, if b_0 is small enough, we can take all the constants independent of b_0. We then write, for $u \, \in \, V_2$

$$\begin{aligned} |\Delta u|^2 \, + \, c |u|^2 \; &= \; |\, div(B_0 \nabla u) - div(B_0^{'} \nabla u)|^2 \, + c\, |u|^2 \\ &\geq \; \frac{1}{2}\, |div(B_0 \nabla u)|^2 \, + \, c\, |u|^2 \, - \, |div(B_0^{'} \nabla u)|^2 \\ &\geq \; c\, \|u\|^2_{H^2(\Omega)}, \end{aligned}$$

$c > 0$ being independent of b_0 (and thus of B_0) if b_0 is small enough.

Now, when $b_0 = 0$, we proceed exactly as in the previous sections, except that when considering the equivalence of norms on V_2, we are not able to prove that the constants do not depend on $|\Omega|$ (except for $n = 1$). Thus, the results of subsection 4.2 will only be valid in one space dimension.

When $b_0 \neq 0$ and is small (in particular as above), we can obtain the same estimates as in subsection 3.1. In particular, we are then able to obtain the existence of solutions for the logarithmic nonlinearity (3). Unfortunately, we are not able to prove the uniqueness of solutions, the reason being that we cannot integrate by parts in the term $(\Delta u,\, v)$ (to obtain $-(\nabla u,\, \nabla v)$).

References

[1] A. Agmon, S. Douglis and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, II. *Comm. Pure Appl. Math. 12 (1959), 623-727 ; 17 (1964), 35-92.*

[2] J.W. Cahn and J.E. Hilliard. Free energy of a nonuniform system, part I : Interfacial free energy. *J. Chem. Phys. 2 (1958), 258-267.*

[3] A. Carrive, M. Miranville and A. Piétrus. The Cahn-Hilliard equation for deformable elastic continua. *Preprint (1998).*

[4] A. Debussche and L. Dettori. On the Cahn-Hilliard equation with a logarithmic free energy. *Nonlinear Analysis*, TMA 24 (10) (1995), 1491-1514.

[5] C. Dupaix. A singularly perturbed phase field model with a logarithmic nonlinearity. *To appear in Nonlinear Analysis.*

[6] C. Eden, A. Foias, B. Nicolaenko, and R. Temam. Exponential attractors for dissipative evolution equations. *Research in Applied Mathematics, Masson (1994).*

[7] C.M. Elliot and S. Luckhauss. A generalized equation for phase separation of a multi-component mixture with interfacial free energy. *Preprint.*

[8] M. Gurtin. Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance. *Physica D 92 (1996), 178-192.*

[9] A. Miranville, A. Piétrus and J.M. Rakotoson. Dynamical aspect of a generalized Cahn-Hilliard equation based on a microforce balance. *Asymptotic Analysis 16 (1998), 315-345.*

[10] R. Temam. Infinite dimensional dynamical systems in mechanics and physics. *Springer-Verlag, New-York, 2nd. ed. (1997).*

Liste des prépublications

99-1 Monique Jeanblanc et Nicolas Privault. A complete market model with Poisson and Brownian components.

99-2 Laurence Cherfils et Alain Miranville. Generalized Cahn-Hilliard equations with a logarithmic free energy.